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ORDER OF UNIVALENCY OF GENERAL INTEGRAL OPERATOR

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Abstract

Let E_β be the integral operator defined by

$$E_\beta(z) = \left[\beta\int_0^z t^{\beta-1}(f_1'(t))^{\alpha_1} P_1^{\zeta_1}(t)\cdots(f_n'(t))^{\alpha_n} P_n^{\zeta_n}(t)\,dt\right]^{\frac{1}{\beta}},$$

where each of the functions f_i, and P_i are, respectively, analytic functions and functions with positive real part defined in the open unit disk for all $i = 1, ..., n.$ The object of this paper is to obtain several univalence conditions for this integral operator. Our main results contain some interesting corollaries as special cases.

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1. Introduction and Definitions

In the past few years, integral operators have been a circle of interests to many young researchers in the area of univalent function theory. Too many developments in creating new operators can cause a lot of confusions as well. However, it is an interesting topic to be discussed and studied. We start by defining a general integral operator based on previous operators.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\} : \mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}$. Also, let P be the class of all functions which are analytic in $\mathbb{U}$ and satisfy $P(0) = 1$, $\Re\{P(z)\} > 0$. Frasin and Darus [3] defined the family $\mathcal{B}(\delta)$, $0 \leq \delta < 1$ so that it consists of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \delta \, (z \in \mathbb{U}). \tag{2}$$

In this paper, we obtain new sufficient conditions for the univalence of the general integral operator $E_\beta(z)$ defined by

$$E_\beta(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=0}^{n} (f_i'(t))^{\alpha_i} P_i^{\zeta_i}(t) dt \right]^{\frac{1}{\beta}}, \tag{3}$$

where $\beta \in \mathbb{C}^*$, $\alpha_i, \gamma_i, \zeta_i \in \mathbb{C}$, $f_i \in \mathcal{A}$ and $P_i \in \mathcal{P}$ for all $i = 1, 2, 3, ..., n$.

Here and throughout in the sequel, every multi-valued function is taken with the principal branch.

Remark 1.1. Note that the integral operator E_β generalizes the following operators introduced and studied by several authors:

(i) For $\alpha_i = 0$ for all $i = 1, ..., n,$ we obtain the integral operator

$$I_{\beta}^{\zeta}(P_i)(z) = \left[\beta\int_0^z t^{\beta-1}\prod_{i=0}^{n} P_i^{\zeta_i}(t)dt\right]^{\frac{1}{\beta}} \tag{4}$$

introduced and studied by Frasin [4].

(ii) For $\beta = 1,$ $\zeta_i = \gamma_i = 0$ for all $i = 1, ..., n,$ we obtain the integral operator

$$F_{\alpha_1, ..., \alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} \cdots (f_1'(t))^{\alpha_n} dt \tag{5}$$

introduced and studied by Breaz et al. [1].

(iii) For $\beta = 1,$ $n = 1,$ $\alpha_i = \alpha,$ $\gamma_i = \zeta_i = 0,$ we obtain the integral operator

$$G(z) = \int_0^z (f'(t))^{\alpha} dt \tag{6}$$

studied in [8].

In order to derive our main results, we have to recall here the following lemmas.

Lemma 1.2 [9]**.** *Let* $\eta \in \mathbb{C}$ *with* $\Re(\eta) > 0$ *if* $h \in \mathcal{A}$ *satisfies*

$$\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)}\left|\frac{zh''(z)}{h'(z)}\right| \le 1 \tag{7}$$

for all $z \in \mathbb{U},$ *then the integral operator*

$$F_{\beta}(z) = \left\{\beta\int_0^z t^{\beta-1}(f'(t))dt\right\}^{\frac{1}{\beta}} \tag{8}$$

is in the class $\mathcal{S}.$

Lemma 1.3 [7]**.** *Let* $\beta \in \mathbb{C}$ *with* $\Re(\beta) > 0$, $c \in \mathbb{C}$ *with* $|c| < 1$, $c \neq -1$ *if* $h \in \mathcal{A}$ *satisfies*

$$\left| c|z|^{2\beta} + (1 + |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \leq 1, \tag{9}$$

for all $z \in \mathbb{U}$, *then the integral operator* $F_{\beta}(z)$ *defined by* (16) *is in the class* $\mathcal{S}$.

Lemma 1.4 [5]**.** *If* $P(z) \in \mathcal{P}$, *then*

$$\left| \frac{zP'(z)}{p(z)} \right| \leq \frac{2|z|}{1 - |z|^2}. \tag{10}$$

Lemma 1.5 [2]**.** *If* $f(z) \in \mathcal{B}(\delta)$, *then*

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{(1 - \delta)(2 + |z|)}{1 - |z|}. \tag{11}$$

Also, we need the following general Schwarz lemma.

Lemma 1.6 [6]**.** *Let the function f be regular in the disk* $U_R = z : |z| < R$, *with* $|M| < R$ *for fixed M. If* $f(z)$ *has one zero with multiplicity order bigger than m for* $z = 0$, *then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \ (z \in \mathbb{U}_R). \tag{12}$$

The equality can hold only if

$$f(z) = e^{i\Theta} \left(\frac{M}{R^m} \right) z^m, \tag{13}$$

where Θ *is constant*.

2. Univalence Conditions for the Operator E_{β}

We first prove the following theorem.

Theorem 2.1. *Let* $f(z) \in \mathcal{B}(\delta_i)$, $0 \leq \delta_i < 1$, $P_i(z) \in \mathcal{P}$ *for all* $i = 1, \ldots, n$, *and* $\eta \in \mathbb{C}$ *with* $\Re(\eta) = a > 0$. *If*

$$\sum_{i=1}^{n} [3|\alpha_i|(1-\delta_i) + 2|\zeta_i|] < \min\left\{a; \frac{1}{2}\right\}, \tag{14}$$

then the integral operator E_β *defined by* (3) *is in the class* $\mathcal{S}$.

Proof. Define the regular function $h(z)$ by

$$h(z) = \int_0^z \prod_{i=0}^{n} [(f_i'(t))^{\alpha_i} P_i^{\zeta_i}(t)] dt. \tag{15}$$

Then it is easy to see that

$$h'(z) = \prod_{i=0}^{n} [(f_i'(t))^{\alpha_i} P_i^{\zeta_i}(t)] \tag{16}$$

and $h(0) = 1 - h'(0) = 0$. Differentiating both sides of equation (16) logarithmically, we obtain

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i''(z)}{f_i'(z)}\right) + \sum_{i=1}^{n} \zeta_i \left(\frac{zP_i'(z)}{P_i(z)}\right). \tag{17}$$

Thus, we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{n} |\alpha_i| \left|\frac{zf_i''(z)}{f_i'(z)}\right| + \sum_{i=1}^{n} |\zeta_i| \left|\frac{zP_i'(z)}{P_i(z)}\right|. \tag{18}$$

Since $f(z) \in \mathcal{B}(\delta_i)$, $P_i(z) \in \mathcal{P}$ for all $i = 1, \ldots, n$, from (18), (10) and (11), we obtain

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{n} |\alpha_i|(1-\delta_i)\left(\frac{(2+|z|)}{1-|z|}\right) + \sum_{i=1}^{n} |\zeta_i| \left(\frac{(2|z|)}{1-|z|^2}\right)$$

$$\leq \sum_{i=1}^{n} |\alpha_i|(1-\delta_i)\left(\frac{3}{1-|z|}\right) + \sum_{i=1}^{n} |\zeta_i| \left(\frac{2}{1-|z|}\right). \tag{19}$$

Multiply both sides of (19) by $\dfrac{1-|z|^{2\Re(\eta)}}{\Re(\eta)}$, we get

$$\frac{1-|z|^{2\Re(\eta)}}{\Re(\eta)}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{1-|z|^{2\Re(\eta)}}{(1-|z|)\Re(\eta)} \times \sum_{i=1}^{n}[3|\alpha_i|(1-\delta_i)+2|\zeta_i|] \quad (20)$$

for all $z \in \mathbb{U}$.

Let us denote $|z| = x$, $x \in [0, 1)$, $\Re(\eta) = a > 0$, and $\Phi(x) = (1-x^{2a})/(1-x)$. It is easy to prove that

$$\Phi(x) \le \begin{cases} 1, & 0 < a < 1, \\ 2a, & \frac{1}{2} < a < \infty. \end{cases} \quad (21)$$

From (20), (21), and the hypothesis (14), we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zh''(z)}{h'(z)}\right| \le \begin{cases} \frac{1}{a}\sum_{i=1}^{n}[3|\alpha_i|(1-\delta_i)+2|\zeta_i|], & 0 < a < 1 \\ 2\sum_{i=1}^{n}[3|\alpha_i|(1-\delta_i)+2|\zeta_i|], & \frac{1}{2} < a < \infty \end{cases}$$

$$\le 1 \quad (22)$$

for all $z \in \mathbb{U}$. Applying Lemma 1.2 for the function $h(z)$, we prove that $E_\beta(z) \in \mathcal{S}$.

Letting $n = 1$, $\delta_1 = \delta$, $\alpha_1 = \alpha$, $\zeta_1 = \zeta$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let* $f(z) \in \mathcal{B}(\delta)$, $0 \le \delta < 1$, $P(z) \in \mathcal{P}$, *and* $\eta, \alpha, \zeta \in \mathbb{C}$ *with* $\Re(\eta) = a > 0$. *If*

$$3|\alpha|(1-\delta)+2|\zeta| < \min\left\{a; \frac{1}{2}\right\}, \quad (23)$$

then the integral operator $E_\beta^{\alpha,\zeta}$ *defined by*

$$E_\beta(z)^{\alpha,\zeta} = \left[\beta\int_0^z t^{\beta-1}(f'(t))^\alpha P^\zeta(t)dt\right]^{\frac{1}{\beta}} \tag{24}$$

is in the class $\mathcal{S}$.

If we set $\delta = 0$ in Corollary 2.2, we have the following.

Corollary 2.3. *Let* $f(z) \in \mathcal{S}$, $P(z) \in \mathcal{P}$, *and* $\eta, \alpha, \zeta \in \mathbb{C}$ *with* $\Re(\eta) = a > 0$. *If*

$$3|\alpha| + 2|\zeta| < \min\left\{a; \frac{1}{2}\right\}, \tag{25}$$

then the integral operator $E_\beta^{\alpha,\zeta}$ *defined by* (33) *is in the class* $\mathcal{S}$.

Next, we prove the following theorem.

Theorem 2.4. *Let* $\alpha_i, \zeta_i \in \mathbb{C}$ *for all* $i = 1, 2, \ldots, n$ *and each* $f_i \in \mathcal{A}$ *satisfies* $\Re(f_i(z)) > 0$, *and*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| \le \Re(\eta) \tag{26}$$

for all $z \in \mathbb{U}$. *If*

$$\sum_{i=1}^{n}(|\alpha_i| + |\zeta_i|) \le 1 \tag{27}$$

then, for any complex number σ, *with* $\Re(\sigma) \ge \Re(\eta) > 0$, *the integral operator* E_β *defined by* (4) *is in the class* $\mathcal{S}$.

Proof. Suppose that $\Re(f_i(z)) > 0$ for all $i = 1, \ldots, n$. Thus, we have

$$f_i'(z) = P_i(z), \tag{28}$$

where $P_i \in \mathcal{P}$ for all $i = 1, \ldots, n$. Differentiating both sides of (27) logarithmically, we obtain

$$\frac{zf_i''(z)}{f_i'(z)} = \frac{zP_i'(z)}{P_i(z)}. \tag{29}$$

Define the regular function $h(z)$ as in (15). Thus, from (17) we have

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} (\alpha_i + \zeta_i)\left(\frac{zf_i''(z)}{f_i'(z)}\right) \tag{30}$$

and so

$$\left|\frac{zh''(z)}{h'(z)}\right| \le \left|\sum_{i=1}^{n} (\alpha_i + \zeta_i)\left(\frac{zf_i''(z)}{f_i'(z)}\right)\right|. \tag{31}$$

Multiplying both sides of (31) by $\frac{1-|z|^{2\Re(\eta)}}{\Re(\eta)}$, from Lemma 1.2 with $\delta = 0,$ we get

$$\begin{aligned}\frac{1-|z|^{2\Re(\eta)}}{\Re(\eta)}\left|\frac{zh''(z)}{h'(z)}\right| &\le \frac{1-|z|^{2\Re(\eta)}}{\Re(\eta)}\sum_{i=1}^{n}(|\alpha_i|+|\zeta_i|)\left|\frac{zf_i''(z)}{f_i'(z)}\right| \\ &\le \frac{1-|z|^{2\Re(\eta)}}{\Re(\eta)}\sum_{i=1}^{n}(|\alpha_i|+|\zeta_i|)\Re(\eta) \\ &\le \sum_{i=1}^{n}(|\alpha_i|+|\zeta_i|).\end{aligned}$$

Using the hypothesis (27) we readily get

$$\left|\frac{zh''(z)}{h'(z)}\right| \le 1. \tag{32}$$

Finally, by applying Lemma 1.2, we conclude that the integral operator E_β defined by (4) is in the class $\mathcal{S}$.

Letting $n = 1,$ $\delta_1 = \delta,$ $\alpha_1 = \alpha,$ $\zeta_1 = \zeta$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.5. *Let* $\alpha, \zeta \in \mathbb{C}$ *for* $f \in \mathcal{A}$ *satisfies* $\Re(f_i(z)) > 0$, *and*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \Re(\eta) \tag{33}$$

for all $z \in \mathbb{U}$. *If*

$$\sum_{i=1}^{n}(|\alpha| + |\zeta|) \leq 1 \tag{34}$$

then, *for any complex number* σ, *with* $\Re(\sigma) \geq \Re(\eta) > 0$, *the integral operator* $E_{\beta}^{\alpha,\zeta}$ *defined by* (4) *is in the class* $\mathcal{S}$.

Using Lemma 1.3, we derive the following theorem.

Theorem 2.6. *Let* $\alpha_i, \zeta_i, \beta \in \mathbb{C}$ *for all* $i = 1, 2, ..., n$, $\Re(\beta) > 0$, $c \in \mathbb{C}(|z| < 1)$ *and each* $f_i \in \mathcal{A}$ *satisfies* $\Re(f_i(z)) > 0$, *and*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| \leq \Re(\eta) \tag{35}$$

for all $z \in \mathbb{U}$. *If*

$$\sum_{i=1}^{n}(|\alpha_i| + |\zeta_i|) \leq 1 - |c| \tag{36}$$

then, *for any complex number* σ, *with* $\Re(\sigma) \geq \Re(\eta) > 0$, *the integral operator* E_{β} *defined by* (4) *is in the class* $\mathcal{S}$.

Proof. From (30), we have

$$\left|c|z|^{2\beta} + (1 - |z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\right| = \left|c|z|^{2\beta} + \frac{1 - |z|^{2\beta}}{\beta}\sum_{i=1}^{n}(\alpha_i + \zeta_i)\left(\frac{zf_i''(z)}{f_i'(z)}\right)\right|$$

$$\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}(|\alpha_i| + |\zeta_i|)|\beta|. \tag{37}$$

Now by using the hypothesis (27) we obtain

$$\left| c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)} \right| \le 1.$$

Finally, by applying Lemma 1.3, we conclude that $E_\beta \in \mathcal{S}$.

Letting $n = 1$, $\delta_1 = \delta$, $\alpha_1 = \alpha$, $\zeta_1 = \zeta$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.7. *Let* $\alpha, \zeta, \beta \in \mathbb{C}$ *for all* $i = 1, 2, \ldots, n$, $\Re(\beta) > 0$, $c \in \mathbb{C}(|z| < 1)$ *and each* $f_i \in \mathcal{A}$ *satisfies* $\Re(f_i(z)) > 0$, *and*

$$\left| \frac{zf''(z)}{f'(z)} \right| \le \Re(\eta) \tag{38}$$

for all $z \in \mathbb{U}$. *If*

$$\sum_{i=1}^{n} (|\alpha| + |\zeta|) \le 1 - |c| \tag{39}$$

then, *for any complex number* σ, *with* $\Re(\sigma) \ge \Re(\eta) > 0$, *the integral operator* $E_\beta^{\alpha, \zeta}$ *defined by* (4) *is in the class* $\mathcal{S}$.

Remark. Other results related to univalence criteria can be read in [10].

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