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APPROXIMATE SOLUTION FOR NONLINEAR SYSTEM OF FREDHOLM-VOLTERRA HAMMERSTEIN INTEGRAL EQUATIONS

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Abstract

A numerical method for solving nonlinear system of Fredholm-Volterra Hammerstein integral equations of second kind is presented. This method is based on replacement of the unknown functions by truncated series of well known Chebyshev expansion of functions. The quadrature formula which we use to calculate integral terms can be estimated by Fast Fourier Transform (FFT). Also, convergence and rate of convergence are given. The numerical examples and the number of operations show the advantages of this method to all other recent works which use operational matrices (with huge number of operations) or solving special case of nonlinear system of integral equations.

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1. Introduction

In this paper, we present a computational method for solving a system of nonlinear Fredholm-Volterra integral equations of Hammerstein type:

$$x_i(s) = y_i(s) + \lambda_1 \sum_{j=1}^{n} \int_0^s K_{ij}(s, t) F(x_j(t)) dt$$

$$+ \lambda_2 \sum_{j=1}^{n} \int_0^1 K'_{ij}(s, t) G(x_j(t)) dt,$$

$$i = 1, \ldots, n, \quad 0 \leq s, t \leq 1. \tag{1}$$

Several numerical methods for approximating the solution of linear and nonlinear integral equations are known [1-18]. The classical method of successive approximation for nonlinear Fredholm integral equations was introduced in [18]. Hat basis functions and Chebyshev wavelets were used for approximate solution of special case of nonlinear system of Volterra or Fredholm integral equations in [4] and [5] with $F(u) = u^m$ and $G(u) = u^k$ for m, k positive integers. Brunner in [7] applied a collocation-type method and Ordokhani in [17] applied rationalized Haar function to nonlinear Volterra-Fredholm integral equations. Some special nonlinear systems of integral equations have been solved in [11-13] and [16]. A collocation type method was developed in [14]. A variation of the Nystrom method was presented in [15]. Also, more recent papers solve the special case of nonlinear systems of integral equations with operational matrices of m power [9]. Borzabadi et al. in [6] converted the nonlinear Fredholm integral equation to an optimal control problem and then used a linear programming to solve the problem. Orthogonal functions and polynomials receive attention in dealing with various problems. One of those is integral equation. The main characteristic of using orthogonal basis is that it reduces these problems to solving a system of nonlinear algebraic equations by truncated approximating series

$$x(t) \simeq x_N(t) = \sum_{i=0}^{N-1} c_i \phi_i(t).$$

We use the above truncated series, where the elements $\phi_0(t)$, $\phi_1(t), \ldots, \phi_{N-1}(t)$ are the orthogonal basis functions defined on a certain interval $[a, b]$. Here we choose $\phi_i(t)$, as Chebyshev functions of the first kind, and also Chebyshev collocation points and the Clenshaw-Curtis quadrature rule [10].

1.1. Chebyshev approximation of solution

We consider the nonlinear system of Fredholm-Volterra Hammerstein integral equation (SFVH), equation (1).

A function $x_i(t) \in L^2([0, 1])$ may be expanded as

$$x_i(t) = \sum_{m=0}^{\infty} c_{im} T_m(t), \tag{2}$$

for $i = 1, \ldots, n$, where $c_{im} = (x_i(t), T_m(t))$, in which $(\cdot, \cdot)$ denotes the inner product in $L^2([0, 1])$. If we consider truncated series in (2), then we obtain

$$x_i(t) \simeq \sum_{m=0}^{N} c_{im} T_m(t) = \mathbf{C}_i^T \mathbf{T}, \tag{3}$$

where $\mathbf{C}_i$ and $\mathbf{T}$ are matrices given by

$$\mathbf{C}_i^T = [c_{i0}, c_{i1}, \ldots, c_{iN}],$$

$$\mathbf{T}(t) = [T_0(t), T_1(t), \ldots, T_N(t)]^T, \tag{4}$$

where $T_n(t)$, $0 \le n \le N$ are Chebyshev polynomials of the first kind of degree n which are orthogonal with respect to the weight function $\omega(t) = 1/\sqrt{1 - t^2}$ on the interval $[-1, 1]$ and satisfy the following recursive formula:

$$T_0(t) = 1,\ T_1(t) = t,\ T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t),\ m = 1, 2, \ldots .$$

2. The Approximate Solution of Nonlinear SFVH Integral Equation

Consider the nonlinear system of integral equations (1). At first, we approximate $x_i(t)$ for $i = 1, \ldots, n$ as

$$x_i(t) \simeq \mathbf{C}_i^T \mathbf{T}(t), \tag{5}$$

then we substitute this approximation into equation (1) to get

$$\mathbf{C}_i^T \mathbf{T}(s) = y_i(s) + \lambda_1 \sum_{j=1}^{n} \int_0^s K_{ij}(s, t) F(\mathbf{C}_j^T \mathbf{T}(t)) dt$$

$$+ \lambda_2 \sum_{j=1}^{n} \int_0^1 K'_{ij}(s, t) G(\mathbf{C}_j^T \mathbf{T}(t)) dt,$$

$$i = 1, \ldots, n, \quad 0 \leq s, t \leq 1. \tag{6}$$

In order to use Gaussian integration formula for equation (6), we transfer the intervals $[0, s_l]$ and $[0, 1]$ into interval $[-1, 1]$ by transformations

$$\tau_1 = \frac{2}{s_l} t - 1, \quad \tau_2 = 2t - 1.$$

For Chebyshev polynomials, we consider the collocation points

$$s_l = \cos\left(\frac{l\pi}{N}\right), \quad l = 0, 1, \ldots, N. \tag{7}$$

Let

$$H_{ij}^f(s, t) = K_{ij}(s, t) F(\mathbf{C}_j^T \mathbf{T}(t)), \quad H_{ij}^g(s, t) = K'_{ij}(s, t) G(\mathbf{C}_j^T \mathbf{T}(t)).$$

Using collocation points (7) in transformed equation (6), we get

$$\mathbf{C}_i^T \mathbf{T}(s_l) = y(s_l) + \lambda_1 \frac{s_l}{2} \int_{-1}^{1} \sum_{j=1}^{n} H_{ij}^f\left(s_l, \frac{s_l(\tau_1 + 1)}{2}\right) d\tau_1$$

$$+ \frac{\lambda_2}{2} \int_{-1}^{1} \sum_{j=1}^{n} H_{ij}^g\left(s_l, \frac{(\tau_2 + 1)}{2}\right) d\tau_2 \tag{8}$$

for $i = 1, \ldots, n.$ Now we use Clenshaw-Curtis quadrature formula [10] to get

$$\mathbf{C}_i^T \mathbf{T}(s_l) = y(s_l)$$

$$+ \sum_{j=1}^{n} \sum_{k=0}^{N}{}'' w_k \left[\lambda_1 \frac{s_l}{2} H_{ij}^f \left(s_l, \frac{s_l (s_k + 1)}{2} \right) + \frac{\lambda_2}{2} H_{ij}^g \left(s_l, \frac{(s_k + 1)}{2} \right) \right] \tag{9}$$

for $l = 0, 1, \ldots, N,$ and where

$$w_k = \frac{4}{N} \sum_{\text{even } n=0}^{N}{}'' \frac{1}{1 - n^2} \cos\left(\frac{nk\pi}{N} \right), \tag{10}$$

and double prime means that the first and the last terms are halved. The system (9) consists of nonlinear equations with unknown vector with elements of $\mathbf{C}_i$ as $\mathbf{C} = [c_{00}, c_{01}, .., c_{0N}, \ldots, c_{N0}, c_{N1}, \ldots, c_{NN}]^T$ which can be solved by usual iterative method such as Newton's method or simplex method. The Fast Fourier Transform (FFT) technique can be used to evaluate the summation part in (9) in $O(N \log N)$ operations. In fact, equation (10) for weights w_k can also be viewed as the discrete cosine transformation of the vector v with entries:

$$v_n = \begin{cases} 2/(1 - n^2), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

The weights w_k can therefore also be computed directly in $O(N \log N)$ operations, this will be the faster computation when we integrate functions on (8) using the same value of N. Therefore, one of the good advantages of this method to all recent methods which use m-power of operational matrices with operation cost of $O(nmN^3)$, [9, 16] (for the special case $(x(t))^m$ as the nonlinear term of integral equations) is that the method is reasonable in cost and also very stable against rounding errors as we will see in the next section.

3. Convergence and Error Analysis

In this section, we discuss the convergence of the Chebyshev polynomial method for the nonlinear system of integral equations (1). The following proposition is fundamental to the convergence analysis:

Proposition 1. *Let* $x(t) \in H^k(-1, 1)$ (*Sobolev space*) *and* $T_N(x(t)) = \sum_{m=0}^{N} c_m T_m(t)$ *be the best approximation polynomial of* $x(t)$ *in* L_2 *norm. Thus, the truncation error is*: $\| x(t) - T_N(x(t)) \|_{L_2[-1,1]} \leq C_0 N^{-k} \| x(t) \|_{H^k(-1,1)}$, *where* C_0 *is a positive constant, which depends on the selected norm and is independent of* $x(t)$ *and N is the degree of Chebyshev polynomials* (*proof* [7]).

From Proposition 1, it is concluded that approximation rate of Chebyshev polynomials is N^{-k}. If $x_i(t)$ is approximated by $x_{iN}(t) = \sum_{m=0}^{N} c_{im} T_m(t)$ and we find $\bar{c}_{im}$ $\left(\bar{c}_{im} \text{ is an approximation of } c_{im} \text{ and } \bar{x}_{iN} = \sum_{m=0}^{N} \bar{c}_{im} T_m(t) \right)$, then for $t \in [-1, 1]$, we have

$$\| x_i(t) - \bar{x}_{iN}(t) \| \leq C_0 N^{-k} \| x(t) \| + C_2 (N+1)^{1/2} N^{-k+1}.$$

Also, from [10] and by using $(N+1)$-point closed Gauss-Chebyshev rule for approximation of c_{im}, we realize [10], $| c_{im} - \bar{c}_{im} | \leq C_1 N^{-k+1}$ and we can easily verify the accuracy of the method. Given that the truncated Chebyshev series in equation (5) is an approximation of (1), it must have approximately satisfied these equations. Thus, for each $s_l \in [0, 1]$:

$$E(s_l) = \mathbf{C}_i^T \mathbf{T}(s_l) - y(s_l) - \lambda_1 \sum_{j=1}^{n} \int_0^{s_l} k_{ij}(s_l, t) F(\mathbf{C}_j^T \mathbf{T}(t)) dt$$

$$- \lambda_2 \sum_{j=1}^{n} \int_0^1 k'_{ij}(s_l, t) G(\mathbf{C}_j^T \mathbf{T}(t)) dt \approx 0, \quad i = 1, \ldots, n.$$

If $\max E(s_l) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(s_l)$ at each point s_l becomes smaller than the prescribed 10^{-k}.

Also, in the $L_\infty[0, 1]$, we can propose as follow:

Let $(C[0, 1], \|\cdot\|)$ be the Banach space of all continuous functions on interval $[0, 1]$ with $\| x_i(t) \|_\infty = \max_{s\in[0,1]} | x_i(t) |$ for $i = 1, \ldots, n$. Assume for $i, j = 1, \ldots, n$, $| K_{ij}(s, t) | \le M_1$ and $| K'_{ij}(s, t) | \le M_2$ and suppose the nonlinear terms $F(t)$ and $G(t)$ are satisfied in Lipschitz conditions:

$$| F(u) - F(v) | \le L_1 | u - v |, \quad | G(u) - G(v) | \le L_2 | u - v |.$$

Moreover, define $\alpha = | \lambda_1 | M_1 L_1 + | \lambda_2 | M_2 L_2$. If $x_i(s)$ and $x_{iN}(s)$ show the exact and approximate solutions of ith equation (1), respectively, then we have:

Theorem 1. *The solution of nonlinear system of Fredholm-Volterra Hammerstein integral equation* (1) *by using Chebyshev polynomials converges if* $0 < \alpha < 1/n$; *in other words, for* $i = 1, \ldots, n$,

$$\lim_{N\to\infty} \| x_i(s) - x_{iN}(s) \| = 0.$$

Proof.

$$\| x_i(s) - x_{iN}(s) \|_\infty$$

$$= \max_{s\in[0,1]} | x_i(s) - x_{iN}(s) |$$

$$= \max_{s\in[0,1]} \left| \lambda_1 \sum_{j=1}^{n} \int_0^s K_{ij}(s, t)(F(x_j(t)) - F(x_{jN}(s)))dt \right|$$

$$+ \max_{s\in[0,1]} \left| \lambda_2 \sum_{j=1}^{n} \int_0^1 K'_{ij}(s, t)(G(x_j(t)) - G(x_{jN}(s)))dt \right|$$

$$\le \sum_{j=1}^{n} | \lambda_1 | M_1 L_1 s \| x_j(s) - x_{jN}(s) \|_\infty$$

$$+ \sum_{j=1}^{n} | \lambda_2 | M_2 L_2 \| x_j(s) - x_{jN}(s) \|_\infty$$

$$= \sum_{j=1}^{n} (|\lambda_1|M_1L_1s + |\lambda_2|M_2L_2)\| x_j(s) - x_{jN}(s) \|_\infty$$

$$\le \sum_{j=1}^{n} (|\lambda_1|M_1L_1 + |\lambda_2|M_2L_2)\| x_j(s) - x_{jN}(s) \|_\infty$$

$$\Rightarrow \| x_i(s) - x_{iN}(s) \|_\infty \le \sum_{j=1}^{n} \alpha \| x_j(s) - x_{jN}(s) \|_\infty.$$

If we write the last relation for $i = 1, ..., n$ and add up them, then we obtain

$$\sum_{i=1}^{n} \| x_i(s) - x_{iN}(s) \|_\infty \le \sum_{j=1}^{n} n\alpha \| x_j(s) - x_{jN}(s) \|_\infty$$

so

$$\sum_{i=1}^{n} (1 - n\alpha)\| x_i(s) - x_{iN}(s) \|_\infty \le 0.$$

According to this equation, if we choose $0 < \alpha < 1/n,$ then we have:

$$\lim_{N \to \infty} \| x_i(s) - x_{iN}(s) \| = 0$$

so the proof is completed.

4. Numerical Examples

In this section, we consider some nonlinear systems of Fredholm-Volterra Hammerstein integral equations which have been solved with other recent methods (with simple nonlinear terms but huge number of basis functions and huge operations) such as Hat basis [4] and Chebyshev wavelet functions [5] and solve them by introduced method.

Example 1. Consider following system of nonlinear Fredholm integral equations:

$$\begin{cases} x_1(s) = y_1(s) + \int_0^1 e^{s+t} x_1^3(t)dt + \int_0^1 te^s x_2^2(t)dt, \\ x_2(s) = y_2(s) + \int_0^1 s\sin(t)dt + \int_0^1 \cos(s-t)x_2^2(t)dt, \end{cases} \tag{11}$$

with suitable $y_1(s), y_2(s)$ and the exact solutions $x_1(s) = e^{-s}$ and $x_2(s) = s.$ Table 1a shows the absolute error $\| x_i - x_{iN} \|_2$ for $i = 1, 2$ in some points of

$[0, 1]$ for method of [4] where x_{iN} is the approximate solution, and N stands for the number of used basis functions in the approximate solution. Also, Table 1b shows the absolute error $\| x_i - x_{iN} \|_2$ in some points of $[0, 1]$ for presented method, where $N+1$ stands for the number of basis functions.

Table 1a. Exact and approximated solution of Example 1 by method [4] ($N =$ the number of Hat basis functions)

s	Exact $x_1(s)$	$x_{1N}(N = 20)$	Exact $x_2(s)$	$x_{2N}(N = 20)$
0.0	1	1.00002	0.0	0.000509
0.2	0.818731	0.818755	0.2	0.199430
0.4	0.670320	0.670349	0.4	0.399392
0.6	0.548812	0.548848	0.6	0.599379
0.8	0.449329	0.449373	0.8	0.799393
1.0	0.367879	0.367933	1.0	0.999432

Table 1b. Absolute error of exact and approximated solution of Example 1 by presented method $e_i = \| x_i(s) - x_{iN}(s) \|$ ($N =$ the number of basis functions)

s	$e_1(N = 4)$	$e_2(N = 4)$	$e_1(N = 8)$	$e_2(N = 8)$
0.0	$0.001e-4$	$0.266e-4$	$0.125e-7$	$0.701e-8$
0.2	$0.156e-3$	$0.325e-4$	$0.135e-7$	$0.772e-8$
0.4	$0.349e-3$	$0.398e-4$	$0.164e-7$	$0.607e-8$
0.6	$0.567e-3$	$0.486e-4$	$0.225e-7$	$0.405e-8$
0.8	$0.797e-3$	$0.594e-4$	$0.298e-7$	$0.571e-8$
1.0	$0.010e-3$	$0.725e-4$	$0.358e-7$	$0.145e-8$

Table 1b shows decreasing absolute error by introduced method.

Example 2. Consider nonlinear system of Volterra integral equations [4]:

$$\begin{cases} x_1(s) = y_1(s) + \int_0^s (s-t)^3 x_1^3(t)\,dt + \int_0^s (s-t)^2 x_2^2(t)\,dt, \\ x_2(s) = y_2(s) + \int_0^s (s-t)^4 x_1^2(t)\,dt + \int_0^s (s-t)^3 x_2^2(t)\,dt \end{cases} \tag{12}$$

with suitable $y_1(s)$, $y_2(s)$ and the exact solutions $x_1(s) = e^{-s}$ and $x_2(s) = s$. Table 2 shows the absolute error $e_i = \| x_i(s) - x_{iN}(s) \|$ in some points of $[0, 1]$ for the presented method and method of [4]. N stands for the number of basis functions.

Table 2. Maximum absolute error of exact and approximated solution of Example 2 by presented method, $\max_{i=1,2} e_i = \| x_i(s) - x_{iN}(s) \|$ ($N =$ the number of basis functions)

s	max e_i new method			Method of [4]
	$(N = 4)$	$(N = 5)$	$(N = 6)$	$N = 32$
0.0	$0.001e-4$	$0.267e-4$	$0.123e-7$	$0.000e-3$
0.2	$0.145e-3$	$0.326e-4$	$0.132e-7$	$0.104e-3$
0.4	$0.323e-3$	$0.389e-4$	$0.146e-7$	$0.086e-3$
0.6	$0.532e-3$	$0.468e-4$	$0.252e-7$	$0.113e-3$
0.8	$0.787e-3$	$0.549e-4$	$0.349e-7$	$0.180e-3$
1.0	$0.009e-3$	$0.752e-4$	$0.354e-7$	$0.250e-3$

The comparison between presented method and the method of [4] shows the accuracy of presented method with using much less basis functions and operations than do by method of [4].

Example 3. Consider nonlinear SFVH integral equation

$$\begin{cases} x_1(s) = y_1(s) - \int_0^1 (s+t)e^{x_1(t)}dt + \int_0^s \sin(s-t)x_2^2(t)dt, \\ x_2(s) = y_2(s) - \int_0^1 (s-t)\cos(x_1(t))dt + \int_0^s \cos(s-t)x_2^3(t)dt \end{cases} \tag{13}$$

with $y_1(s) = s - \cos(s) + s(e-1) - s^2 + 3$ and $y_2(s) = 2\sin(s) + (s-1)\sin(s) - s + 2\sin^2(1/2)$ and the exact solution $x_1(s) = x_2(s) = s$. Table 3 shows the maximum absolute error $|x_i - x_{iN}|$ in some points of $[0, 1]$, N stands for the number of basis functions in the approximate solution.

Table 3. Maximum absolute error of exact and approximated solution of Example 3 by introduced method $\max_{i=1,2} e_i = \| x_i(s) - x_{iN}(s) \|$ ($N =$ the number of basis functions)

s	$N = 5$	$N = 6$	$N = 7$	$N = 8$
0.0	$0.423e-3$	$0.302e-5$	$0.306e-6$	$0.011e-9$
0.1	$0.324e-4$	$0.321e-5$	$0.305e-6$	$0.201e-9$
0.2	$0.203e-4$	$0.527e-5$	$0.304e-6$	$0.311e-9$
0.3	$0.311e-4$	$0.248e-5$	$0.311e-6$	$0.421e-9$
0.4	$0.012e-4$	$0.301e-5$	$0.336e-6$	$0.133e-9$
0.5	$0.203e-4$	$0.801e-5$	$0.391e-6$	$0.518e-9$
0.6	$0.581e-4$	$0.625e-5$	$0.485e-6$	$0.419e-9$
0.7	$0.620e-4$	$0.207e-5$	$0.620e-6$	$0.255e-9$
0.8	$0.522e-4$	$0.302e-5$	$0.785e-6$	$0.307e-9$
0.9	$0.701e-4$	$0.522e-5$	$0.953e-6$	$0.388e-9$
1.0	$0.347e-4$	$0.301e-5$	$0.107e-6$	$0.244e-9$

Table 3 shows that by increasing the number of basis functions we obtain better results.

5. Conclusion

As shown by numerical examples, the method introduced here can be simply implemented to general nonlinear system of Hammerstein integral equations of the second kind. The advantages are much less implementations and fast computations which is comparable with all huge cost early methods with simple nonlinear terms. Moreover, convergence and the rate of the convergence are shown either.

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