Far East Journal of Mathematical Sciences (FJMS)

Published Online: December 2014
Available online at http://pphmj.com/journals/fjms.htm
Volume 94, Number 1, 2014, Pages 65-88

A CHARACTERIZATION OF GORENSTEIN TORIC FANO *n*-FOLDS WITH INDEX *n* AND FUJITA'S CONJECTURE

Shoetsu Ogata and Huai-Liang Zhao

Mathematical Institute
Tohoku University
Sendai 980-8578, Japan
e-mail: ogata@math.tohoku.ac.jp

Abstract

We give a characterization of Gorenstein toric Fano varieties of dimension n with index n among toric varieties. As an application, we give a stronger version of Fujita's freeness conjecture and also give a simple proof of Fujita's very ampleness conjecture on Gorenstein toric varieties.

0. Introduction

A nonsingular projective variety X is called *Fano* if its anti-canonical divisor $-K_X$ is ample. For a Fano variety X, the number $i_X := \max\{i \in \mathbb{N};\ -K_X = iD$ for ample divisor $D\}$ is called the *Fano index*, or simply *index*. Kobayashi and Ochiai [6] showed that a Fano variety X of dimension n with index $n+1$ is the projective n-space and that a Fano variety with index n is the hyperquadric.

Received: April 28, 2014; Revised: June 19, 2014; Accepted: September 6, 2014
2010 Mathematics Subject Classification: Primary 14M25; Secondary 14J45, 52B20.
Keywords and phrases: Fano variety, toric variety.

First, we give a characterization of the projective space among toric varieties.

Theorem 1. *Let X be a projective toric variety of dimension* n $(n \geq 2)$. *We assume that there exists an ample line bundle L on X with* $\dim \Gamma(X, L^{\otimes n} \otimes \omega_X) = 0$, *where* ω_X *is the dualizing sheaf of X. Then X is the projective space of dimension* n.

This is not a new result. If we consider the lattice polytope corresponding to a polarized toric variety (X, L), then this is a characterization of the basic lattice simplex treated by Batyrev and Nill [2, Proposition 1.4]. In this article, we use the normality of some multiple of lattice polytope in order to characterize Gorenstein toric Fano varieties. In particular, we use this characterization in the form stated as Lemma 3 in Section 1. Theorem 1 is obtained as Corollary 1 in Section 1.

We also give a characterization of a Gorenstein toric Fano variety of dimension *n* with index *n* among toric varieties.

Theorem 2. *Let X be a projective toric variety of dimension* n $(n \geq 2)$. *We assume that there exists an ample line bundle L on X with* $\dim \Gamma(X, L^{\otimes n} \otimes \omega_X) = 1$. *Then X is a Gorenstein toric Fano variety of dimension n with index* n.

Batyrev and Nill obtained a classification of lattice polytope P of dimension *n* such that the multiple $(n-1)P$ does not contain lattice points in its interior [2, Theorem 2.5]. This can be interpreted as a classification of polarized toric variety (X, L) with $\Gamma(X, L^{\otimes(n-1)} \otimes \omega_X) = 0$. We do not use their classification but do an elementary argument about the shape of polytopes.

In this paper, we call a variety of dimension *n*, simply an *n-fold*. We see that a Gorenstein toric Fano surface with index 2 is the quadratic surface $\mathbb{P}^1 \times \mathbb{P}^1$ or the weighted projective plane $\mathbb{P}(1, 1, 2)$. For higher dimension $n \geq 3$, we see that the weighted projective *n*-space $\mathbb{P}(1, 1, 2, ..., 2)$ is a

Gorenstein toric Fano n-fold with index n. We also have other toric Fano n-folds with index n.

Theorem 3. *Let X be a Gorenstein toric Fano variety of dimension n $(n \geq 2)$ with index n. Let D be an ample Cartier divisor on X with $-K_X = nD$. Then D is very ample and (X, D) is a quadratic hypersurface in $\mathbb{P}^{n+1}$ which is a cone over a plane conic, i.e., the weighted projective space $\mathbb{P}(1, 1, 2, ..., 2)$, or a cone over the quadratic surface $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.*

We may expect the same characterization of a Gorenstein toric Fano n-fold with index $n-1$ among toric varieties for $n \geq 3$. We give a special case as Proposition 4 in Section 5.

Theorem 4. *Let X be a projective toric variety of dimension n $(n \geq 3)$. We assume that there exists an ample line bundle L on X with $\dim \Gamma(X, L) = n+1$ and $\dim \Gamma(X, L^{\otimes(n-1)} \otimes \omega_X) = 1$. Then X is a Gorenstein toric Fano variety of dimension n with index $n-1$.*

Batyrev and Juny [1] classified Gorenstein toric Fano n-folds with index $n-1$. For the proof of Theorem 4, we do not use the classification of Batyrev and Juny. We also remark that there exists a non-Gorenstein toric n-fold X with the ample line bundle L such that $\dim \Gamma(X, L) = n+1$ and $\dim \Gamma(X, L^{\otimes(n-2)} \otimes \omega_X) = 1$ for every $n \geq 4$. We give examples in the last of this article.

As a corollary of above theorems, we obtain a strong version of Fujino's Theorem [3] for Gorenstein toric varieties. This is given in Section 4. The theorem of this type is called "*Fujita's freeness conjecture*" [4], in general.

Theorem 5. *Let X be a Gorenstein projective toric variety of dimension n $(n-2)$. We assume that X is not the projective space, a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ nor a Gorenstein toric Fano n-fold with index n. If an ample line bundle L on X satisfies that the intersection number with every irreducible invariant curve is at least $n-1$, then the adjoint bundle $L + K_X$ is nef.*

We also give a simple proof of the following theorem, which is a special case of theorem of Payne [10] for Gorenstein toric varieties. This is given in Section 5. The theorem of this type is also called "*Fujita's very ampleness conjecture*", in general.

Theorem 6. *Let X be a Gorenstein projective toric variety of dimension n not isomorphic to the projective space. If an ample line bundle L on X satisfies that the intersection number with every irreducible invariant curve is at least* $n+1$, *then the adjoint bundle* $L + K_X$ *is very ample.*

We note that the condition on the intersection number "at least $n+1$" is trivially best possible for Gorenstein toric Fano n-folds with index n. We remark, however, that even if we make an exception on X as in Theorem 5, we cannot weaken the condition "at least $n+1$". We give examples in Section 5 for all dimensions $n \geq 3$.

1. Toric Varieties and Lattice Polytopes

In this section, we recall the correspondence between polarized toric varieties and lattice polytopes and give criterions for polytopes to be basic.

Let $M = \mathbb{Z}^n$ be a free abelian group of rank n and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ the extension of coefficients into real numbers. We define a *lattice polytope P* in $M_{\mathbb{R}}$ as the convex hull $P := \mathrm{Conv}\{m_1, \ldots, m_r\}$ of a finite subset $\{m_1, \ldots, m_r\}$ of M. We define the dimension of a lattice polytope P as that of the smallest affine subspace containing P. A polytope of dimension n is often called an *n-polytope*.

The space $\Gamma(X, L)$ of global sections of an ample line bundle L on a toric variety X of dimension n is parametrized by the set of lattice points in a lattice polytope P of dimension n (see, for instance, Oda's book [9, Section 2.2] or Fulton's book [5, Section 3.5]). And k times tensor product $L^{\otimes k}$ corresponds to the polytope $kP := \{kx \in M_{\mathbb{R}};\ x \in P\}$. Furthermore, the surjectivity of the multiplication map

$$\Gamma(X, L^{\otimes k}) \otimes \Gamma(X, L) \to \Gamma(X, L^{\otimes(k+1)})$$

is equivalent to the equality

$$(kP) \cap M + P \cap M = ((k+1)P) \cap M. \tag{1}$$

In particular, the dimension of the space of global sections of $L \otimes \omega_X$ is equal to the number of lattice points contained in the interior of P, i.e.,

$$\dim \Gamma(X, L \otimes \omega_X) = \sharp\{(\mathrm{Int}\, P) \cap M\}.$$

A lattice polytope P is called *normal* if the equality (1) holds for all $k \geq 1$.

By using the generalized Castelnuovo's Lemma of Mumford [7], we can prove the following lemma. We can also prove this in terms of lattice polytopes.

Lemma 1. *Let P be a lattice polytope of dimension n. If there exists an integer r with $1 \leq r \leq n-1$ satisfying the condition that the multiple rP does not contain lattice points in its interior*, *then the equality*

$$(kP) \cap M + P \cap M = ((k+1)P) \cap M$$

holds for all integers $k \geq n-r$.

Even if P contains lattice points in its interior, then we have a result of Nakagawa [8].

Theorem 7 (Nakagawa [8])**.** *If a lattice polytope P is dimension n*, *then the equality*

$$(kP) \cap M + P \cap M = ((k+1)P) \cap M$$

holds for all integers $k \geq n-1$.

Let $\{e_1, ..., e_n\}$ be a $\mathbb{Z}$-basis of M. A lattice polytope is called *basic* if it is isomorphic to the n-simplex

$$\Delta_0^n := \mathrm{Conv}\{0, e_1, ..., e_n\}$$

by a unimodular transformation of M. For a basic n-simplex P, we see that $\operatorname{Int}(nP) \cap M = \emptyset$ and $\sharp \operatorname{Int}((n+1)P) \cap M = 1$. A lattice polytope P of dimension n is called an *empty lattice n-simplex* if the set of lattice points in P are only $n+1$ vertices.

Lemma 2. *Let P be an empty lattice n-simplex. If the multiple $(n-1)P$ does not contain lattice points in its interior, then it is basic.*

Proof. From Lemma 1, we see that P is normal, that is, the equality (1) holds for all $k \geq 1$.

By taking a parallel transformation of M so that the origin is a vertex of P, set $P = \operatorname{Conv}\{0, m_1, ..., m_n\}$. Set the cone

$$C(P) := \mathbb{R}_{\geq 0} m_1 + \cdots + \mathbb{R}_{\geq 0} m_n \subset M_{\mathbb{R}}.$$

Then the normality of P implies that the semi-group $C(P) \cap M$ is generated by $m_1, ..., m_n$. This proves the lemma. □

Lemma 3. *Let P be a lattice n-polytope. If the multiple nP does not contain lattice points in its interior, then it is basic.*

Proof. By a parallel transformation of M, we may assume that the origin coincides with a vertex of P.

First, we treat the case that P is a lattice n-simplex $\operatorname{Conv}\{0, m_1, ..., m_n\}$. If P is an empty lattice simplex, then it is basic by Lemma 2. We assume that a face $F = \operatorname{Conv}\{0, m_1, ..., m_r\}$ has a lattice point m in its relative interior. Then $m + m_{r+1} + \cdots + m_n$ is contained in the interior of nP. This contradicts to the assumption. If a face $F' = \operatorname{Conv}\{m_1, ..., m_r\}$ contains a lattice point m in its interior, then $m + m_{r+1} + \cdots + m_n$ is contained in the interior of nP. Thus, we see that P is an empty lattice simplex if it is a lattice simplex.

Next, we treat the general case. Let $C_0(P) := \mathbb{R}_{\geq 0}(P)$ be the cone of P with apex 0. If a lattice point in P is contained in the interior of the cone $C_0(P)$, then it is contained in the interior of $2P$.

When $n = 2,$ we see P is a triangle, hence, it is basic by the first part of the proof.

In general, set $n \geq 3.$ Let $\{m_1, ..., m_t\}$ be the set of the end points of all edges of P through the origin. We may assume that $\{m_1, ..., m_n\}$ is linearly independent. Set $Q := \mathrm{Conv}\{0, m_1, ..., m_n\}.$ Then Q is basic by the first part because $\dim Q = n$ and $\mathrm{Int}(nQ) \cap M = \emptyset.$

If $t > n,$ that is, $P \neq Q,$ then we have a facet, say

$$F = \mathrm{Conv}\{0, m_1, ..., m_{n-1}\}$$

of Q which is not a facet of P. Since $\mathrm{Int}(n)F \cap M$ contains a lattice point, it would be a lattice point in the interior of nP. This contradicts to the assumption. Hence, we see that $t = n,$ that is, P is a lattice n-simplex. By the first part of this proof, it is basic. □

We may consider Lemma 3 as a characterization of the projective n-space among toric varieties.

Corollary 1. *Let X be a projective toric variety of dimension* n $(n \geq 2)$. *We assume that there exists an ample line bundle L on X with* $\dim \Gamma(X, L^{\otimes n} \otimes \omega_X) = 0.$ *Then X is the projective space* $\mathbb{P}^n$ *and* $L = \mathcal{O}(1).$

2. Gorenstein Fano with Index n

In this section, we will give a characterization of a Gorenstein toric Fano n-fold with index n among toric varieties.

Let M be a free abelian group of rank $n \geq 2$ and $P \subset M_{\mathbb{R}}$ a lattice n-polytope. For a vertex v of P, make the cone

$$C_v(P) := \mathbb{R}_{\geq 0}(P - v) = \{r(x - v) \in M_{\mathbb{R}};\ r \geq 0 \text{ and } x \in P\}.$$

We call P *Gorenstein* at v if there exists a lattice point m_0 in $C_v(P)$ such that the equality

$$(\mathrm{Int}\, C_v(P)) \cap M = m_0 + C_v(P) \cap M$$

holds. We call P *Gorenstein* if it is Gorenstein at all vertices.

We also define the notion of very ampleness. P is called *very ample* at v if the semi-group $C_v(P) \cap M$ is generated by $(P - v) \cap M$ and P is called *very ample* if it is very ample at all vertices.

Let $\{e_1, \ldots, e_n\}$ be a $\mathbb{Z}$-basis of M. Set

$$P_n := \mathrm{Conv}\{0, 2e_1, e_2, \ldots, e_n\},$$

$$Q_n := \mathrm{Conv}\{0, e_1, e_2, e_1 + e_2, e_3, \ldots, e_n\}.$$

Then they are very ample Gorenstein polytopes and $\sharp\{\mathrm{Int}(nP_n) \cap M\} = \sharp\{\mathrm{Int}(nQ_n) \cap M\} = 1.$ The polytope P_n corresponds to the polarized variety $(\mathbb{P}(1, 1, 2, \ldots, 2), \mathcal{O}(2))$. The polygon Q_2 corresponds to $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 1))$, and Q_n corresponds to a cone over the quadratic surface $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.

Proposition 1. *Let $P \subset M_{\mathbb{R}}$ be a lattice n-polytope for $n \geq 2$ satisfying the condition $\sharp\{\mathrm{Int}(nP) \cap M\} = 1$. Then P is isomorphic to P_n or Q_n by a unimodular transformation of M.*

For the proof, we fix notations. Take a vertex $v \in P$. We may set so that the origin coincides with v by a parallel transformation of M. Let $\{m_1, \ldots, m_t\}$ be the set of the end points of all edges through the origin and define as

$$Q := \mathrm{Conv}\{0, m_1, \ldots, m_t\}. \tag{2}$$

Set $\{\widetilde{m}\} = \mathrm{Int}(nP) \cap M$. We note that $(n-1)P$ does not contain lattice points in its interior because if $m' \in M$ is in the interior of $(n-1)P$, then $\{m' + m_i;\ i = 1, \ldots, t\}$ is in the interior of nP. Hence, $\mathrm{Int}(n-1)Q \cap M = \varnothing$.

First, consider the case when $n = 2$. Then $t = n = 2$. Since $\mathrm{Int}P \cap M = \varnothing$, we may set $m_1 = ae_1$, $m_2 = be_2$ with positive integers $a \geq b$.

When P is a triangle, we see that $a = b = 2$ or $b = 1$, and the condition $\sharp\{\mathrm{Int}(2P) \cap M\} = 1$ implies $a = 2$, $b = 1$. Thus, $P \cong P_2$.

If P is not a triangle, then $b = 1$ and P is a quadrangle with the other vertex $ce_1 + e_2$ for $c \geq 1$. Since $2P$ contains only one lattice point in its interior, we see $a = c = 1$. This P coincides with Q_2.

In the following, we set $n \geq 3$.

Lemma 4. *Let Q be the n-polytope defined in* (2). *Assume that* $\mathrm{Int}(nQ) \cap M = \{\widetilde{m}\}$. *Then Q is isomorphic to P_n or Q_n.*

Proof. Since $\widetilde{m}$ is contained in the interior of $C_v(P) = C_v(Q)$, by the Carathéodory's theorem, we may choose n elements from $m_1, \ldots, m_t$ such that the simplicial cone spanned by them contains $\widetilde{m}$ and furthermore we can choose $m_1, \ldots, m_r$ so that the subcone spanned by them contains $\widetilde{m}$ in its relative interior by renumbering.

Set $G := \mathrm{Conv}\{m_1, \ldots, m_r\}$ and $\widetilde{G} := \mathrm{Conv}\{0, G\}$. Then $\dim \widetilde{G} = r$. Since $\mathrm{Int}(n-1)\widetilde{G} \cap M = \emptyset$ and $\mathrm{Int}(n+1)\widetilde{G} \cap M \neq \emptyset$, we have $n - 1 < r + 1$, hence, $r \geq n - 1$. We separate our argument into two cases:

Case (a): $r = n - 1$. Since $\mathrm{Int}(n-1)\widetilde{G} \cap M = \emptyset$, we see that $\widetilde{G}$ is basic from Lemma 3. Let H be the hyperplane in $M_{\mathbb{R}}$ containing $\widetilde{G}$. We can divide Q into the union of two n-polytopes $Q_{(1)}$ and $Q_{(2)}$ separated by H. Since $\widetilde{m} \in n\widetilde{G}$ and $\sharp\{\mathrm{Int}(nQ) \cap M\} = 1$, we see that $\mathrm{Int}(nQ_{(i)}) \cap M = \emptyset$ for $i = 1, 2$. Thus, $Q_{(i)}$ is basic. We may set $m_j = e_j$ for $j = 1, \ldots, n$ so that $Q_{(1)} = \mathrm{Conv}\{\widetilde{G}, e_n\}$, $Q_{(2)} = \mathrm{Conv}\{\widetilde{G}, m_{n+1}\}$ and $t = n + 1$. Furthermore, we set

$$m_{n+1} = a_1 e_1 + \cdots + a_{n-1} e_{n-1} - e_n.$$

Since Q is convex, we see that $a_i \geq 0$ and $1 \leq \sum_i a_i \leq 2$. If $a_2 = \cdots = a_{n-1}$

$= 0,$ then $m_1 = e_1$ would not be a vertex of Q. Hence, we have $a_1 = a_2 = 1,$ $a_3 = \cdots = a_{n-1} = 0.$ If we set $e_i' = e_i - m_{n+1}$ for $i = 1, ..., n-1$ and $e_n' = 0 - m_{n+1},$ then $e_n - m_{n+1} = e_1' + e_2'$ and $\{e_1', ..., e_n'\}$ is a $\mathbb{Z}$-basis of M. Thus, we see that Q is isomorphic to Q_n.

Case (b): $r = n.$ If $t > n,$ then there is a facet, say $F := \mathrm{Conv}\{0, m_1, ..., m_{n-1}\}$ of $\widetilde{G}$ which is not facet of Q, that is, the relative interior of F is contained in the interior of Q. Since $\mathrm{Int}(nF) \cap M \neq \emptyset,$ the interior of nQ would contain lattice points more than one. Thus, we have $t = n.$

Take the primitive elements $m_i' \in M$ for $i = 1, ..., n$ so that $m_i = a_i m_i'$ with positive integers a_i. Set $Q' := \mathrm{Conv}\{0, m_1', ..., m_n'\}.$ We may set $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1.$ For $2 \leq r \leq n-1,$ set

$$F_r := \mathrm{Conv}\{0, m_1, ..., m_r\}.$$

If F_2 contains lattice points in its relative interior, then the interior of $(n-1)Q$ would contain lattice points. Thus, $a_1 = a_2 = 2,$ or $a_2 = 1.$ If $a_1 = a_2 = 2,$ then $2F_2$ contains lattice points more than two in its relative interior, hence, the interior of nQ would contain lattice points more than two. Thus, we have $a_2 = 1.$

When $a_1 \geq 2,$ set

$$G := \mathrm{Conv}\{m_1', m_2, ..., m_n\}.$$

Since the relative interior of G is contained in the interior of Q and $\dim G = n-1,$ the interior of $(n-1)G$ does not contain lattice points, hence, G is basic from Lemma 3. Since the number of lattice points in the interior of nG is one, we see $a_1 = 2.$ Set

$$\widetilde{G} := \mathrm{Conv}\{0, m_1', m_2, ..., m_n\}.$$

Since $n\widetilde{G}$ could not contain lattice points in its interior, it is basic. Thus, Q is isomorphic to P_n.

When $a_1 = 1$, we see that Q is not basic because of the assumption $\mathrm{Int}(nQ) \cap M \neq \varnothing$. If Q is an empty lattice simplex, then the condition $\mathrm{Int}(n-1)Q \cap M = \varnothing$ implies that Q is basic by Lemma 2. Thus, there exists a lattice point v in Q other than its vertices. After renumbering, assume that F_r contains v with the smallest dimension. We note F_r does not contain lattice points in its interior. Set

$$F_r' \coloneqq \mathrm{Conv}\{m_1, ..., m_r\}.$$

Since v is contained in the relative interior of F_r', it is contained in the relative interior of $2F_r$, hence, it is contained in that of $(2 + n - r)Q$. Thus, we have $r = 2$. In this case, we see that Q is isomorphic to P_n. □

Proof of Proposition 1. We separate our argument into two cases according to the existence of interior lattice points of nQ:

Case I: $\mathrm{Int}(nQ) \cap M = \varnothing$. Then Q is basic by Lemma 3 and $P \neq Q$. Let v' be a vertex of P not contained in Q. If v' is contained in the interior of the cone $C_v(P)$, then it would be contained in the interior of $2P$. Thus, v' is contained in the relative interior of a face cone

$$\mathbb{R}_{\geq 0}m_1 + \cdots + \mathbb{R}_{\geq 0}m_r$$

with $2 \leq r \leq n-1$ by renumbering, if necessary. Let $G \subset P$ be the face containing $m_1, ..., m_r$ and v'. Then v' is contained in the relative interior of $2G$ and $v' + m_{r+1} + \cdots + m_n$ is also contained in the interior of $(n - r + 2)P$, hence, $r \leq 2$ and $r = 2$. Since $(\mathrm{Int}(n-1)P) \cap M = \varnothing$, the face G does not contain lattice points in its relative interior. Since $\{m_1, ..., m_n\}$ is a $\mathbb{Z}$-basis of M, we can write as

$$v' = am_1 + m_2$$

with a positive integer a. If $a \geq 2$, then the number of lattice points in the interior of nP would be more than one. Thus, $a = 1$ and P is isomorphic to Q_n.

Case II: $\operatorname{Int}(nQ) \cap M \neq \emptyset$. Since $\sharp\{\operatorname{Int}(nP) \cap M\} = 1$, we have $\operatorname{Int}(nQ) \cap M = \{\widetilde{m}\}$. If $P \neq Q$, then there exists a facet F of Q which is not a facet of P. Since $\operatorname{Int}(nF) \cap M \neq \emptyset$, they would be contained in the interior of nP and do not coincide with $\widetilde{m}$, which contradicts to the assumption. Thus, we have $P = Q$. This case has been proved in Lemma 4. □

We note that Proposition 1 is an interpretation of Theorems 2 and 3 in terms of lattice polytopes.

3. A Certain Class of Gorenstein n-polytopes

In this section, for $n \geq 3$, we determine an n-polytope P such that $(n-1)P$ does not contain lattice points in its interior. This polytope will play an important role in the next section.

We recall that a basic n-simplex is isomorphic to

$$\Delta_0^n = \operatorname{Conv}\{0, e_1, \ldots, e_n\}.$$

Now, for positive integers $a_1 \leq a_2 \leq \cdots \leq a_n$, set

$$R_n := \operatorname{Conv}\{\Delta_0^{n-1}, e_1 + a_1 e_n, \ldots, e_{n-1} + a_{n-1}e_n, a_n e_n\}.$$

Then R_n is a nonsingular n-polytope and $(\operatorname{Int}(n-1)R_n) \cap M = \emptyset$. We also know $(\operatorname{Int}(n-1)P_n) \cap M = (\operatorname{Int}(n-1)Q_n) \cap M = \emptyset$.

Proposition 2. *Let $P \subset M_{\mathbb{R}}$ be a Gorenstein n-polytope for $n \geq 3$ satisfying the conditions $(\operatorname{Int}(n-1)P) \cap M = \emptyset$ and $(\operatorname{Int}(nP)) \cap M \neq \emptyset$. Then P is isomorphic to P_n, Q_n or R_n by a unimodular transformation of M.*

As in the proof of Proposition 1, we take a vertex v of P so that v coincides with the origin 0 of M. Since P is Gorenstein, the cone $C_v(P) = \mathbb{R}_{\geq 0}(P - v)$ contains the lattice point m_0 in its interior such that

$$(\operatorname{Int} C_v(P)) \cap M = m_0 + C_v(P) \cap M. \tag{3}$$

Let $\{\rho_1, ..., \rho_t\}$ be the set of all rays of the cone $C_v(P)$. For each i, let m_i be the generator of the semigroup $\rho_i \cap M$, that is, $\rho_i \cap M = \mathbb{Z}_{\geq 0} m_i$. Let $Q := \operatorname{Conv}\{0, m_1, ..., m_t\}$.

Let $\{F_1, ..., F_s\}$ be the set of all facets containing 0 of Q. For an F_j, set H the hyperplane containing F_j. Take a basis $\{e_1, ..., e_n\}$ so that $\{e_1, ..., e_{n-1}\}$ are contained in H and that the cone $C_v(P)$ is contained in the part of nonnegative coefficient of e_n. Then, from the equality (3), we see that

$$\operatorname{Conv}\{rF_j, m_0\} \cap (\operatorname{Int} C_v(P) \cap M) = \{m_0\}$$

for any positive integer r.

From this observation, we have a useful lemma.

Lemma 5. *Let Q be the n-polytope defined above. If m_0 is not contained in the interior of $2Q$, then $\operatorname{Conv}\{m_1, ..., m_t\}$ is a facet of Q.*

Proof. Let F be a facet of Q such that it does not contain 0 and it meets with the half line $\mathbb{R}_{\geq 0} m_0$. We assume that F does not contain all m_i's. For a vertex m_i of F, let $\{F_{i_1}, ..., F_{i_r}\}$ be the set of all facets of Q containing m_i. We may assume that a vertex $m_j \in F_{i_1}$ is not contained in F. We can choose a vertex $m_k \in F_{i_k}$ so that the relative interior of the segment $[m_j, m_k]$ is contained in the interior of Q from the convexity. If m_0 is not contained in the interior of the convex hull of the union of cones $C(F_{i_1}), ..., C(F_{i_r})$, then the lattice point $m_j + m_k$ is contained in the union of $\operatorname{Conv}\{2F_{i_j}, m_0\}$, which contradicts to our observation.

If m_0 is contained in the convex hull of the union of $C(F_{i_1}), \ldots, C(F_{i_r})$, then we can choose another vertex $m_{i'}$ of F instead of m_i outside of the convex hull of the union such that there is a facet $F_{i_1'}$ containing $m_{i'}$ with a vertex $m_{j'}$ not contained in F. By the same reason, this does not occur. Thus, Q has a facet containing all m's. □

Corollary 2. *Let Q be the n-polytope defined in Lemma* 5. *Assume that* $\operatorname{Int}(rQ) \cap M \neq \emptyset$ *for some* $2 \leq r \leq n$. *Then* m_0 *is contained in the interior of* rQ.

Proof. First, we assume that Q' has a facet F containing all $m_1, \ldots, m_t$. Let $\nu \in M_{\mathbb{Q}}^{\star}$ be the rational point in the dual space to $M_{\mathbb{Q}}$ such that the hyperplane $H_1 := \{x \in M_{\mathbb{R}}; \langle \nu, x \rangle = 1\}$ contains F. We note that the parallel hyperplane $H_0 = \{\langle \nu, x \rangle = 0\}$ touches the cone $C_v(Q)$ at one point 0, in particular, we have $\langle \nu, x \rangle \geq 0$ for all $x \in C_v(Q)$.

Set $\widetilde{m} \in \operatorname{Int}(rQ) \cap M$. Then $\langle \nu, \widetilde{m} \rangle < r$. If m_0 is not contained in the interior of rQ, then $\langle \nu, m_0 \rangle \geq r$. Since $C_v(Q)$ is Gorenstein at 0, there exists $m' \in (nQ) \cap M$ such that $\widetilde{m} = m_0 + m'$ by the equality (3). But $\langle \nu, m' \rangle < 0$. This is a contradiction. Thus, we have $m_0 \in \operatorname{Int}(rQ)$.

If m_0 is not contained in the interior of $2Q$, then Q has a facet F containing all m_i's. This proves Corollary 2. □

Remark 1. When $n = 2$, the statement for $r = 1$ in Corollary 2 also trivially holds because Q is a triangle.

We also define a Gorenstein Fano n-polytope Q_n' isomorphic to Q_n as

$$Q_n' := \operatorname{Conv}\{0, e_1, \ldots, e_n, e_1 + e_2 - e_n\}.$$

Lemma 6. *Let Q be the n-polytope defined above. Assume that* $\operatorname{Int}(n-1)Q \cap M = \emptyset$ *and that* $\operatorname{Int}(nQ) \cap M \neq \emptyset$. *Then Q is isomorphic to* P_n *or* Q_n'.

Proof. Since m_0 is contained in the interior of $C_v(P)$, as in the proof of Lemma 4, we can choose $m_1, \ldots, m_r$ so that the simplicial cone spanned by them contains m_0 in its relative interior by renumbering.

Set $G := \mathrm{Conv}\{m_1, \ldots, m_r\}$ and $\widetilde{G} := \mathrm{Conv}\{0, \ldots, G\}$. Then we have $r = n-1$ or $r = n$. We separate our argument into two cases:

Case (a): $r = n-1$. Since $\mathrm{Int}(n-1)\widetilde{G} \cap M = \emptyset$, we see that $\widetilde{G}$ is basic from Lemma 3. We note that $m_0 = m_1 + \cdots + m_{n-1} \in \mathrm{Int}(n\widetilde{G}) \subset \mathrm{Int}(nQ)$. As in the proof of Lemma 4, take the hyperplane H in $M_{\mathbb{R}}$ containing $\widetilde{G}$, and divide Q into the union of two n-polytopes $Q_{(1)}$ and $Q_{(2)}$ separated by H.

If $\mathrm{Int}(nQ_{(i)}) \cap M = \emptyset$ for $i = 1, 2$, then $Q = Q'_n \cong Q_n$ as in the proof of Lemma 4.

Assume that $\mathrm{Int}(nQ_{(1)}) \cap M \neq \emptyset$. Set $\widetilde{m} \in \mathrm{Int}(nQ_{(1)}) \cap M$. Since Q is Gorenstein at 0 and m_0 is contained in the boundary of $(n-1)\widetilde{G}$, hence, in that of $(n-1)Q_{(1)}$, there exists $m' \in (Q_{(1)} \backslash H) \cap M$ with $\widetilde{m} = m_0 + m'$. Consider the n-polytope $R = \mathrm{Conv}\{m', \widetilde{G}\}$. We see that $\widetilde{m}$ is not contained in the interior of nR. Since $\widetilde{m}$ is in the interior of $nQ_{(1)}$, the lattice point m' must be contained in the interior of $Q_{(1)}$. This is a contradiction. Thus, we see that $Q_{(i)}$ are basic.

Case (b): $r = n$. From Corollary 2, we have $m_0 \in \mathrm{Int}(n\widetilde{G})$. We note that $\mathrm{Int}(n-1)\widetilde{G} \cap M = \emptyset$ by assumption.

First, consider the case that $t = n$, that is, $Q = \widetilde{G}$. We claim that $\mathrm{Int}(n\widetilde{G}) \cap M = \{m_0\}$. If $\widetilde{m} \in M$ is a lattice point in the interior of $n\widetilde{G}$ other than m_0, then there would exist a lattice point $m' \in \widetilde{G}$ with $\widetilde{m} = m_0 + m'$ and m' would be in the interior of $\widetilde{G}$. This contradicts to the assumption. We confirm the claim. In this case, we know that $Q = \widetilde{G} = P_n$ from Lemma 4.

Next, consider the case $t > n$. Then there is a facet, say $F_{n-1} := \mathrm{Conv}\{0, m_1, ..., m_{n-1}\}$ of $\widetilde{G}$ which is not a facet of Q and is basic because of $\mathrm{Int}(n-1)F_{n-1} \cap M = \emptyset$. We can choose a new $\mathbb{Z}$-basis $\{e'_1, ..., e'_n\}$ of M as $m_i = e'_i$ for $i = 1, ..., n-1$ and $m_n = a_1 e'_1 + \cdots + a_n e'_n$ with $a_i \geq 0$. We note that $a_n \geq 2$ because $\widetilde{G}$ is not basic. Since $\mathrm{Int}(n-1)\widetilde{G} \cap M = \emptyset$, we see that $\widetilde{G}$ is normal from Lemma 1. Thus, $\widetilde{G}$ contains a lattice point u whose nth coordinate is one.

Let $\widetilde{F}'_r := \mathrm{Conv}\{0, m_1, ..., m_{r-1}, m_n\}$ be the face containing u with the smallest dimension. If u is in the relative interior, then we see $r = 1$ by the assumption. Then $m_n = a_n e'_n$ and $u = e'_n$. In this case, $\widetilde{G}$ is nonsingular at 0 and $m_0 = m_1 + \cdots + m_{n-1} + e'_n \in \mathrm{Int}(nG) \cap M$. The point $m_1 + \cdots + m_{n-1}$ is contained in the interior of nQ. Equation (3) implies that $-e'_n \in Q$, which contradicts to 0 is a vertex of Q.

If u is in the relative interior of the face $F'_r := \mathrm{Conv}\{m_1, ..., m_{r-1}, m_n\}$ of $\widetilde{F}'_r$, then $r = 2$ and $m_n = e'_1 + a_n e'_n$, $u = e'_1 + e'_n$. When $a_n = 2$, $\widetilde{G}$ is Gorenstein at 0 and coincides with P_n, hence, $m_0 = m_2 + \cdots + m_{n-1} + u \in \mathrm{Int}(n\widetilde{G}) \cap M$. Since $m_1 + \cdots + m_{n-1}$ is in $\mathrm{Int}(nQ)$, the point $-e'_n$ is contained in Q. This contradicts to m_1 is a vertex of Q.

In the case that $a_n \geq 3$, we can decompose $\widetilde{G}$ into a union of $a_n - 1$ basic n-simplices with vertices $\{e'_2, ..., e'_{n-1}, e'_1 + (j-1)e'_n, e'_1 + je'_n\}$ for $j = 1, ..., a_n - 1$. If m_0 is in the relative interior of the cone of dimension $n-1$, say $C(e'_2, ..., e'_{n-1}, e'_1 + je'_n)$, then it contradicts by the above reason. If m_0 is contained in the interior of the cone of one on these n-simplices, say $C(e'_2, ..., e'_{n-1}, e'_1 + (j-1)e'_n, e'_1 + je'_n)$, then $m_0 = e'_2 + \cdots + e'_{n-1} + (e'_1 + (j-1)e'_n) + (e'_1 + je'_n)$. In this case, it contradicts to that the cone is strictly convex. Thus, the case that $t > n$ does not occur. □

Proof of Proposition 2. We separate our argument into two cases according to the existence of interior lattice points of nQ:

Case I: $\operatorname{Int}(nQ) \cap M = \varnothing$. Then Q is basic and $P \neq Q$. Let v' be a vertex of P not contained in Q. As in the proof of Proposition 1, we may set $m_i = e_i$ for $i = 1, ..., n$ and we can write as

$$v' = ae_1 + e_2$$

with a positive integer a. We note that $m_0 = e_1 + \cdots + e_n$.

Set

$$G := \operatorname{Conv}\{v', e_2, ..., e_n\} \quad \text{and} \quad \widetilde{G} := \operatorname{Conv}\{0, G\}.$$

Then m_0 is contained in the relative interior of $(n-1)G$ and in the interior of $n\widetilde{G}$. In other words, P is included in the prism written as $x_1 \geq 0, ..., x_n \geq 0, x_2 + \cdots + x_n \leq 1$ by using the coordinates $(x_1, ..., x_n)$ of $M_{\mathbb{R}}$.

If v' is the only vertex of P other than $m_1, ..., m_n$, then $a = 1$ and P is isomorphic to Q_n otherwise the vertex m_3 is not Gorenstein.

Let $v_j = a_j e_1 + e_j$ for $j = 2, ..., n$ be vertices of P. We may set $a_2 \geq \cdots \geq a_n \geq 0$ and $a_3 \geq 1$. If $a_{n-1} \geq 1$ and $a_n = 0$, then the vertex m_n is not Gorenstein. Thus, $P \cong R_n$.

Case II: $\operatorname{Int}(nQ) \neq \varnothing$. Lemma 6 shows that $m_0 \in \operatorname{Int}(nQ)$ and that Q coincides with Q'_n (Case (a)) or P_n (Case (b)). We have to consider the case that $P \neq Q$.

In Case (a), we have

$$Q = Q'_n = \operatorname{Conv}\{0, e_1, ..., e_n, e_1 + e_2 - e_n\}.$$

We note that $n+1$ vertices of Q'_n except 0 are on a hyperplane. Set $\widetilde{F} = \operatorname{Conv}\{e_1, ..., e_n, e_1 + e_2 - e_n\}$. Since it is not simplex, $\operatorname{Int}(n-1)\widetilde{F}$

$\bigcap M \neq \emptyset$. If $P \neq Q$, then $\widetilde{F}$ is not a facet of P. This contradicts to $\mathrm{Int}(n-1)P \bigcap M = \emptyset$. Thus, $P = Q$.

In Case (b), we have

$$Q = P_n = \mathrm{Conv}\{0, 2e_1, e_2, ..., e_n\}.$$

We note that $m_0 = e_1 + \cdots + e_n \notin (n-1)P_n$. Consider the prism

$$(x_1 \geq 0, ..., x_n \geq 0, x_2 + \cdots + x_n \leq 1)$$

by using the coordinates $(x_1, ..., x_n)$ of $M_{\mathbb{R}}$. The point m_0 is contained in the boundary of the $(n-1)$-tuple of the prism. We see that if $P \neq Q$, then $P \cong R_n$ as in Case I. □

4. Fujita's Freeness Conjecture

In this section, we give a proof of Theorem 5, which is a strong version of Fujino's theorem [3] but restricted to Gorenstein toric varieties.

We recall the construction of the polarized toric n-fold (X, L) from a lattice n-polytope P (see, for instance, [9] or [5]). For simplicity, we assume that all toric varieties are defined over the complex number field $\mathbb{C}$. Let N be a free abelian group of rank n and M be the dual with the natural pairing $\langle\, , \rangle : M \times N \to \mathbb{Z}$. Let $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^{\times}$ be the algebraic torus of dimension n. Then the group of characters $\mathrm{Hom}_{\mathrm{gr}}(T_N, \mathbb{C}^{\times})$ can be identified with M and we have $T_N = \mathrm{Spec}\, \mathbb{C}[M]$. Let $P \subset M_{\mathbb{R}}$ be a lattice n-polytope. From P, we construct a polarized toric n-fold (X, L) satisfying the equality

$$\Gamma(X, L) \cong \bigoplus_{m \in P \bigcap M} \mathbb{C}e(m), \tag{4}$$

where we write as $e(m)$ the character corresponding to a lattice point $m \in M$. Since X contains T_N as an open subset, we can consider $e(m)$ as a rational function on X.

For a vertex v of P, let $\sigma(v) \subset N_{\mathbb{R}}$ be the cone dual to the cone $C_v(P) = \mathbb{R}_{\geq 0}(P - v) \subset M_{\mathbb{R}}$. Let Φ be the set of all faces of cones $\sigma(v)$ for all vertices of P. The Φ is a complete fan in N and defines a toric variety X of dimension n. We note that X is covered by affine open sets $U_v :=$ $\operatorname{Spec} \mathbb{C}[M \cap C_v(P)]$. Here we define a line bundle L so that

$$\Gamma(U_v, L) = e(v)\mathbb{C}[M \cap C_v(P)].$$

Then L is generated by global sections and ample. By definition, L satisfies (4).

Furthermore, we assume that X is Gorenstein. For each vertex v, the cone $C_v(P)$ contains the lattice point m_v satisfying the equality

$$(\operatorname{Int} C_v(P)) \cap M = m_v + C_v(P) \cap M. \tag{5}$$

Thus, we see that $L + K_X$ is generated by global sections if P contains all m_v in its interior.

Proposition 3. *Let X be a projective Gorenstein toric n-fold with $n \geq 2$. Let L be an ample line bundle on X satisfying the condition that $\Gamma(X, L + K_X) \neq 0$ and that the intersection number $L \cdot C \geq n - 1$ for all irreducible invariant curves C. Then $L + K_X$ is nef.*

Proof. Let $P \subset M_{\mathbb{R}}$ be the lattice n-polytope corresponding to L. The condition $\Gamma(X, L + K_X) \neq 0$ is interpreted to $\operatorname{Int} P \cap M \neq \emptyset$. Since P is Gorenstein, for each vertex v of P, there exists the lattice point m_v satisfying equation (5). From the above observation, it suffices to show that the lattice points m_v are contained in the interior of P for all vertices v.

As in the proof of Proposition 1, we take a vertex v of P so that v coincides with the origin 0 of M. Let $\{m_1, ..., m_t\}$ be the set of the nearest points on all edges through the vertex $v = 0$ and $Q := \operatorname{Conv}\{0, m_1, ..., m_t\}$.

Then $(n-1)Q$ is contained in P because all edges of P have length at least $n-1$. If $\operatorname{Int}(n-1)Q \cap M \neq \emptyset$, then we see that $m_v \in \operatorname{Int}(n-1)Q \subset \operatorname{Int}P$ by Corollary 2. We assume that $\operatorname{Int}(n-1)Q \cap M = \emptyset$.

If $\operatorname{Int}(nQ) \cap M = \emptyset$, then Q is basic, hence, $t = n$ and we may set $m_i = e_i$ with a $\mathbb{Z}$-basis $\{e_1, ..., e_n\}$ of M. Then we have $m_v = e_1 + \cdots + e_n$. Let m' be a lattice point in the interior of P different from m_v. Then since $\operatorname{Conv}\{(n-1)e_1, ..., (n-1)e_n, m'\}$ contains m_v, the polytope P contains m_v in its interior.

If $\operatorname{Int}(nQ) \cap M \neq \emptyset$, then Q is not basic and $m_v \in \operatorname{Int}(nQ)$ by Corollary 2. From Lemma 6, we see that Q is Q_n', that is, $n \geq 3$, $t = n+1$ and $m_i = e_i$ for $i = 1, ..., n$ and $m_{n+1} = e_1 + e_2 - e_n$ for a $\mathbb{Z}$-basis $\{e_1, ..., e_n\}$ of M. In particular, Q has the facet F containing all m_i's and $m_v = e_1 + \cdots + e_{n-1}$. Since $m_v \in (n-1)F$, the condition $\operatorname{Int}P \cap M \neq \emptyset$ implies that $(n-1)Q \neq P$ and $m_v \in \operatorname{Int}P$. □

By combining Propositions 2 and 3 and Lemma 3, we obtain the proof of Theorem 5.

5. Fujita's Very Ampleness Conjecture

In this section, we will give a proof of Theorem 6. Let $P \subset M_{\mathbb{R}}$ be the Gorenstein lattice n-polytope corresponding to L. As in the previous section, for a vertex v of P denote m_v the lattice point satisfying equation (4). As in the proof of Proposition 3, we take a vertex v of P so that v coincides with the origin 0 of M. Let $\{m_1, ..., m_t\}$ be the set of the nearest points on all edges through the vertex $v = 0$ and $Q := \operatorname{Conv}\{0, m_1, ..., m_t\}$. We know that m_v is contained in the interior of $(n+1)Q$.

If $\operatorname{Int}(r-1)Q \cap M = \emptyset$ and $\operatorname{Int}(rQ) \cap M \neq \emptyset$ for some r with $2 \leq r \leq n$, then $m_v \in \operatorname{Int}(rQ)$ by Corollary 2 and $(n+1-r)Q$ is normal by

Lemma 1 and

$$m_v + (n+1-r)Q \subset \mathrm{Int}(n+1)Q \subset \mathrm{Int}P.$$

When $\mathrm{Int}(Q) \cap M \neq \emptyset,$ we see that m_v is contained in the interior of $2Q$ from Corollary 2 and $(n-1)Q$ is normal from Theorem 7, hence, $m_v + (n-1)Q$ is contained in the interior of $(n+1)Q.$ This implies that $L + K_X$ is very ample on the affine open set $U_v.$

Assume that $\mathrm{Int}(nQ) \cap M = \emptyset.$ Then Q is basic, $t = n$ and $m_i = e_i$ for a $\mathbb{Z}$-basis $\{e_1, ..., e_n\}$ of M. In particular, Q is normal and $m_v = e_1 + \cdots + e_n.$ Let F be the facet of Q containing all $e_i.$ Then all $m_v + e_i$ are contained in the relative interior of $(n+1)F.$ Since $P \neq (n+1)Q$ by assumption, all $m_v + e_i$ are contained in the interior of P. This implies that $L + K_X$ is very ample on this $U_v.$ This completes the proof of Theorem 6.

We remark that the condition $L \cdot C \geq n+1$ is best possible for all dimensions $n \geq 2$. A Gorenstein toric Fano n-fold with index n corresponding to P_n or Q_n trivially attains the bound for all $n \geq 2.$

Besides Gorenstein toric Fano n-fold with index n, we also have a Gorenstein toric Fano n-fold with index $n-1$ which attains the bound "$n+1$".

For $n \geq 3,$ we define a lattice n-simplex as

$$D_n := \mathrm{Conv}\{0, e_1, e_2, e_1 + e_2 + 2e_3, e_4, ..., e_n\}.$$

Then it is not very ample and satisfies $\mathrm{Int}(n-2)D_n \cap M = \emptyset$ and $\sharp(\mathrm{Int}(n-1)D_n) \cap M = 1,$ hence, $2D_n$ is normal. If we denote by m_0 the unique interior lattice point of $(n-1)D_n,$ then we have

$$m_0 + (2D_n) \cap M = \mathrm{Int}(n+1)D_n \cap M.$$

Payne has pointed out the case $n = 3$ in [10].

We can characterize D_n among polarized toric varieties as the following way.

Proposition 4. *Let D be an empty lattice n-simplex for* $n \geq 3$. *Assume that* $\sharp(\mathrm{Int}(n-1)D) \cap M = 1$. *Then D is isomorphic to* D_n.

Proof. We note that D is not basic.

Consider the case $n = 3$. An empty lattice 3-simplex D is written as

$$D = \mathrm{Conv}\{0, e_1, e_2, e_1 + pe_2 + qe_3\}$$

with $1 \leq p < q$ and $\gcd(p, q) = 1$ by [9]. Then $\sharp\mathrm{Int}(2D) \cap M = q - 1$. Hence, $q = 2$ and $D \cong D_3$.

Set $n \geq 4$. We may write as

$$D = \mathrm{Conv}\{0, m_1, ..., m_n\}$$

for linearly independent $m_1, ..., m_n \in M$.

First, we assume that the facet $F = \mathrm{Conv}\{0, m_1, ..., m_{n-1}\}$ is basic, that is, $m_i = e_i$ for $i = 1, ..., n-1$. Then $\mathrm{Int}(n-2)F \cap M = \emptyset$. We may write as

$$m_n = a_1 e_1 + \cdots + a_n e_n$$

with $a_i \geq 0$ for $i = 1, ..., n-1$ and $a_n \geq 1$. Set H the hyperplane containing F. Moreover, we assume that all the facets of D are basic. Then the nth coordinates of lattice points in $(n-2)D \backslash H$ are at least a. On the other hand, since $\sharp(\mathrm{Int}(n-1)D) \cap M = 1$, $(n-2)D$ does not contain lattice points in its interior, hence, $2D$ is normal by Lemma 1. Since $2 \leq n-2$, $(n-2)D$ is also normal, hence, there exists a lattice point in it whose nth coordinate is 1. Then $a = 1$. This implies that D is basic and contradicts to the assumption. Then we see that at least one facet of D is not basic.

Next, if the facet F is not basic, then $(n-2)F$ contains lattice points in its interior, in fact, interior lattice point is unique point in $\mathrm{Int}(n-1)D$. By induction on n, we see that the facet F is isomorphic to D_{n-1} and $\mathbb{Z}(F \cap M) = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{n-1}$. Since $2D$ is normal, we can take $m_n = e_n$, hence, $D \cong D_n$. □

We remark that "$n-1$" in Proposition 4 is essential. For $n \geq 4$, we define a lattice n-simplex as

$$P := \mathrm{Conv}\{0, e_1, ..., e_{n-1}, e_1 + e_2 + e_3 + 3e_n\}.$$

We note that P is not Gorenstein. Set $m' := e_1 + \cdots + e_{n-1} + 2e_n$. Then we see that $\mathrm{Int}(n-2)P \cap M = \{m'\}$.

References

[1] V. Batyrev and D. Juny, Classification of Gorenstein toric Del Pezzo varieties in arbitrary dimension, Mosc. Math. J. 10 (2010), 285-316.

[2] V. Batyrev and B. Nill, Multiples of lattice polytopes without interior lattice points, Mosc. Math. J. 7 (2007), 195-207.

[3] O. Fujino, Notes on toric varieties from Mori theoretic view point, Tohoku Math. J. 55 (2003), 551-564.

[4] T. Fujita, On polarized manifolds whose adjoint bundles are not semi-positive, Algebraic Geometry Sendai, 1985, pp. 167-178, Adv. Stud. Pure Math. 10, North-Holland, Amsterdam, 1987.

[5] W. Fulton, Introduction to Toric Varieties, Ann. Math. Studies No. 131, Princeton Univ. Press, 1993.

[6] S. Kobayashi and T. Ochiai, Characterizations of complex projective spaces and hyperquadrics, J. Math. Kyoto Univ. 13 (1973), 31-47.

[7] D. Mumford, Varieties defined by quadratic equations, Questions on Algebraic Varieties, Corso C.I.M.E. 1969, Cremonese, Rome, 1970, pp. 29-100.

[8] K. Nakagawa, Generators for the ideal of a projectively embedded toric varieties, Thesis, Tohoku University, 1994.

[9] T. Oda, Convex bodies and algebraic geometry, Ergebnisse der Math., 15, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1988.

[10] S. Payne, Fujita's very ampleness conjecture for singular toric varieties, Tohoku Math. J. 58 (2006), 447-459.