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$\mathcal{X}$-INJECTIVE AND GORENSTEIN $\mathcal{X}$-INJECTIVE MODULES

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Abstract

In this paper, we study the notion of Gorenstein $\mathcal{X}$-injective module with respect to its complete injective resolution and give its characterization, where $\mathcal{X}^{\perp} = \mathcal{U}_{\mathcal{X}}$ and $\mathcal{X} = \{N \in R\text{-}Mod \mid N$ is finitely presented and $Ext_R^1(N, M) = 0, \forall M \in \mathcal{U}_{\mathcal{X}}\}$. We prove that (i) the class of all $\mathcal{X}$-injective modules and the class of all Gorenstein $\mathcal{X}$-injective modules are injectively resolving, respectively, (ii) if R is a coherent ring, then every R-module M has an $\mathcal{X}$-injective cover. Finally, we show that if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of R-modules with M_2 and M_3 Gorenstein $\mathcal{X}$-injective R-modules, then M_1 is Gorenstein $\mathcal{X}$-injective if and only if $Ext_R^1(G', M_1) = 0$ for all $\mathcal{X}$-injective R-modules G'.

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1. Introduction

Throughout this paper, R denotes an associative ring with identity and all R-modules, if not specified otherwise, are left R-modules. R-*Mod* denotes the category of left R-modules.

Let $\mathscr{C}$ be a class of left R-modules. Following [2], we say that a map $f \in Hom_R(C, M)$ with $C \in \mathscr{C}$ is a $\mathscr{C}$*-precover* of M, if the group homomorphism $Hom_R(C', f) : Hom_R(C', C) \to Hom_R(C', M)$ is surjective for each $C' \in \mathscr{C}$. A $\mathscr{C}$-precover $f \in Hom_R(C, M)$ of M is called a $\mathscr{C}$*-cover* of M if f is right minimal. That is, $fg = f$ implies that g is an automorphism for each $g \in End_R(C)$. $\mathscr{C} \subseteq R\text{-}Mod$ is a precovering class (resp. covering class) provided that each module has a $\mathscr{C}$-precover (resp. $\mathscr{C}$-cover). Dually, we have the definition of $\mathscr{C}$ preenvelope (resp. $\mathscr{C}$ envelope).

Given a class $\mathscr{C}$ of left R-modules, we write

$$\mathscr{C}^{\perp} = \{N \in R - Mod \mid Ext_R^1(M, N) = 0, \forall M \in \mathscr{C}\},$$

$$^{\perp}\mathscr{C} = \{N \in R - Mod \mid Ext_R^1(N, M) = 0, \forall M \in \mathscr{C}\}.$$

A class $\mathscr{C}$ of left R-modules is said to be *injectively resolving* [5] if $\mathscr{C}$ contains all injective modules and if given an exact sequence of left R-modules

$$0 \to A \to B \to C \to 0$$

with $A \in \mathscr{C}$, the conditions $B \in \mathscr{C}$ and $C \in \mathscr{C}$ are equivalent. The $\mathcal{U}_{\mathcal{X}}$-coresolution dimension of M, denoted by $cores \cdot dim_{\mathcal{U}_{\mathcal{X}}}(M)$, is defined to be the smallest nonnegative integer n such that $Ext_R^{n+1}(A, M) = 0$ for all R-modules $A \in \mathcal{X}$ (if no such n exists, set $cores \cdot dim_{\mathcal{U}_{\mathcal{X}}}(M) = \infty$).

The notion of Gorenstein injective module was introduced by Enochs and Jenda in [3]. An R-module M is called *Gorenstein injective* if there is an exact sequence $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ of injective R-modules such that $M = \ker(E^0 \to E^1)$ and the functor $Hom_R(I, -)$ preserves the exact sequence whenever I is an injective R-module.

In this paper, we define a class $\mathcal{U}_{\mathcal{X}} = \{M \in R\text{-}Mod \mid$ there exists a finitely presented R-module N such that $Ext_R^1(N, M) = 0\}$. It follows that we can define a class $\mathcal{X} = \{N \in \mathcal{N} \mid Ext_R^1(N, M) = 0, \forall M \in \mathcal{U}_{\mathcal{X}}\}$, where $\mathcal{N}$ is the class of all finitely presented R-modules. We get $\mathcal{X}^{\perp} = \mathcal{U}_{\mathcal{X}}$, that is, $\mathcal{U}_{\mathcal{X}}$ is the class of all $\mathcal{X}$-injective R-modules. Thus $({}^{\perp}\mathcal{U}_{\mathcal{X}}, \mathcal{U}_{\mathcal{X}})$ is a *cotorsion theory*.

This paper is organized as follows: In Section 2, we prove that the class $\mathcal{U}_{\mathcal{X}}$ of all $\mathcal{X}$-injective R-modules is injectively resolving and if R is a coherent ring, then every R-module M has an $\mathcal{X}$-injective cover.

In the last section, we define a Gorenstein $\mathcal{X}$-injective module with respect to its complete injective resolution and give its characterization. Finally, we show that if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of R-modules with M_2 and M_3 Gorenstein $\mathcal{X}$-injective R-modules, then M_1 is Gorenstein $\mathcal{X}$-injective if and only if $Ext_R^1(G', M_1) = 0$ for all $\mathcal{X}$-injective R-modules G'.

2. Covers and Envelopes of $\mathcal{X}$-injective Modules

In this section, we define a class $\mathcal{X} = \{N \in \mathcal{N} \mid Ext_R^1(N, M) = 0, \forall M \in \mathcal{U}_{\mathcal{X}}\}$, where $\mathcal{N}$ is the class of all finitely presented R-modules. We begin with the following definition.

Definition 2.1 [7]**.** A left R-module M is called $\mathcal{X}$-*injective* if $Ext_R^1(X, M) = 0$ for all left R-modules $X \in \mathcal{X}$.

Proposition 2.2. *The class* $\mathcal{U}_{\mathcal{X}}$ *of all* $\mathcal{X}$*-injective modules is closed under pure submodules*.

Proof. Let A be a pure submodule of an $\mathcal{X}$-injective module M. Then there is a pure exact sequence $0 \to A \to M \to M/A \to 0$ and a functor $Hom_R(N, -)$ preserves this sequence is exact whenever N is finitely presented. This implies that the sequence

$$0 \to Hom_R(N, A) \to Hom_R(N, M) \to Hom_R(N, M/A)$$

$$\to Ext_R(N, A) \to 0$$

is also exact for all $N \in \mathcal{X}$. It follows that $Ext_R^1(N, A) = 0$ for all $N \in \mathcal{X}$, as desired. □

Theorem 2.3. *Every R-module has an* $\mathcal{X}$*-injective preenvelope*.

Proof. Let M be an R-module. By [4, Lemma 5.3.12], there is a cardinal number $\aleph_\alpha$ such that for any R-homomorphism $\phi : M \to G$ with G an $\mathcal{X}$-injective R-module, there exists a pure submodule A of G such that $|A| \leq \aleph_\alpha$ and $\phi(M) \subset A$. Clearly, $\mathcal{U}_{\mathcal{X}}$ is closed under direct products and by Proposition 2.2, A is $\mathcal{X}$-injective. Hence the theorem follows by [4, Proposition 6.2.1]. □

Proposition 2.4. *The class* $\mathcal{U}_{\mathcal{X}}$ *of all* $\mathcal{X}$*-injective R-modules is injectively resolving*.

Proof. Let $0 \to M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3 \to 0$ be an exact sequence of left R-modules with $M_1 \in \mathcal{U}_{\mathcal{X}}$. If $M_3 \in \mathcal{U}_{\mathcal{X}}$, then $M_2 \in \mathcal{U}_{\mathcal{X}}$ since the class $\mathcal{U}_{\mathcal{X}}$ is closed under extension. Let $M_2 \in \mathcal{U}_{\mathcal{X}}$ and $G \in \mathcal{X}$. By [5, Theorem 3.2.1(a)], G has a special $\mathcal{U}_{\mathcal{X}}$-precover. Then there exists an exact sequence $0 \to K \to A \to G \to 0$ with $A \in \mathcal{U}_{\mathcal{X}}$ and $K \in \mathcal{U}_{\mathcal{X}}$. We prove that M_3 is $\mathcal{X}$-injective, i.e., to prove that $Ext_R^1(G, M_3) = 0$. For this, it suffices to extend any $\alpha \in Hom_R(K, M_3)$ to an element of $Hom_R(A, M_3)$. Clearly, K has $\mathcal{U}_{\mathcal{X}}$ -precover,

$$0 \to K' \xrightarrow{f} A' \xrightarrow{g} K \to 0,$$

where $K, K' \in \mathcal{U}_{\mathcal{X}}$ and $A' \in \mathcal{U}_{\mathcal{X}}$. As the class $\mathcal{U}_{\mathcal{X}}$ is closed under extensions, $A' \in \mathcal{U}_{\mathcal{X}}$. Since $\alpha \circ g : A' \to M_3$ with $A' \in \mathcal{U}_{\mathcal{X}}$ and M_1 an $\mathcal{X}$ -injective R-module, then there exists $\beta : A' \to M_2$ such that $\psi \circ \beta = \alpha \circ g$. That is the following diagram is commutative:

$$\begin{tikzcd}
A' \arrow[r, "g"] \arrow[d, dashed, "\beta"] & K \arrow[d, "\alpha"] \\
M_2 \arrow[r, "\psi"] & M_3
\end{tikzcd}$$

Now, we define $\beta \restriction_{im\phi} : A' \to im\phi$, where $\restriction$ is a restriction map. Then there exists $\gamma : K' \to M_1$ such that $\beta \restriction_{im\phi}(f(K')) = \phi\gamma(K')$. Hence, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r, "f"] \arrow[d, dashed, "\gamma"] & A' \arrow[r, "g"] \arrow[d, dashed, "\beta"] & K \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "\phi"] & M_2 \arrow[r, "\psi"] & M_3 \arrow[r] & 0
\end{tikzcd}$$

The $\mathcal{X}$ -injectivity of M_1 yields a homomorphism $\gamma_1 : A' \to M_1$ such that $\gamma = \gamma_1 \circ f$. So for each $k' \in K'$, we get $(\beta \circ f)(k') = (\phi \circ \gamma)(k') = (\phi \circ (\gamma_1 \circ f))(k')$. Then there exists a map $\beta_1 \in Hom_R(K, M_2)$ such that $\beta = \beta_1 \circ g$ and we get $\alpha = \psi \circ \beta_1$. That is, the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r, "f"] \arrow[d, dashed, "\gamma"'] & A' \arrow[r, "g"] \arrow[d, dashed, "\beta"'] \arrow[dl, dashed, "\gamma_1"'] & K \arrow[r] \arrow[d, "\alpha"] \arrow[dl, dashed, "\beta_1"'] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "\phi"] & M_2 \arrow[r, "\psi"] & M_3 \arrow[r] & 0
\end{tikzcd}$$

Since M_2 is $\mathcal{X}$-injective, there exists $\rho \in Hom_R(A, M_2)$ such that $\beta_1 = \rho \circ f$. Thus $\alpha = \psi \circ \beta_1 = \psi \circ (\rho \circ f)$, where $\psi \circ \rho \in Hom_R(A, M_3)$. Hence M_3 is $\mathcal{X}$-injective. □

Proposition 2.5. *The class $\mathcal{U}_{\mathcal{X}}$ is injectively resolving if and only if for every pure submodule A of M, M/A is also $\mathcal{X}$-injective.*

Proof. Assume that $\mathcal{U}_{\mathcal{X}}$ is injectively resolving. Let $M \in \mathcal{U}_{\mathcal{X}}$ and *A* be a pure submodule of *M*. By Proposition 2.2, *A* is $\mathcal{X}$-injective. From the short exact sequence $0 \to A \to M \to M/A \to 0$, we get M/A is $\mathcal{X}$-injective. Conversely, assume that for every $M \in \mathcal{U}_{\mathcal{X}}$ and every pure submodule *A* of *M*, M/A is $\mathcal{X}$-injective. Let $0 \to M_1 \xrightarrow{f} M_2 \to M_3 \to 0$ be an exact sequence of *R*-modules with $M_1, M_2 \in \mathcal{U}_{\mathcal{X}}$. By the $\mathcal{X}$-injectivity of M_1, we get $imf \subset M_2$ is also $\mathcal{X}$-injective. Thus *imf* is pure in M_2. By assumption, $M_2/imf \in \mathcal{U}_{\mathcal{X}}$. Since $M_2/imf \cong M_3$, $M_3 \in \mathcal{U}_{\mathcal{X}}$. Hence $\mathcal{U}_{\mathcal{X}}$ is injectively resolving since $E \in \mathcal{U}_{\mathcal{X}}$ for all injective *R*-modules *E*. □

Theorem 2.6. *Let R be a coherent ring. Then every R-module M has an $\mathcal{X}$-injective cover.*

Proof. Let *M* be an *R*-module with $|M| = \alpha$. Let κ be a cardinal as in [1, Theorem 5]. By [4, Lemma 5.3.12], we show that any homomorphism $G \to M$ with $G \in \mathcal{U}_{\mathcal{X}}$ factors through an $\mathcal{X}$-injective *R*-module G' with $|G'| \leq \kappa$.

Let $\phi \in Hom_R(G, M)$ and $G \in \mathcal{U}_{\mathcal{X}}$. If $|G| \leq \kappa$, then the assertion holds by taking $G' = G$. So we may assume that $|G| \geq \kappa$. Let $K = \ker \phi$. Thus $|G/K| \leq \lambda$ since G/K can be embedded in *M*. By [1, Theorem 5], *K* contains a nonzero submodule *H* which is pure in *G*. It follows that *H* is $\mathcal{X}$-injective by Proposition 2.2. Hence G/H is $\mathcal{X}$-injective by Proposition 2.4.

If $|G/H| \leq \kappa$, then the desired result follows immediately since ϕ factors through G/H. Now, we assume that $|G/H| \geq \kappa$. Define a class $\mathcal{C} = \{C \mid H \leq C \leq K \text{ and } G/C \in \mathcal{U}_{\mathcal{X}}\}$. Since $H \in \mathcal{C}$, $\mathcal{C}$ is a nonempty class. Let $\{C_\beta \in \mathcal{C} \mid \beta \in I\}$ be a sequence of R-modules. Then we have $H \leq \bigcup C_\beta \leq K$ and $G/\bigcup C_\beta = G/\varinjlim C_\beta = \varinjlim(G/C_\beta)$ is $\mathcal{X}$ -injective since direct limit of $\mathcal{X}$ -injective R-modules is $\mathcal{X}$ -injective. Thus $\bigcup C_\beta \in \mathcal{C}$. Hence G' is a maximal element of the class $\mathcal{C}$ by the Zorn's Lemma.

Suppose $|G/G'| \geq \kappa$. Since $G' \subset K$, there exists $\gamma : G/G' \to M$ such that $\ker\gamma = K/G'$. Now, $|(G/G')/(K/G')| = |G/K| \leq \lambda$. By Lemma [1, Theorem 5], K/G' contains a nonzero submodule F/G' which is pure in G/G'. Then F/G' is $\mathcal{X}$ -injective by Proposition 2.2. Therefore, $G/F \cong (G/G')/(F/G')$ is $\mathcal{X}$ -injective by Proposition 2.5. This implies that $F \in \mathcal{C}$, which is a contradiction to the maximality of G'.

Finally, it is clear that G/G' is $\mathcal{X}$ -injective and ϕ factors through G/G'. Clearly, $\mathcal{U}_{\mathcal{X}}$ is closed under direct sums. It follows that M has an $\mathcal{X}$ -injective precover. Since R is a coherent ring, $\mathcal{U}_{\mathcal{X}}$ is closed under direct limits. Therefore, M has an $\mathcal{X}$ -injective cover. □

3. Gorenstein $\mathcal{X}$ -injective Modules

We define the following definition with respect to a complete injective resolution of a module:

Definition 3.1. An R-module M is called *Gorenstein $\mathcal{X}$ -injective* if there is an exact sequence

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

of injective R-modules such that $M = \ker(I^0 \to I^1)$ and such that $Hom_R(G, -)$ leaves the above sequence exact whenever G is an $\mathcal{X}$ -injective R-module.

Remark 3.2. (1) The following implications are straightforward. Every injective module is Gorenstein $\mathcal{X}$-injective and every Gorenstein $\mathcal{X}$-injective is Gorenstein injective.

(2) If $\mathbb{I} : \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$ is a sequence of injective R-modules and $Hom_R(G, -)$ preserves the exact sequence whenever G is an $\mathcal{X}$-injective R-module, then by symmetry, all the images, the kernels and the cokernels of $\mathbb{I}$ are Gorenstein $\mathcal{X}$-injective.

(3) The class of Gorenstein $\mathcal{X}$-injective R-modules is closed under direct products.

Lemma 3.3. *Let M be a Gorenstein $\mathcal{X}$-injective R-module. Then*

(1) $Ext_R^i(G, M) = 0$ *for all $\mathcal{X}$-injective R-modules G and all* $i \geq 1$.

(2) *The injective dimension of M is zero or infinite.*

Proof. (1) Let $0 \to G \to I^0 \to I^1 \to \cdots$ be an injective resolution of G. By hypothesis, the functor $Hom_R(G', -)$ preserves the exact sequence whenever G' is an $\mathcal{X}$-injective R-module. It follows that $Ext_R^i(G', M) = 0$ for all $\mathcal{X}$-injective R-modules G' and all $i \geq 1$ by definition.

(2) Assume that $id(M) < \infty$. Then there is an exact sequence $0 \to M \to I^0 \to I^1 \to \cdots \to I^n \to 0$ of injective R-modules I^i such that $Hom_R(G', -)$ preserves the exact sequence whenever G' is an $\mathcal{X}$-injective R-module. It follows that M is injective. □

Now we give some characterizations of Gorenstein $\mathcal{X}$-injective R-modules.

Theorem 3.4. *Let R be a coherent ring. Then the following are equivalent for an R-module M*:

(1) *M is Gorenstein $\mathcal{X}$-injective.*

(2) *M has an exact left $\mathcal{X}$-injective resolution and* $Ext_R^i(G, M) = 0$ *for all $\mathcal{X}$-injective R-modules G and all* $i \geq 1$.

(3) *M has an exact left $\mathcal{X}$-injective resolution and $Ext_R^i(G, M) = 0$ for all R-modules G with $cores \cdot dim_{\mathcal{U}_{\mathcal{X}}}(G) < \infty$ and all $i \geq 1$.*

Proof. (1) $\Rightarrow$ (2) follows by Lemma 3.3 and (2) $\Rightarrow$ (3) holds by dimension shifting.

(3) $\Rightarrow$ (1) Let $\phi : G_0 \to M$ be an $\mathcal{X}$-injective cover of *M*. Then there is an exact sequence $0 \to G_0 \xrightarrow{i} E \to L \to 0$ with E an injective envelope of G_0. By Proposition 2.4, L is $\mathcal{X}$-injective. Thus, there exists $\overline{\phi} \in Hom_R(E, M)$ such that $\phi = \overline{\phi}i$ since $Ext_R^1(L, M) = 0$. It follows that there is $\phi' \in Hom_R(E, G_0)$ such that $\phi\phi' = \overline{\phi}$ since ϕ is an $\mathcal{X}$-injective cover of *M*. Therefore, $\phi = \phi(\phi' i)$, and hence $\phi' i$ is an isomorphism. It follows that G_0 is injective. For any $\mathcal{X}$-injective *R*-module *G*, it is easy to verify that $Hom_R(G, G_0) \to Hom(G, im(f)) \to 0$ is exact since ϕ is an $\mathcal{X}$-injective cover of *M*. However, the exactness of $0 \to \ker(\phi) \to G_0 \to im(\phi) \to 0$ yields the exact sequence $Hom_R(G, G_0) \to Hom(G, im(\phi)) \to Ext_R^1(G, \ker(\phi)) \to 0$. Thus $Ext_R^1(G, \ker(\phi)) = 0$. So $\ker(\phi)$ has an $\mathcal{X}$-injective cover $G_1 \to \ker(\phi)$ with G_1 injective by the proof above. Continuing this process, we can get a complex $\cdots \to G_1 \to G_0 \to M \to 0$ with each G_i an injective module such that $Hom(G, -)$ preserves the exact sequence. Note that $Ext_R^i({}_R R, M) = 0$ for all $i \geq 1$ and $Ext_0({}_R R, M) \cong M$ since M has an exact left $\mathcal{X}$-injective resolution. So the complex $\cdots \to G_1 \to G_0 \to M \to 0$ is exact. On the other hand, we have an exact sequence $0 \to M \to G^0 \to G^1 \to \cdots$ of with each G^i an injective module such that $Hom(G, -)$ preserves the exact sequence since $Ext_R^i(G, M) = 0$ for all $\mathcal{X}$-injective *R*-modules G and all $i \geq 1$. Now, we get an exact sequence $\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$ of injective *R*-modules with $M = \ker(G^0 \to G^1)$. So *M* is Gorenstein $\mathcal{X}$-injective. □

Lemma 3.5. *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be an sequence of R-modules. If* M_1 *and* M_3 *are Gorenstein* $\mathcal{X}$*-injective R-modules, then so is* M_2. *That is, the class of all Gorenstein* $\mathcal{X}$*-injective modules is closed under extensions.*

Proof. It is straightforward. □

Lemma 3.6. *An R-module M is Gorenstein* $\mathcal{X}$*-injective if and only if there exists an exact sequence of modules* $0 \to N \to I \to M \to 0$ *such that I is injective and N is Gorenstein* $\mathcal{X}$*-injective.*

Proof. The direct implication is clear by the definition of Gorenstein $\mathcal{X}$-injective module. Conversely, let

$$0 \to N \xrightarrow{\alpha} I \to M \to 0 \tag{1}$$

be an exact sequence of modules with I an injective module and N be a Gorenstein $\mathcal{X}$-injective module. Then for any $\mathcal{X}$-injective R-module G', we get $Ext_R^i(G', N) = 0$ for all $i \geq 1$ by Lemma 3.3, and hence we have $Ext_R^i(G', M) = 0$ for all $i \geq 1$ by using the long exact sequence $Ext_R^i(G', N) \to Ext_R^i(G', I) \to Ext_R^{i+1}(G', N)$. Since N is Gorenstein $\mathcal{X}$-injective, there exists an exact sequence of injective R-modules

$$\cdots \to I_1 \to I_0 \xrightarrow{\beta} N \to 0 \tag{2}$$

such that the functor $Hom_R(G', -)$ preserves the exact sequence whenever G' is an $\mathcal{X}$-injective R-module. From the sequences (1) and (2), we get an exact sequence

$$\cdots \to I_1 \to I_0 \xrightarrow{\alpha\beta} I \to M \to 0 \tag{3}$$

with each I_i and I are injective R-modules and such that the functor $Hom_R(G', -)$ preserves the exact sequence whenever G' is an X-injective R-module. Consider an injective resolution of M,

$$0 \to M \to I^0 \to I^1 \to \cdots. \tag{4}$$

Since $Ext_R^i(G', M) = 0$ for all $\mathcal{X}$ -injective R-modules G' and all $i \geq 1$, the functor $Hom_R(G', -)$ preserves the above exact sequence (4). From the sequences (3) and (4), we get an exact sequence of injective R-modules

$$\cdots \to I_1 \to I_0 \xrightarrow{\alpha\beta} I \to I^0 \to I^1 \to \cdots$$

such that $M = \ker(I^0 \to I^1)$ which remains exact after applying the functor $Hom_R(G', -)$ for any $\mathcal{X}$ -injective R-module G'. So M is Gorenstein $\mathcal{X}$ -injective. □

Theorem 3.7. *The class $\mathcal{GXI}$ of Gorenstein $\mathcal{X}$ -injective R-modules is injectively resolving.*

Proof. Consider an exact sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ with M_1 a Gorenstein $\mathcal{X}$ -injective R-module. If M_3 is Gorenstein $\mathcal{X}$ -injective, then by Lemma 3.5, M_2 is Gorenstein $\mathcal{X}$ -injective. If M_2 is Gorenstein $\mathcal{X}$ -injective, then by Lemma 3.6, there exists a short exact sequence of R-modules $0 \to N \to I \to M_2 \to 0$ with I an injective R-module and N a Gorenstein $\mathcal{X}$ -injective R-module. Consider the following pullback diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & N \arrow[r, equal] \arrow[d] & N \arrow[d] & & \\
0 \arrow[r] & X \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

By Lemma 3.5, we get from the left vertical sequence that X is Gorenstein $\mathcal{X}$ -injective. Therefore, by the middle horizontal sequence and

Lemma 3.6, M_3 is Gorenstein $\mathcal{X}$-injective. This proves that the class $\mathcal{GXI}$ is injectively resolving. □

Corollary 3.8. *The class* $\mathcal{GXI}$ *is closed under direct summands*.

Proof. It follows from Theorem 3.7 and [6, Proposition 1.4]. □

Theorem 3.9. *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be an exact sequence of R-modules*. *If* M_2 *and* M_3 *are Gorenstein* $\mathcal{X}$*-injective*, *then* M_1 *is Gorenstein* $\mathcal{X}$*-injective if and only if* $Ext_R^1(G', M_1) = 0$ *for all* $\mathcal{X}$*-injective R-modules* G'.

Proof. The direct implication follows by Lemma 3.3. We now prove the sufficiency. Let $0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0$ be a short exact sequence of modules such that M_2 and M_3 are Gorenstein $\mathcal{X}$-injective and $Ext_R^1(G', M_1) = 0$ for all $\mathcal{X}$-injective *R*-modules G'. Then there exist exact sequences of *R*-modules

$$\mathbb{N} : \cdots \to I_1 \to I_0 \to N \to 0$$

$$\mathbb{M} : \cdots \to J_1 \to J_0 \to M \to 0$$

with each I_i and J_i injective, which remain exact after applying $Hom_R(G', -)$ for any $\mathcal{X}$-injective *R*-module G'. It is easy to see that the homomorphism $\beta \in Hom(M_2, M_3)$ can be lifted to a chain map $\overline{\beta} : \mathbb{N} \to \mathbb{M}$. Define a class $\mathcal{C}$ is a mapping cone of $\overline{\beta} : \mathbb{N} \to \mathbb{M}$. Since $\overline{\beta}$ is a quasi-isomorphism (both $\mathbb{M}$ and $\mathbb{N}$ are exact), the long exact sequence of homology for the mapping cone shows that $\mathcal{C}$ is exact. Also, the sequence $\mathcal{C}$ of modules remains exact after applying the functor $Hom_R(G', -)$ for any $\mathcal{X}$-injective *R*-module G' since both $\mathbb{N}$ and $\mathbb{M}$ are so. Consider the following commutative diagram:

$$\begin{array}{ccccccccccc}
\mathcal{A}\colon \cdots & \longrightarrow & J_2 \oplus I_1 & \longrightarrow & J_1 \oplus I_0 & \longrightarrow & K' & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \downarrow & & \\
\mathcal{B}\colon \cdots & \longrightarrow & J_2 \oplus I_1 & \longrightarrow & J_1 \oplus I_0 & \longrightarrow & J_0 \oplus N & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
\mathcal{D}\colon \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where $K' = \ker(J_0 \oplus N \to M)$. Clearly, the sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{D} \to 0$ is exact. Since the sequences $\mathcal{B}$ and $\mathcal{D}$ are exact, we get that $\mathcal{A}$ is exact. Moreover, $\mathcal{A}$ remains exact after applying the functor $Hom_R(G', -)$ for any $\mathcal{X}$-injective R-module G' since both $\mathcal{B}$ and $\mathcal{D}$ are so. It is easy to see that there exists a homomorphism $k \in Hom_R(K, K')$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle i} & & \| & & \\
0 & \longrightarrow & K' & \longrightarrow & J_0 \oplus M_2 & \longrightarrow & M_3 & \longrightarrow & 0
\end{array}$$

where $i \in Hom_R(M_2, J_0 \oplus M_2)$ is the canonical injection. By the five lemma, we get that k is monomorphic. By the snake lemma, we have $im(k) \cong im(i) = M_2$. Thus, the sequence of R-modules $0 \to M_1 \to K' \to J_0 \to 0$ is exact. Moreover, this short exact sequence splits by assumption $Ext^1_R(J_0, M_1) = 0$. Hence $K' \cong J_0 \oplus M_1$. On the other hand, one can check that $Ext^i_R(G', M_1) = 0$ for all $\mathcal{X}$-injective R-modules G' and all $i \geq 1$, and so $Ext^i_R(G', K') \cong Ext^1_R(G', J_0 \oplus M_1) = 0$ for all $\mathcal{X}$-injective R-modules G' and all $i \geq 1$, the injective resolution

$$\mathbb{K}' : 0 \to K' \to I^0 \to I^1 \to \cdots$$

of K' remains exact after applying $Hom_R(G', -)$ for any $\mathcal{X}$-injective R-module G'. Assembling the sequences $\mathbb{K}'$ and $\mathcal{A}$, we get the strongly complete injective resolution of K',

$$\cdots \to J_2 \oplus I_1 \to J_1 \oplus I_0 \to I^0 \to I^1 \to \cdots,$$

where $K' \cong J_0 \oplus K$ is Gorenstein $\mathcal{X}$-injective, as desired. □

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