



THE SIRD EPIDEMIAL MODEL

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Abstract

We modified the classical SIR model of Kermack and McKendrick by assuming that an individual can be born infected. We analysed the model from the point of view of symmetry and singularity analysis. In this study, we provide a demonstration of the integrability of the model to present an explicit solution.

1. Introduction

Using the classical (but still highly relevant) models, we considered here the total population was deemed to be constant. If a small group of infected individuals was introduced into a large population, then the basic problem was to describe the spread of the infection within the population as a function of time. This, of course, depended on a variety of circumstances, including the actual disease involved. Notably, as a first attempt towards modelling

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directly transmitted diseases, we expounded general assumptions which under the circumstances were reasonable.

When considering a disease which, after recovery, conferred immunity and which, if lethal, included deaths, the dead individuals were still counted. We assumed that the disease was such that the population could be divided into three distinct classes: the susceptible, S , who can catch the disease; the infected, I , who has the disease and can transmit it; and the removed class, R , namely, those who have had the disease or had recovered immune or isolated until re-covered; and the death class, D , who died. Most childhood diseases, such as measles, have a removed and death class. The disease can be represented diagrammatically by

$$S \rightarrow I \rightarrow R \rightarrow D.$$

In this section, we looked at the simplest model in this class, which dates back to Kermack and McKendrick's classic paper of 1927 [4]. The results are basic to the mathematical epidemial modelling. We modelled the movement between the classes as follows:

$$\begin{aligned}\dot{S} &= -\beta SI + \gamma I - \mu_1 S + \nu_1 S, \\ \dot{I} &= \beta SI - (\alpha + \gamma)I - \mu_2 I + \nu_2 I, \\ \dot{R} &= \alpha I - \mu_3 R + \nu_3 R, \\ \dot{D} &= \mu_1 S + \mu_2 I + \mu_3 R\end{aligned}\tag{1}$$

in which the overdot denoted differentiation with respect to time, $S(t)$ was the susceptible component of the population, $I(t)$ was the infected component of the population, $R(t)$ was the recovery component of the population, $D(t)$ was the dead component of the population, μ_1 was the proportionate death rate of the susceptibles, μ_2 was the proportionate death rate of the infected, μ_3 was the proportionate death rate of the recovered, ν_1 was the proportionate birth rate of the susceptibles, ν_2 was the proportionate

birth rate of the infected, v_3 was the proportionate birth rate of the recovered, α was the rate of those who had recovered were becoming immune, γ was the rate of susceptibles becoming ill and β was the rate of the infected who were becoming susceptibles.

In Section 2, we reduced the four dimensional first order differential equations to one dimensional second order ordinary differential equation. In Section 3, we subjected the reduced equation to the Painlevé analysis in its present state as a raw dynamical system, as it were, and also as a single second-order ordinary differential equation which had ‘sanitised’ by the removal of mathematically distracting parameters. We performed a Lie symmetry analysis of the reduced equation in Section 4. In Section 5, we established the general solutions of this ordinary differential equation and we then performed numerical simulation of the SIRD epidemial model.

2. Reduction to a Second Order Ordinary Differential Equation

In this section, we reduced the model equation (1) to a second order ordinary differential equation. Notably, the first two equations in system (1) were independent of R and D , and one could solve R once I was known, hence it could be excluded in any subsequent analysis of the system.

From (1b), we had

$$S = \frac{1}{\beta} \frac{\dot{I}}{I} + \frac{\alpha + \gamma + \mu_2 - v_2}{\beta}. \quad (2)$$

The derivative of (2) gave

$$\dot{S} = \frac{1}{\beta} \left(\frac{\ddot{I} - \dot{I}^2}{I^2} \right). \quad (3)$$

The substitution of (2) and (3) into (1a) gave

$$\ddot{I} - \dot{I}^2 = -\beta \dot{I}^2 - \beta(\alpha + \gamma + \mu_2 - v_2)I^3 + \gamma\beta I^3 - \mu_1 \dot{I}$$

$$\begin{aligned}
& -\mu_1(\alpha + \gamma + \mu_2 - \nu_2)I^2 + \nu_1\dot{I} \\
& + \nu_1(\alpha + \gamma + \mu_2 - \nu_2)I^2.
\end{aligned} \tag{4}$$

We have after some arrangement

$$\begin{aligned}
\ddot{I} - \dot{I}^2 + \beta\dot{I}^2 &= -\beta(\alpha + 2\gamma + \mu_2 - \nu_2)I^3 + (\nu_1 - \mu_1)\dot{I} \\
&+ (\nu_1 - \mu_1)(\alpha + \gamma + \nu_2 - \mu_2)I^2.
\end{aligned} \tag{5}$$

We may achieve a cosmetic simplification by means of the rescaling

$$I = \frac{u}{\beta}. \tag{6}$$

The substitution of (6) into (5) gave

$$u\ddot{u} - \dot{u}^2 + \dot{u}u^2 + a\dot{u}u + (b + \gamma)u^3 + abu^2 = 0 \tag{7}$$

with

$$\mu_1 - \nu_1 = a, \tag{8}$$

$$\alpha + \gamma + \mu_2 - \nu_2 = b. \tag{9}$$

3. Singularity Analysis

Singularity analysis was initiated by Kowalevski [5] in her determination of the third integrable case of the Euler equations for the top and was in large measure developed by the French School founded by Paul Painlevé during the period of the La Belle Époque [9]. There have been significant contributions since then. For less technical works devoted to the methodology the interested reader is referred to the text of Tabor [11] and the report of Ramani et al. [10]. The essence of the singularity analysis of a differential equation (system of ordinary differential equations) was the determination of the existence of isolated movable polelike singularities with which one could develop a Laurent expansion containing arbitrary constants

equal in number to the order of the system. The location of then singularity was determined by the initial conditions of the system. An equation of moderately, or more, complicated structure can possess more than one polelike singularity – a fortiori in the case of systems of differential equations which can have many patterns of singularities. Notably, it is conventional wisdom that a Laurent expansion with the requisite number of arbitrary constants must exist for all possibilities. However, there exists a counter example [6] for which one pattern of singular behaviour possessed a Laurent expansion with the correct number of arbitrary constants and the second does not, but had a ‘peculiar’ solution [7] of the type already discussed by Ince [3] many years ago. Nevertheless, the closed form general solution of the system was manifestly analytic.

The application of the analysis is usually quite algorithmic. Indeed, it is standard practice to apply the ARS algorithm [1], although there are instances, of particular relevance to the analysis of systems of first-order ordinary linear differential equations typically encountered in the mathematical modelling of epidemics, in which the subtler approach advocated by Hua et al. [2] is preferred. First, we outlined the standard algorithm and secondly the alternative approach. We considered an autonomous system of first-order ordinary differential equations

$$\Phi_i(x, \dot{x}, \sigma) = 0, \quad i = 1, n, \quad (10)$$

where x represented the n dependent variables, the overdot differentiation with respect to the independent variable t and σ the set of parameters which invariably seemed to accompany a system arising in the course of the mathematical modelling of natural phenomena. We assume that the n functions Φ_i were polynomials in the dependent variables x and linear in the first derivatives. The assumptions which made were not completely necessary, but they do reflect the reality found in models and simplified our theoretical discussion.

The first step was to determine the leading-order behaviour of the dependent variables of the system. We substituted $x_i = \sigma_i \tau^{P_i}$, $i = 1, n$, where $\tau = t - t_0$ and t_0 was the putative location of the movable singularity, into the system (10) and compared the resulting power.

The formal Laurent series may be obtained by substituting

$$u = \sum_{i=0}^{\infty} \sigma_i \tau^{i-1} \quad (11)$$

into (7) to obtain

$$\begin{aligned} & \sigma_i \sigma_j (i-1)(i-2) \tau^{i+j-4} - (i-1)(i-1) \sigma_i \sigma_j \tau^{i+j-4} \\ & + (i-1) \sigma_i \sigma_j \sigma_k \tau^{i+j+k-4} \\ & = a(i-1) \sigma_i \sigma_j \tau^{i+j-3} - (b+\gamma) \sigma_i \sigma_j \sigma_k \tau^{i+j+k-3} + ab \sigma_i \sigma_j \tau^{i+j-2} \end{aligned} \quad (12)$$

for $i = j = k = 0, 1, 2, \dots$. At τ^{-4} , we required

$$2\sigma_0^2 - \sigma_0^2 - \sigma_0^3 = 0.$$

Therefore,

$$\sigma_0 = 1.$$

We moved to the next power, τ^{-3} , and established that

$$2\sigma_0 \sigma_1 - \sigma_0 \sigma_1 - \sigma_0^2 \sigma_1 = -a\sigma_0^2 - (b+\gamma)\sigma_0^3.$$

Since $\sigma_0 = 1$, this gave an arbitrary σ_1 only if

$$a = \gamma + b. \quad (13)$$

From (9) and (13), we had:

$$v_1 - \mu_1 = v_2 - \mu_2 + \alpha + 2\gamma. \quad (14)$$

4. Symmetry Analysis

There are four standard approaches to the analysis of nonlinear ordinary or partial differential equations. The approaches comprise numerical computation, dynamical systems analysis, singularity analysis and symmetry analysis, all of which possess extensive literatures.

The Lie group analysis is the most powerful tool to establish the general solution of ordinary differential equations. Any known integration technique can be shown to be a particular case of general integration method based on the derivation of the continuous group of symmetries admitted by differential equation, that is, the Lie symmetry algebra [8].

An n th order ordinary differential equation

$$N(x, y, y', \dots, y^{(n)}) = 0 \quad (15)$$

admits the one-parameter Lie group of transformations

$$\bar{x} = x + \varepsilon \xi, \quad (16)$$

$$\bar{y} = y + \varepsilon \eta \quad (17)$$

with infinitesimal generator

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \quad (18)$$

if

$$G^{[n]} N|_{N=0} = 0, \quad (19)$$

where $G^{[n]}$ is the n th extension of G given by

$$G^{[n]} = G + \sum_{i=1}^n \left\{ \eta^{(i)} - \sum_{j=0}^{i-1} \binom{i}{j} y^{(j+1)} \xi^{(i-j)} \right\} \frac{\partial}{\partial y^{(i)}}. \quad (20)$$

We say that the equation *possesses* the symmetry (group generator)

$$G = \xi \partial_x + \eta \partial_y \quad (21)$$

iff (19) holds.

Equation (7) does not possess the rescaling symmetry, we may achieve a simplification by means of the rescaling

$$u = ay, \quad t = \frac{x}{a} \quad (22)$$

so that equation (7) is considered as

$$yy'' - y'^2 + y'y^2 + y'y + \frac{b+\gamma}{a} y^3 + \frac{b}{a} y^2 = 0. \quad (23)$$

The case $a = b$ ($\gamma = 0$) reduces equation (23) to

$$yy'' - y'^2 + y'y^2 + y'y + y^3 + y^2 = 0. \quad (24)$$

Nucci and Leach [8] obtained two Lie point symmetries in the special case $a = b$ ($\gamma = 0$) [8]

$$G_1 = \partial_x, \quad (25)$$

$$G_2 = e^x (\partial_x - y \partial_y). \quad (26)$$

The Lie bracket is given by $[G_1, G_2]_{LB} = G_2$, so that the reduction of (23) should be by G_2 rather than the more usual G_1 . The associated Lagrange's system for the zeroth and first-order invariants of G_2 in (23) representation is [8]

$$\frac{dx}{1} = \frac{dy}{-y} = \frac{dy'}{-2y' - y}$$

so that

$$X = x + \log y, \quad Y = \frac{y'}{y^2} + \frac{1}{y} \quad (27)$$

with X and Y the new dependent and independent variables, respectively. Therefore, equation (24) becomes

$$\frac{dY}{dX} + Y + 1 = 0$$

which can be easily integrated to give

$$(Y + 1) \exp[X] = A. \quad (28)$$

The substitution of (27) into (28) gave

$$\frac{y'}{y} + y + 1 = A \exp[-x]. \quad (29)$$

The integration of (29) gave

$$y = \frac{C \exp[-x]}{D \exp[A \exp[-x]] + B}. \quad (30)$$

Substituting (8) and (30) into (22) gave

$$u(t) = \frac{(\mu_1 - \nu_1) C \exp[-(\mu_1 - \nu_1)t]}{D \exp[A \exp[-(\mu_1 - \nu_1)t]] + B}. \quad (31)$$

The number of infected population is obtained by substituting (31) into (6)

$$I(t) = \frac{(\mu_1 - \nu_1) C \exp[-(\mu_1 - \nu_1)t]}{\beta [D \exp[A \exp[-(\mu_1 - \nu_1)t]] + B]}. \quad (32)$$

The substitution of (33) into (2) gave

$$S(t) = \frac{1}{\beta} (\mu_1 - \nu_1) + \frac{1}{\beta} (\mu_1 - \nu_1) \exp[-(\mu_1 - \nu_1)t] + \frac{\alpha + \mu_2 - \nu_2}{\beta}. \quad (33)$$

5. The General Solutions

The solution obtained using the Lie symmetry analysis may not be realistic since the rate of susceptible individual becoming sick was set to be zero. In this section, we obtained the explicit solution by setting the total

number of population to be the sum of the susceptible population plus the number of infected population.

Combining (1a) and (1b), we obtained:

$$\begin{aligned}\dot{S} + \dot{I} &= (\nu_1 - \mu_1)S + (\nu_2 - \mu_2 - \alpha)I \\ &= (\nu_1 - \mu_1)(S + I).\end{aligned}\tag{34}$$

Let

$$N = S + I.\tag{35}$$

Equation (34) became

$$\dot{N} = (\nu_1 - \mu_1)N.\tag{36}$$

The solution of (36) was

$$N(t) = N(0) \exp[(\nu_1 - \mu_1)t].\tag{37}$$

From (35) and (37), we had

$$S = N(0) \exp[(\nu_1 - \mu_1)t] - I.\tag{38}$$

Equation (1b) became

$$\begin{aligned}\dot{I} &= -\beta I^2 + \beta N(0) \exp[(\nu_1 - \mu_1)t] - (\alpha + \gamma + \mu_2 - \nu_2)I, \\ \frac{\dot{I}}{I^2} &= -\beta + \frac{\beta}{I} N(0) \exp[(\nu_1 - \mu_1)t] - \frac{(\alpha + \gamma + \mu_2 - \nu_2)}{I}.\end{aligned}\tag{39}$$

With the use of the transformation

$$u = \frac{1}{I},\tag{40}$$

equation (39) became:

$$\dot{u} = \beta - \beta N(0) \exp[(\nu_1 - \mu_1)t]u + (\alpha + \gamma + \mu_2 - \nu_2)u.\tag{41}$$

Since

$$\nu_1 - \mu_1 = a$$

and

$$\alpha + \gamma + \mu_2 - \nu_2 = b,$$

equation (41) gave

$$\dot{u} + (\beta N(0) \exp[at] - b)u = \beta \quad (42)$$

which has the integrating factor

$$\exp\left[\int (\beta N(0) \exp[at] - b) dt\right].$$

The solution of (42) was

$$\begin{aligned} u = & A \exp\left[-\int (\beta N(0) \exp[at] - b) dt\right] \\ & + \exp\left[-\int (\beta N(0) \exp[at] - b) dt\right] \\ & \times \int \beta \exp\left[\int (\beta N(0) \exp[at] - b) dt\right] dt, \end{aligned}$$

and from (40), we had

$$I(t) = \frac{\exp\left[\int (\beta N(0) \exp[at] - b) dt\right]}{A + \int \beta \exp\left[\int (\beta N(0) \exp[at] - b) dt\right]}. \quad (43)$$

From (38), we had

$$\begin{aligned} S(t) = & N(0) \exp[(\nu_1 - \mu_1)t] \\ & - \frac{\exp\left[\int (\beta N(0) \exp[at] - b) dt\right]}{A + \int \beta \exp\left[\int (\beta N(0) \exp[at] - b) dt\right]}. \end{aligned} \quad (44)$$

From (1c), we obtained

$$\dot{R} - (v_3 - \mu_3)R = \alpha I. \quad (45)$$

Equation (45) has the integrating factor $\exp[(v_3 - \mu_3)t]$. Therefore,

$$R(t) = \exp[(v_3 - \mu_3)t] \times \left[B + \int \frac{\alpha \exp[-(v_3 - \mu_3)t] \exp\left[\int (\beta N(0) \exp[at] - b) dt\right]}{A + \int \beta \exp\left[\int (\beta N(0) \exp[at] - b) dt\right]} dt \right] \quad (46)$$

and from (1d), we obtain the death component of the population

$$\begin{aligned} D(t) = & \mu_1 N(0) \exp[(v_1 - \mu_1)t] \\ & + (\mu_2 - \mu_1) \frac{\exp\left[\int (\beta N(0) \exp[at] - b) dt\right]}{A + \int \beta \exp\left[\int (\beta N(0) \exp[at] - b) dt\right]} \\ & + \mu_3 \exp[(v_3 - \mu_3)t] \\ & \times \left[B + \int \frac{\alpha \exp[-(v_3 - \mu_3)t] \exp\left[\int (\beta N(0) \exp[at] - b) dt\right]}{A + \int \beta \exp\left[\int (\beta N(0) \exp[at] - b) dt\right]} dt \right]. \end{aligned}$$

6. Numerical Result

In this section, we give a numerical result based on the susceptibles and infected component of the population. The parameters are chosen as $v_1 = 0.0003$, $v_2 = 0.0001$, $v_3 = 0.0003$, $\mu_1 = 0.0002$, $\mu_2 = 0.0003$, $\mu_3 = 0.0002$, $\alpha = 0.01$, $\beta = 0.04$, $\gamma = 0.04$. Figure 1 suggests that the solution is globally asymptotically stable.

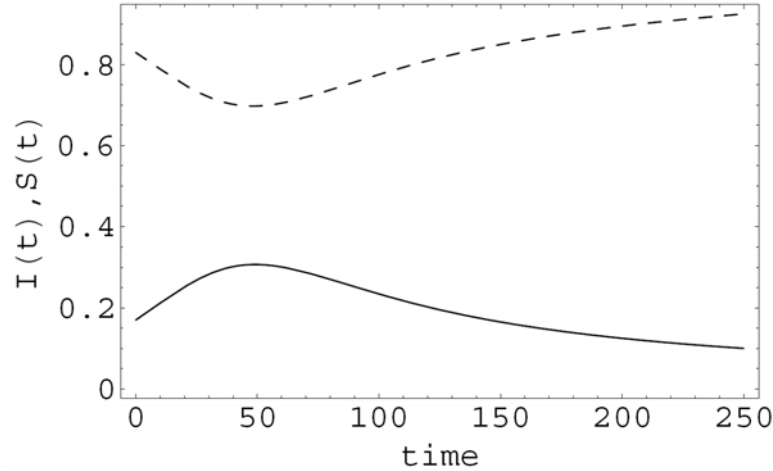


Figure 1. Susceptibles (dashed line) and infected (solid line). The computer package Mathematica is used with the initial values $I(0) = 0.17$ and $S(0) = 0.83$.

7. Conclusion

We formulated the SIRD epidemic model and studied its dynamical behaviour. We have shown that the SIRD epidemic model is integrable from the viewpoint of singularity analysis. We have performed numerical simulation on the asymptotical behaviour of the SIRD model.

It is notable, therefore, that the mathematical models can aid in the understanding of the processes which govern a system. This fact notwithstanding, the confidence placed in such models should always be questioned. Thus, critically it should be remembered that the ultimate purpose of a model is to present a simplification of a more complex system that may ultimately prove to be too complex to model accurately. Models must, therefore, be used with care and their results should be interpreted with due caution. Margins of uncertainty should be attached to the use of model results.

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