Far East Journal of Mathematical Sciences (FJMS)

Published Online: December 2014
Available online at http://pphmj.com/journals/fjms.htm
Volume 94, Number 1, 2014, Pages 33-45

β-OPEN SETS IN BICLOSURE SPACES

Jeeranunt Khampakdee

Department of Mathematics
Faculty of Science
Mahasarakham University
Mahasarakham 44150, Thailand
e-mail: jeeranunt.k@msu.ac.th

Abstract

The purpose of this paper is to introduce the concept of β-open sets in biclosure spaces and investigate their fundamental properties. Moreover, we give the concepts of β-open maps and β-continuous maps and their characterizations.

1. Introduction

The concept of Čech closure spaces was introduced and studied in [3]. In 2003, Šlapal [6] introduced generalized Čech closure operators which are called *closure operators*. Later, some properties of closed sets in closure spaces were studied by Boonpok and Khampakdee [2]. In 2009, Boonpok [1] introduced biclosure spaces and provided a characterization of closed sets of some kind in the product of biclosure spaces. In [5], Khampakdee and Boonpok introduced the notions of semi-open sets and semi-closed sets in biclosure spaces.

In this paper, we introduce and study β-open sets in biclosure spaces and

Received: June 10, 2014; Accepted: September 9, 2014
2010 Mathematics Subject Classification: 54A05.
Keywords and phrases: β-open sets, biclosure spaces, β-open maps, β-continuous maps.

investigate some of their fundamental properties. Then we use β-open sets to define β-open maps and β-continuous maps. We obtain certain properties of β-openness and β-continuity in biclosure spaces.

2. Preliminary Notes

In this section, we recall some basic definitions and notations of closure spaces and biclosure spaces in [1, 2] and [5].

A map $u : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* [2] on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied:

(A1) $u\varnothing = \varnothing$,

(A2) $A \subseteq uA$ for every $A \subseteq X$,

(A3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A subset $A \subseteq X$ is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement is closed. The empty set and the whole space are both open and closed.

If (X, u) and (Y, v) are closure spaces, then a map $f : (X, u) \to (Y, v)$ is called:

(i) *open* (respectively, *closed*) if the image of each open (respectively, closed) set in (X, u) is open (respectively, closed) in (Y, v),

(ii) *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$. One can see that, if f is continuous, then the inverse image under f of each open set in (Y, v) is open in (X, u).

The product [1] of a family $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ of closure spaces, denoted by $\prod_{\alpha \in J} (X_\alpha, u_\alpha)$, is the closure space $\left(\prod_{\alpha \in J} X_\alpha, u\right)$, where

$\prod_{\alpha \in J} X_\alpha$ denotes the Cartesian product of the sets X_α, $\alpha \in J$ and u is the closure operator generated by the projections $\pi_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$, $\alpha \in J$, i.e., is defined by $uA = \prod_{\alpha \in J} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod_{\alpha \in J} X_\alpha$. Clearly, π_α is continuous for each $\alpha \in J$.

Since $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ is continuous whenever $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$ is the product of a family $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ of closure spaces, $G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is open in $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$ for every open set G_β in (X_β, u_β), $\beta \in J$.

A *biclosure space* [5] is a triple (X, u_1, u_2), where X is a set and u_1, u_2 are two closure operators on X.

A subset A of a biclosure space (X, u_1, u_2) is called *closed* if $u_1 u_2 A = A$. The complement of closed set is called *open*.

One can see that, A is open in (X, u_1, u_2) if and only if A is open in both (X, u_1) and (X, u_2).

Let (X, u_1, u_2) be a biclosure space. A biclosure space (Y, v_1, v_2) is called a *subspace* of (X, u_1, u_2) if $Y \subseteq X$ and $v_i A = u_i A \cap Y$ for all $i \in \{1, 2\}$ and every subset A of Y.

Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i \in \{1, 2\}$. Then a map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called:

(i) *i-open* (respectively, *i-closed*) if the map $f : (X, u_i) \to (Y, v_i)$ is open (respectively, closed) for all $i \in \{1, 2\}$,

(ii) *open* (respectively, *closed*) if f is i-open (respectively, i-closed) for all $i \in \{1, 2\}$,

(iii) *i-continuous* if the map $f : (X, u_i) \to (Y, v_i)$ is continuous for all $i \in \{1, 2\}$,

(iv) *continuous* if f is *i-continuous* for all $i \in \{1, 2\}$.

Proposition 2.1 [5]. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. If f is open, then $f(G)$ is open in (Y, v_1, v_2) for every open subset G of (X, u_1, u_2).*

Proposition 2.2 [5]. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. If f is continuous, then $f^{-1}(H)$ is open in (X, u_1, u_2) for every open subset H of (Y, v_1, v_2).*

3. Main Results

In this section, we introduce the concept of β-open sets in biclosure spaces and investigate some of their properties.

Definition 3.1. A subset A of a biclosure space (X, u_1, u_2) is called β-*open* if there exists an open subset G of (X, u_1) such that $G \subseteq A$ and $u_2A \subseteq u_2G$. The complement of a β-open subset of X is called β-*closed*.

Clearly, if (X, u_1, u_2) is a biclosure space and A is open (respectively, closed) in (X, u_1), then A is β-open (respectively, β-closed) in (X, u_1, u_2).

The converse is not true as shown in the following example.

Example 3.2. Let $X = \{1, 2, 3\}$ and define a closure space u_1 on X by $u_1\varnothing = \varnothing$, $u_1\{1\} = u_1\{2\} = u_1\{1, 2\} = \{1, 2\}$, $u_1\{3\} = \{2, 3\}$, and $u_1\{1, 3\} = u_1\{2, 3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\varnothing = \varnothing$, $u_2\{1\} = \{1, 3\}$, $u_2\{2\} = \{1, 2\}$, $u_2\{3\} = u_2\{1, 2\} = u_2\{1, 3\} = u_2\{2, 3\} = u_2X = X$.

It follows that $\{3\}$ is an open subset of (X, u_1) such that $\{3\} \subseteq \{1, 3\}$ and $u_2\{1, 3\} \subseteq u_2\{3\}$. Hence, $\{1, 3\}$ is β-open in (X, u_1, u_2). But $\{1, 3\}$ is not open in (X, u_1). Moreover, $\{2\}$ is β-closed in (X, u_1, u_2) but $\{2\}$ is not closed in (X, u_1).

Theorem 3.3. *Let (X, u_1, u_2) be a biclosure space and let $A \subseteq X$. Then A is β-closed if and only if there exists a closed subset F of (X, u_1) such that $A \subseteq F$ and $u_2(X - A) \subseteq u_2(X - F)$.*

Proof. Let A be β-closed in (X, u_1, u_2). Then there exists an open set G in (X, u_1) such that $G \subseteq X - A$ and $u_2(X - A) \subseteq u_2G$. Since G is open in (X, u_1), there exists a closed subset F of (X, u_1) such that $G = X - F$. But $A \subseteq X - G$ and $u_2(X - A) \subseteq u_2G$. Thus, $A \subseteq F$ and $u_2(X - A) \subseteq u_2(X - F)$.

Conversely, by the assumption, there is a closed subset F of (X, u_1) such that $A \subseteq F$ and $u_2(X - A) \subseteq u_2(X - F)$. Since F is closed in (X, u_1), there exists an open subset G of (X, u_1) such that $F = X - G$. It follows that $A \subseteq X - G$ and $u_2(X - A) \subseteq u_2(X - (X - G)) = u_2G$. Hence, $G \subseteq X - A$ and $u_2(X - A) \subseteq u_2G$. Thus, $X - A$ is β-open in (X, u_1, u_2). Therefore, A is β-closed in (X, u_1, u_2). □

Theorem 3.4. *Let (X, u_1, u_2) be a biclosure space and A be a β-open subset in (X, u_1, u_2). If $A \subseteq B$ and $u_2B \subseteq u_2A$, then B is β-open.*

Proof. Let A be β-open in (X, u_1, u_2). Then there exists an open set G in (X, u_1) such that $G \subseteq A$ and $u_2A \subseteq u_2G$. Since $A \subseteq B$ and $u_2B \subseteq u_2A$, $G \subseteq A \subseteq B$ and $u_2B \subseteq u_2A \subseteq u_2G$. Therefore, B is β-open in (X, u_1, u_2). □

Theorem 3.5. *Let* (Y, v_1, v_2) *be a biclosure subspace of* (X, u_1, u_2) *and let* A *be a subset of* Y. *If* A *is* β-*open in* (X, u_1, u_2), *then* A *is* β-*open in* (Y, v_1, v_2).

Proof. Let A be a β-open set in (X, u_1, u_2). Then there exists an open set G in (X, u_1) such that $G \subseteq A$ and $u_2 A \subseteq u_2 G$. It follows that $u_2 A \cap Y \subseteq u_2 G \cap Y$. Hence, $G \subseteq A$ and $v_2 A \subseteq v_2 G$. Since G is open in (X, u_1), $v_1(Y - G) = u_1(Y - G) \cap Y \subseteq u_1(X - G) \cap Y = (X - G) \cap Y = Y - G$. Thus, $Y - G$ is closed in (Y, v_1), i.e., G is open in (Y, v_1). Therefore, A is β-open in (Y, v_1, v_2). □

Definition 3.6. Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of subsets in a set X. A closure operator u on X is called *generalized additive* if $u \bigcup_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} u A_\alpha$.

Example 3.7. Let $X = \{1, 2, 3\}$ and define a closure operator u on X by $u\emptyset = \emptyset$, $u\{1\} = \{1\}$, $u\{2\} = u\{3\} = u\{1, 2\} = u\{1, 3\} = u\{2, 3\} = uX = X$. Then u is generalized additive on X.

Theorem 3.8. *Let* $\{A_\alpha\}_{\alpha \in J}$ *be a collection of* β-*open sets in a biclosure space* (X, u_1, u_2). *If* u_2 *is generalized additive*, *then* $\bigcup_{\alpha \in J} A_\alpha$ *is a* β-*open set in* (X, u_1, u_2).

Proof. Let A_α be β-open in (X, u_1, u_2) for all $\alpha \in J$. Hence, there exists an open set G_α in (X, u_1) such that $G_\alpha \subseteq A_\alpha$ and $u_2 A_\alpha \subseteq u_2 G_\alpha$ for each $\alpha \in J$. Thus, $\bigcup_{\alpha \in J} G_\alpha \subseteq \bigcup_{\alpha \in J} A_\alpha$ and $\bigcup_{\alpha \in J} u_2 A_\alpha \subseteq \bigcup_{\alpha \in J} u_2 G_\alpha$. Since u_2 is additive,

$$u_2 \bigcup_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} u_2 A_\alpha \subseteq \bigcup_{\alpha \in J} u_2 G_\alpha = u_2 \bigcup_{\alpha \in J} G_\alpha.$$

As G_α is open in (X, u_1) for all $\alpha \in J$, $u_1 \bigcap_{\alpha \in J} (X - G_\alpha) \subseteq u_1(X - G_\alpha)$

$= X - G_\alpha$ for each $\alpha \in J$. Thus, $u_1 \bigcap_{\alpha \in J} (X - G_\alpha) \subseteq \bigcap_{\alpha \in J} (X - G_\alpha)$. It follows that $\bigcap_{\alpha \in J} (X - G_\alpha)$ is closed in (X, u_1), i.e., $\bigcup_{\alpha \in J} G_\alpha$ is open in (X, u_1). Therefore, $\bigcup_{\alpha \in J} A_\alpha$ is β-open in (X, u_1, u_2). □

If $\{A_\alpha\}_{\alpha \in J}$ is a collection of β-open sets in a biclosure space (X, u_1, u_2) but u_2 is not generalized additive, then $\bigcup_{\alpha \in J} A_\alpha$ need not be a β-open set in (X, u_1, u_2) as shown in the following example.

Example 3.9. Let $X = \{1, 2, 3, 4\}$ and define a closure operator u_1 on X by $u_1\varnothing = \varnothing$, $u_1\{1\} = \{1\}$, $u_1\{2\} = u_1\{3\} = u_1\{4\} = u_1\{2, 3\} = u_1\{2, 4\} = u_1\{3, 4\} = u_1\{2, 3, 4\} = \{2, 3, 4\}$ and $u_1\{1, 2\} = u_1\{1, 3\} = u_1\{1, 4\} = u_1\{1, 2, 3\} = u_1\{1, 2, 4\} = u_1\{1, 3, 4\} = u_1 X = X$. Define a closure operator u_2 on X by $u_2\varnothing = \varnothing$, $u_2\{2\} = \{2\}$, $u_2\{3\} = \{1, 3\}$, $u_2\{4\} = \{1, 4\}$, $u_2\{1\} = u_2\{1, 2\} = u_2\{1, 3\} = \{1, 2, 3\}$ and $u_2\{1, 4\} = u_2\{2, 3\} = u_2\{2, 4\} = u_2\{3, 4\} = u_2\{1, 2, 3\} = u_2\{1, 2, 4\} = u_2\{1, 3, 4\} = u_2\{2, 3, 4\} = u_2 X = X$.

Since $u_2(\{1, 2\} \cup \{1, 3\}) = X$ is not a subset of $u_2\{1, 2\} \cup u_2\{1, 3\} = \{1, 2, 3\}$, u_2 is not generalized additive. By the definitions of u_1 and u_2, $\{1, 2\}$ and $\{1, 3\}$ are β-open in (X, u_1, u_2). But $\{1, 2\} \cup \{1, 3\} = \{1, 2, 3\}$ is not β-open in (X, u_1, u_2).

Corollary 3.10. *Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of β-closed sets in a biclosure space (X, u_1, u_2). If u_2 is generalized additive, then $\bigcap_{\alpha \in J} A_\alpha$ is a β-closed set in (X, u_1, u_2).*

If $\{A_\alpha\}_{\alpha \in J}$ is a collection of β-closed sets in a biclosure space (X, u_1, u_2) but u_2 is not generalized additive, then $\bigcap_{\alpha \in J} A_\alpha$ need not be a β-closed set in (X, u_1, u_2) as shown in the following example.

Example 3.11. By Example 3.9, u_2 is not generalized additive. By the definitions of u_1 and u_2, $\{2, 4\}$ and $\{3, 4\}$ are β-closed in (X, u_1, u_2). But $\{2, 4\} \cap \{3, 4\} = \{4\}$ is not β-closed in (X, u_1, u_2).

Theorem 3.12. *Let* $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in J\}$ *be a family of biclosure spaces and let* $\beta \in J$. *If* A_β *is* β-*open in* $(X_\beta, u_\beta^1, u_\beta^2)$, *then* $A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ *is* β-*open in* $\prod_{\alpha \in J} (X_\alpha, u_\alpha^1, u_\alpha^2)$.

Proof. Let $\beta \in J$ and let A_β be β-open in $(X_\beta, u_\beta^1, u_\beta^2)$. Then there exists an open set G_β in (X_β, u_β^1) such that $G_\beta \subseteq A_\beta$ and $u_\beta^2 A_\beta \subseteq u_\beta^2 G_\beta$. Hence, $G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \subseteq A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ and $u_\beta^2 A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \subseteq u_\beta^2 G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$.

Since

$$u_\beta^2 A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha = u_\beta^2 A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} u_\alpha^2 X_\alpha$$

$$= \prod_{\alpha \in J} u_\alpha^2 \pi_\alpha \left(A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \right) = u_2 \left(A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \right)$$

and

$$u_\beta^2 G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha = u_\beta^2 G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} u_\alpha^2 X_\alpha$$

$$= \prod_{\alpha \in J} u_\alpha^2 \pi_\alpha \left(G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \right) = u_2 \left(G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \right),$$

$$u_2\left(A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha\right) \subseteq u_2\left(G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha\right).$$

As π_β is continuous, $\pi_\beta^{-1}(G_\beta) = G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is open in $\prod_{\alpha \in J} (X_\alpha, u_\alpha^1)$.

Therefore, $A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is β-open in $\prod_{\alpha \in J} (X_\alpha, u_\alpha^1, u_\alpha^2)$. □

Theorem 3.13. *Let* $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in J\}$ *be a family of closure spaces and let* $\beta \in J$. *If* A_β *is* β-*closed in* $(X_\beta, u_\beta^1, u_\beta^2)$, *then* $A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ *is* β-*closed in* $\prod_{\alpha \in J} (X_\alpha, u_\alpha^1, u_\alpha^2)$.

Proof. Let A_β be β-closed in $\prod_{\alpha \in J} (X_\alpha, u_\alpha^1, u_\alpha^2)$, $\beta \in J$. Then $X_\beta - A_\beta$ is β-open in $(X_\beta, u_\beta^1, u_\beta^2)$. Since $(X_\beta - A_\beta) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is a β-open set in $\prod_{\alpha \in J} (X_\alpha, u_\alpha^1, u_\alpha^2)$. But $(X_\beta - A_\beta) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha = \prod_{\alpha \in J} X_\alpha - \left(A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha\right)$.
It follows that $A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is β-closed in $\prod_{\alpha \in J} (X_\alpha, u_\alpha^1, u_\alpha^2)$. □

Definition 3.14. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called β-*open* (respectively, β-*closed*) if $f(A)$ is β-open (respectively, β-closed) in (Y, v_1, v_2) for every open (respectively, closed) subset A of (X, u_1, u_2).

Clearly, if f is open (respectively, closed), then f is β-open (respectively, β-closed). The converse need not be true in general as can be seen from the following example.

Example 3.15. Let $X = \{1, 2, 3\} = Y$ and define a closure operator u_1 on X by $u_1\varnothing = \varnothing$, $u_1\{2\} = \{2\}$ and $u_1\{1\} = u_1\{3\} = u_1\{1, 2\} = u_1\{1, 3\} = u_1\{2, 3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\varnothing = \varnothing$, $u_2\{1\} = \{1, 2\}$, $u_2\{2\} = \{2\}$, $u_2\{3\} = \{1, 3\}$ and $u_2\{1, 2\} = u_2\{1, 3\} = u_2\{2, 3\} = u_2X = X$. Define a closure operator v_1 on Y by $v_1\varnothing = \varnothing$, $v_1\{1\} = v_1\{2\} = v_1\{1, 2\} = \{1, 2\}$, $v_1\{3\} = v_1\{1, 3\} = v_1\{2, 3\} = v_1Y = Y$ and define a closure operator v_2 on Y by $v_2\varnothing = \varnothing$ and $v_2\{1\} = \{1, 3\}$, $v_2\{2\} = \{1, 2\}$ and $v_2\{3\} = v_2\{1, 2\} = v_2\{1, 3\} = v_2\{2, 3\} = v_2Y = Y$. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be an identity map. It is easy to see that f is β-open but not open because $f(\{1, 3\})$ is not open in (Y, v_1, v_2) while $\{1, 3\}$ is open in (X, u_1, u_2). Moreover, we can see that f is β-closed but not closed because $f(\{2\})$ is not closed in (Y, v_1, v_2) while $\{2\}$ is closed in (X, u_1, u_2).

Theorem 3.16. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. Then $g \circ f$ is β-open if f is open and g is β-open.*

Proof. Let G be an open subset of (X, u_1, u_2). Since f is open, $f(G)$ is open in (Y, v_1, v_2). As g is β-open, $g(f(G)) = g \circ f(G)$ is β-open in (Z, w_1, w_2). Therefore, $g \circ f$ is β-open. □

Theorem 3.17. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. If $g \circ f$ is β-open and f is a continuous surjection, then g is β-open.*

Proof. Let H be an open set in (Y, v_1, v_2). Since f is continuous, $f^{-1}(H)$ is open in (X, u_1, u_2). Since $g \circ f$ is β-open, $g \circ f(f^{-1}(H))$ is β-open in (Z, w_1, w_2). But f is a surjection, hence $g \circ f(f^{-1}(H)) = g(f(f^{-1}(H))) = g(H)$. Thus, $g(H)$ is β-open in (Z, w_1, w_2). Therefore, g is β-open. □

Definition 3.18. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called β-*continuous* if $f^{-1}(H)$ is a β-open subset of (X, u_1, u_2) for every open subset H of (Y, v_1, v_2).

Clearly, if f is continuous, then f is β-continuous. The converse need not be true as can be seen from the following example.

Example 3.19. Let $X = \{1, 2, 3\} = Y$ and define a closure operator u_1 on X by $u_1\varnothing = \varnothing$, $u_1\{1\} = \{1, 2\}$, $u_1\{2\} = u_1\{3\} = u_1\{2, 3\} = \{2, 3\}$, $u_1\{1, 2\} = u_1\{1, 3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\varnothing = \varnothing$, $u_2\{2\} = \{2\}$ and $u_2\{1\} = u_2\{3\} = u_2\{1, 2\} = u_2\{1, 3\} = u_2\{2, 3\} = u_2X = X$. Define a closure operator v_1 on X by $v_1\varnothing = \varnothing$, $v_1\{3\} = \{3\}$, $v_1\{1\} = \{1, 3\}$, $v_1\{2\} = \{2, 3\}$, $v_1\{1, 2\} = v_1\{1, 3\} = v_1\{2, 3\} = v_1Y = Y$ and define a closure operator v_2 on Y by $v_2\varnothing = \varnothing$, $v_2\{1\} = v_2\{1, 3\} = \{1, 3\}$, $v_2\{3\} = \{3\}$, $v_2\{2\} = v_2\{1, 2\} = u_1\{2, 3\} = v_2Y = Y$. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be an identity map. It is easy to see that f is β-continuous but not continuous because $f^{-1}(\{1, 2\})$ is not open in (X, u_1, u_2) while $\{1, 2\}$ is open in (Y, v_1, v_2).

Theorem 3.20. *Let* (X, u_1, u_2) *and* (Y, v_1, v_2) *be biclosure spaces. A map* $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ *is* β-*continuous if and only if* $f^{-1}(F)$ *is a* β-*closed subset of* (X, u_1, u_2) *for every closed subset* F *of* (Y, v_1, v_2).

Proof. Let F be a closed subset of (Y, v_1, v_2). As f is β-continuous, $f^{-1}(Y-F)$ is β-open in (X, u_1, u_2). But $X - f^{-1}(F) = f^{-1}(Y-F)$. Hence, $f^{-1}(F)$ is β-closed in (X, u_1, u_2).

Conversely, let H be an open subset of (Y, v_1, v_2). By the assumption, $f^{-1}(Y-H) = X - f^{-1}(H)$ is β-closed in (X, u_1, u_2). Hence, $f^{-1}(H)$ is β-open in (X, u_1, u_2). Thus, f is β-continuous. □

Theorem 3.21. *Let* (X, u_1, u_2), (Y, v_1, v_2) *and* (Z, w_1, w_2) *be biclosure spaces. If* $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ *is* β-*continuous and* $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ *is continuous, then* $g \circ f$ *is* β-*continuous.*

Proof. Let G be an open subset of (Z, w_1, w_2). Since g is continuous, $g^{-1}(G)$ is open in (Y, v_1, v_2). As f is β-continuous, $f^{-1}(g^{-1}(G))$ is β-open in (X, u_1, u_2). But $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$. Therefore, $g \circ f$ is β-continuous. □

Corollary 3.22. *Let* (X, u_1, u_2) *be a biclosure space,* $\{(Y_\alpha, v_\alpha^1, v_\alpha^2) : \alpha \in J\}$ *be a family of biclosure spaces and*

$$f : (X, u_1, u_2) \to \prod_{\alpha \in J} (Y_\alpha, v_\alpha^1, v_\alpha^2)$$

be a map. If f is β-*continuous and* π_α *is a projection map, then* $\pi_\alpha \circ f$ *is* β-*continuous for each* $\alpha \in J$.

Definition 3.23. A biclosure space (X, u_1, u_2) is said to be a β-*open space* if every β-open set in (X, u_1, u_2) is open in (X, u_1, u_2).

Example 3.24. Let $X = \{1, 2, 3\}$ and define a closure operator u_1 on X by $u_1\varnothing = \varnothing$, $u_1\{1\} = u_1\{3\} = u_1\{1, 3\} = \{1, 3\}$, $u_1\{2\} = u_1\{1, 2\} = u_1\{2, 3\} =$

$u_1 X = X$. Define a closure operator u_2 on X by $u_2\varnothing = \varnothing$, $u_2\{1\} = u_2\{2\} = \{1, 2\}$, $u_2\{3\} = u_2\{1, 3\} = \{1, 3\}$ and $u_2\{1, 2\} = u_2\{2, 3\} = u_2 X = X$. It is easy to see that (X, u_1, u_2) is a β-open space.

Theorem 3.25. *Let* (X, u_1, u_2) *and* (Z, w_1, w_2) *be biclosure spaces and let* (Y, v_1, v_2) *be a* β*-open space. If* $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ *and* $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ *are* β*-continuous, then* $g \circ f$ *is* β*-continuous.*

Proof. Let H be an open subset of (Z, w_1, w_2). Since g is β-continuous, $g^{-1}(H)$ is β-open in (Y, v_1, v_2). But (Y, v_1, v_2) is a β-open space. It follows that $g^{-1}(H)$ is open in (Y, v_1, v_2). As f is β-continuous, $f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H)$ is β-open in (X, u_1, u_2). Thus, $g \circ f$ is β-continuous. □

Acknowledgement

The author would like to thank the Faculty of Science, Mahasarakham University for financial support.

References

[1] C. Boonpok, ∂-closed sets in biclosure spaces, Acta Math. Univ. Ostrav. 17 (2009), 51-66.

[2] C. Boonpok and J. Khampakdee, Between closed sets and generalized closed sets in closure spaces, Acta Math. Univ. Ostrav. 16 (2008), 3-14.

[3] E. Čech, Topological spaces, Topological Papers of Eduard Čech, Academia, Prague, 1968, pp. 436-472.

[4] J. C. Kelly, Bitopological spaces, Proc. London Math. Soc. 3(13) (1969), 71-79.

[5] J. Khampakdee and C. Boonpok, Semi-sets in biclosure spaces, Discuss. Math. Gen. Algebra Appl. 29 (2009), 181-210.

[6] J. Šlapal, Closure operations for digital topology, Theoret. Comput. Sci. 305 (2003), 457-471.