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TOTALLY PARANORMAL OPERATORS WITH PROPERTY (δ) HAVE INVARIANT SUBSPACES

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Abstract

Suppose H is a complex Hilbert space and $T \in L(H)$ is a bounded linear operator. In this paper, we show that if $T \in L(H)$ is a totally paranormal with no eigenvalues, then $H_T(\{\lambda\}) = \mathcal{N}(\lambda I - T)$ for all $\lambda \in \mathbb{C}$. We also show that if $T \in L(H)$ is a totally paranormal with no eigenvalues, then T is algebraic if and only if the spectrum $\sigma(T)$ is finite. Finally, it is shown that that if $T \in L(H)$ is a totally paranormal operator on a complex Hilbert space H with property (δ), then T has a nontrivial invariant subspace. As a corollary, every hyponormal operator with (δ) has a nontrivial closed invariant subspace.

1. Introduction and Preliminaries

Throughout the paper, H and K stand for infinite-dimensional complex Hilbert spaces, and $L(H, K)$ denotes for the Banach space of all bounded linear operators from H to K. For a bounded linear operator $T \in L(H) :=$

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$L(H, H)$, let $\mathcal{N}(T) \subseteq H$ and $\mathcal{R}(T) \subseteq K$ denote the kernel and range of $T \in L(H, K)$, respectively. For $T \in L(H)$, we denote by $\sigma_p(T)$, $\sigma(T)$ and $\rho(T)$ the point spectrum, the spectrum and the resolvent set of T, respectively, and let $Lat(T)$ stand for the collection of all T-invariant closed linear subspaces of H. For $T \in Lat(T)$, let $T|Y$ denote the operator given by the restriction of T to Y. A closed subspace $\mathcal{M}$ of H is called an *invariant* subspace of T if $T(\mathcal{M}) \subseteq \mathcal{M}$. Also, $\mathcal{M}$ is called *nontrivial* if $\{0\} \neq \mathcal{M} \neq H$. A linear subspace $\mathcal{M}$ of H is said to be *T-hyperinvariant* if $S(\mathcal{M}) \subseteq \mathcal{M}$ for every bounded linear operator S on H that commutes with T.

The invariant subspace problem asks whether every operator on a complex separable Hilbert space has a nontrivial invariant subspace. In 1966, Bernstein and Robinson [3] proved that if $T \in L(H)$ is a polynomially compact operator, i.e., $p(T)$ is compact for some non-zero polynomial p, then T has a nontrivial invariant subspace. Recall that a subnormal operator on a Hilbert space is the restriction of a normal operator to an invariant subspace. In 1978, Brown [4] proved that every subnormal operator has a nontrivial invariant subspace. In 1988, Brown et al. [6] proved that every contraction on the Hilbert space whose spectrum contains the unit circle has a nontrivial invariant subspace. It is well known that every subnormal operator is hyponormal. In 1987, Brown proved [5] that every hyponormal operator $T \in L(H)$ has a nontrivial invariant subspace whenever $C(\sigma(T)) \neq R(\sigma(T))$, where $C(\sigma(T))$ denotes the continuous functions on $\sigma(T)$ and $R(\sigma(T))$ denotes the closure in the $C(\sigma(T))$ norm of the rational functions on $\sigma(T)$ with poles outside of $\sigma(T)$. An example of Banach space that admits an operator without any nontrivial invariant subspace was given by Enflo [9, 10] and Read [20].

An operator $T \in L(H)$ on the Hilbert space H is said to be *paranormal* if

$$\| Tx \|^2 \le \| T^2 x \| \| x \| \text{ for all } x \in H.$$

If $\lambda I - T$ is paranormal for every $\lambda \in \mathbb{C}$, then we say that T is *totally paranormal*.

The following alternative definition is well known. An operator $T \in L(H)$ is paranormal if and only if

$$0 \le T^{2*}T^2 - 2\lambda T^* T + \lambda^2 I \text{ for all } \lambda > 0,$$

see [1] and [23]. Equivalently, T is paranormal if and only if

$$\lambda \| Tx \|^2 \le \frac{1}{2} (\| T^2 x \|^2 + \lambda^2 \| x \|^2) \text{ for every } x \in H \text{ and for all } \lambda > 0.$$

The class of paranormal operators is a generalization of the class of normal operators and several interesting properties have been proved by many authors. Every hyponormal operator (in particular, subnormal, quasinormal and normal) is paranormal. It is well known that if $T \in L(H)$ is paranormal, then T^n is paranormal for all positive integer $n \in \mathbb{N}$. Moreover, if $T \in L(H)$ is paranormal, then $\mathcal{N}(T) = \mathcal{N}(T^2)$ and $\| T \| = r(T)$, the spectral radius of T. It is easily seen that all hyponormal operators are totally paranormal.

In this paper, we prove that if $T \in L(H)$ is a totally paranormal with $\sigma_p(T) = \varnothing$, then $H_T(\{\lambda\}) = \mathcal{N}(\lambda I - T)$ for all $\lambda \in \mathbb{C}$. We also prove that if $T \in L(H)$ is a totally paranormal with $\sigma_p(T) = \varnothing$, then T is algebraic if and only if the spectrum $\sigma(T)$ is finite. Finally, we prove that if T is a totally paranormal operator with property (δ), then T has a nontrivial closed invariant subspace. As a corollary, every hyponormal operator with (δ) has a nontrivial closed invariant subspace.

We begin with several basic definitions and notations. Let H be an infinite-dimensional complex separable Hilbert space, and let $T \in L(H)$. Let U be an open subset of the complex numbers $\mathbb{C}$, and we denote by $O(U, H)$

the Fréchet space of analytic H-valued functions on U endowed with the topology of local uniform convergence. If $T \in L(H)$, then we define T_U : $O(U, H) \to O(U, H)$ by

$$(T_U f)(\lambda) = (\lambda I - T) f(\lambda) \text{ for each } f \in O(U, H) \text{ and } \lambda \in U.$$

For each closed $F \subseteq \mathbb{C}$, we will denote by $H_T(F)$ the linear manifold of all vectors $x \in H$ such that $(\lambda I - T) f(\lambda) = x$, $\lambda \in \mathbb{C} \backslash F$, for some $f \in O(\mathbb{C} \backslash F, H)$. For an open set $G \subseteq \mathbb{C}$, let $H_T(G)$ denote the union of all spaces $H_T(K)$, where K runs through all compact subsets of G.

We say that $T \in L(H)$ has the *single-valued extension property* (abbreviated *SVEP*) provided that T_U is injective for every open $U \subseteq \mathbb{C}$. In the case when T has the SVEP, the *local spectrum* of a vector $x \in H$, denoted by $\sigma_T(x)$, is defined as the smallest compact set K with $x \in H_T(K)$. In this case, the local resolvent of x is the unique function $f \in O(\mathbb{C} \backslash \sigma_T(x), H)$ satisfying $(\lambda I - T) f(\lambda) = x$, $\lambda \in \mathbb{C} \backslash \sigma_T(x)$. Obviously, we have $\sigma_T(x) \subseteq \sigma(T)$ and $\sigma_T(x)$ is closed. The *local resolvent set* $\rho_T(x)$ of T at the point $x \in H$ is the set defined by $\rho_T(x) := \mathbb{C} \backslash \sigma_T(x)$.

It is easily seen from definition that for each closed $F \subseteq \mathbb{C}$, $H_T(F)$ is a linear subspace T-invariant of H, but need not closed in general, see [8]. Recall that an operator $T \in L(H)$ is said to have *Dunford's property* (C) (for short, property (C)) if, for each closed sets $F \subseteq \mathbb{C}$, $H_T(F)$ is closed.

The following proposition is found in [17].

Proposition 1.1. *Let* $T \in L(H)$ *be a bounded linear operator on a Hilbert space H and every set* $F \subseteq \mathbb{C}$, *the following assertions hold*:

(a) $H_T(F)$ *is a T-invariant linear subspace of H.*

(b) $(\lambda I - T) H_T(F) = H_T(F)$ *for all* $\lambda \in \mathbb{C} \backslash F$.

(c) $\mathcal{N}(\lambda I - T)^n \subseteq H_T(\{\lambda\})$ *for all* $\lambda \in \mathbb{C}$ *and* $n \in \mathbb{N}$.

(d) *T has SVEP if and only if* $H_T(\varnothing) = \{0\}$.

(e) *If* T *has SVEP and* $F, G \subseteq \mathbb{C}$ *are closed and disjoint*, *then the decomposition* $H_T(F \cup G) = H_T(F) + H_T(G)$ *holds as an algebraic direct sum*.

Recall that an operator $T \in L(H)$ is said to have *Bishop*'*s property* (β) provided that for every open $U \subseteq \mathbb{C}$, the mapping T_U is injective and has dense range. An operator $T \in L(H)$ has *decomposition property* (δ) if $H = H_T(\overline{U}) + H_T(\overline{V})$, where $\{U, V\}$ is an open cover of $\sigma(T)$. It is not difficult to see that decompositions property (δ) is inherited by quotients. Recall from Proposition 1.2.19 of [17] that property (β) implies property (C), and in turn implies SVEP. It can be shown that the converse implications do not hold in general as can be seen from [8] and [18].

In 1992, Laursen [16] proved that all totally paranormal operators have property (C) and moreover, if $T \in L(H)$ is a totally paranormal with no eigenvalues, then

$$H_T(F) = \bigcap_{\lambda \in \mathbb{C} \setminus F} \mathcal{R}(\lambda I - T) \text{ for all closed sets } F \subseteq \mathbb{C}.$$

Recently, Uchiyama and Tanahashi [24] proved that all totally paranormal operators have Bishop's property (β).

Recall that an operator $T \in L(H)$ is called *decomposable* provided that for each open cover $\{U, V\}$ of the complex plane $\mathbb{C}$, there exist $Y, Z \in Lat(T)$ for which

$$H = Y + Z, \quad \sigma(T \,|\, Y) \subseteq U \quad \text{and} \quad \sigma(T \,|\, Z) \subseteq V.$$

This class is quite general, containing for example all compact operators, normal operators on the Hilbert spaces, Dunford's spectral operators and generalized scalar operators. Moreover, a simple application of the Riesz functional calculus shows that all operators with totally disconnected spectrum are decomposable, see the monographs by [8], [17] and [25].

As shown in [2], an operator $T \in L(H)$ has property (β) if and only if it is similar to the restriction of a decomposable operator to an invariant subspace. Also, an operator $T \in L(H)$ has property (δ) if and only if it is the quotient of a decomposable operator. Moreover, properties (β) and (δ) are dual to each other, in the sense that an operator $T \in L(H)$ satisfies (β) if and only if its adjoint has property (δ), and conversely, T has property (δ) if and only if its adjoint has property (β). Clearly, an operator $T \in L(H)$ is decomposable if and only if it has both properties (β) and (δ).

It is easily seen that if T is the unilateral shift on ℓ^2, then T is paranormal, and its adjoint T^* is normaloid, i.e., $\| T \| = r(T)$. Moreover, T^* does not have the SVEP and hence T is not decomposable.

The following proposition is found in [17].

Proposition 1.2. *If $T \in L(H)$ is decomposable, then T has property (β), and for each closed $F \subseteq \mathbb{C}$, $H_T(F)$ is a T-hyperinvariant subspace with $\sigma(T \mid H_T(F)) \subseteq F$.*

2. Invariant Subspaces for Totally Paranormal Operators

The following theorem is found in [24].

Theorem 2.1. *If $T \in L(H)$ is paranormal, then T has property (β). In particular, every totally paranormal operator has property (β).*

The following theorem is found in [16].

Theorem 2.2. *If $T \in L(H)$ is a totally paranormal with no eigenvalues, then*

$$H_T(F) = \bigcap_{\lambda \in \mathbb{C} \setminus F} \mathcal{R}(\lambda I - T) \text{ for all closed sets } F \subseteq \mathbb{C}.$$

An operator $T \in L(H)$ on a complex Hilbert space H is said to be *algebraic* if $p(T) = 0$ for some non-zero complex polynomial *p*. It is clear that if $T \in L(H)$ is algebraic, then by the spectral mapping theorem, the spectrum $\sigma(T)$ is finite.

Proposition 2.3. *Let* $T \in L(H)$ *be a totally paranormal with no eigenvalues. Then the following assertions hold*:

(a) $H_T(\{\lambda\}) = \mathcal{N}(\lambda I - T)$ *for all* $\lambda \in \mathbb{C}$.

(b) *T is algebraic if and only if the spectrum* $\sigma(T)$ *is finite*.

Proof. (a) By Proposition 1.1(c), $\mathcal{N}(\lambda I - T) \subseteq H_T(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. It follows from Theorem 2.2 and Proposition 1.1(d) that

$$\bigcap_{\lambda \in \mathbb{C}} \mathcal{R}(\lambda I - T) = H_T(\varnothing) = \{0\}.$$

We want to show that

$$(\lambda I - T) H_T(\{\lambda\}) = \{0\}.$$

It suffices to show that $(\lambda I - T) H_T(\{\lambda\}) \subseteq R(\mu I - T)$ for all $\mu \in \mathbb{C}$. Clearly, this is true if $\mu = \lambda$. Assume $\mu \neq \lambda$. By Proposition 1.1(b),

$$H_T(\{\lambda\}) = (\mu I - T) H_T(\{\lambda\}) \text{ for all } \mu \in \mathbb{C} \setminus \{\lambda\}.$$

Thus we have

$$(\lambda I - T) H_T(\{\lambda\}) \subseteq H_T(\{\lambda\}) = (\mu I - T) H_T(\{\lambda\}) \subseteq \mathcal{R}(\mu I - T).$$

(b) Clearly, if *T* is algebraic, then $\sigma(T)$ is finite. Now assume that the spectrum $\sigma(T)$ is finite. Since *T* has property (β), *T* has SVEP. Then by Proposition 1.1(e), $\sigma(T) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ implies

$$H = H_T(\sigma(T)) = H_T(\{\lambda_1\}) + \cdots + H_T(\{\lambda_n\}).$$

Clearly, by (a), we have $H = \mathcal{N}(\lambda_1 I - T) + \cdots + \mathcal{N}(\lambda_n I - T)$. Hence $(\lambda_1 I - T)(\lambda_2 I - T)\cdots(\lambda_n I - T) = 0$ on *H*. This shows that *T* is algebraic. This completes the proof. □

In 1997, Read [22] proved that there exist decomposable operators on a complex Banach space without a nontrivial closed invariant subspace.

We can now prove the main result of this section.

Theorem 2.4. *Let $T \in Ł(H)$ be a totally paranormal operator on a complex Hilbert space H of dimension greater than* 1. *Suppose that T has property* (δ). *Then T has a nontrivial invariant subspace.*

Proof. Let $T \in L(H)$ be a totally paranormal operator with property (δ). Then *T* is decomposable. We consider the following two different cases $\sigma_p(T) \neq \emptyset$ and $\sigma_p(T) = \emptyset$.

Case I. $\sigma_p(T) \neq \emptyset$.

Let $\lambda \in \sigma_p(T)$. Then $\mathcal{N}(\lambda I - T) \neq \{0\}$ and $\mathcal{N}(\lambda I - T)$ is a closed invariant subspace of *H*. Obviously, either $\mathcal{N}(\lambda I - T) \neq H$ or $\mathcal{N}(\lambda I - T) = H$. Suppose $\mathcal{N}(\lambda I - T) \neq H$. Then $\mathcal{N}(\lambda I - T)$ is a nontrivial closed invariant subspace of *H*. Suppose $\mathcal{N}(\lambda I - T) = H$. Then $T = \lambda I$. Since $\dim H \geq 2$, the linear subspace generated by for some non-zero $x \in H$ is a one-dimensional *T*-invariant subspace of *H*.

Case II. $\sigma_p(T) = \emptyset$.

By Theorem 2.2, *T* is decomposable with property

$$H_T(F) = \bigcap_{\lambda \in \mathbb{C} \setminus F} \mathcal{R}(\lambda I - T) \text{ for all closed sets } F \subseteq \mathbb{C}.$$

At first, suppose that $\sigma(T)$ contains at least two points. Let $\lambda \in \sigma(T)$. Then $H_T(\{\lambda\})$ is a closed *T*-hyperinvariant subspace of *H* and $\sigma(T \mid H_T(\{\lambda\}))$

$\subseteq \{\lambda\}$, by Proposition 1.2. We claim that $H_T(\{\lambda\})$ is nontrivial. Let U be an arbitrary open neighborhood of λ in $\mathbb{C}$. We choose another open set $V \subseteq \mathbb{C}$ such that $\lambda \notin V$ and $\{U, V\}$ is an open covering of $\mathbb{C}$. Since T is decomposable, we obtain

$$H = H_T(\{\lambda\}) + H_T(V), \quad \sigma(T \mid H_T(\{\lambda\})) \subseteq U \quad \text{and} \quad \sigma(T \mid H_T(V)) \subseteq V.$$

It remains to show that $H_T(\{\lambda\})$ is nontrivial. Suppose $H_T(\{\lambda\}) = \{0\}$. Then $\sigma(T) = \sigma(T \mid H_T(V)) \subseteq V$, which contradicts $\lambda \notin V$. Suppose $H_T(\{\lambda\}) = H$. Then $\sigma(T) = \sigma(T \mid H_T(\{\lambda\})) \subseteq \{\lambda\}$, which contradicts that $\sigma(T)$ contains at least two points. Hence $H_T(\{\lambda\})$ is a nontrivial invariant subspace.

Finally, we assume that $\sigma(T)$ is a singleton. It follows from Proposition 2.3 that $T = \lambda I + N$ for some $\lambda \in \mathbb{C}$ and some nilpotent operator $N \in L(H)$. Let $q \in \mathbb{N}$ be the smallest integer for which $N^q = 0$, and choose an $x \in H$ for which $N^{q-1}x \neq 0$. The linear subspace generated by $N^{q-1}x$ is a one-dimensional T-invariant linear subspace of H. Hence T has a nontrivial invariant subspace. □

It is well known that every hyponormal operator is a totally paranormal operator. As an immediate application of Theorem 2.4, we obtain

Corollary 2.5. *Every hyponormal operator with property* (δ) *has a nontrivial invariant subspace*.

Clearly, every subnormal operator is hyponormal and every subnormal operator has property (δ). We have the following:

Corollary 2.6. *Every subnormal* (*in particular*, *normal*) *operator has a nontrivial invariant subspace*.

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