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A COMPUTATIONAL STUDY OF GAMMA PARAMETER ESTIMATES

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Abstract

Assuming a two parameter gamma distribution to model the lifetime of a system we study the sampling distribution of three types of estimates of the model parameters, namely - (i) the maximum likelihood estimator (MLE); (ii) a bias corrected version of the MLE (BMLE); and (iii) a modified version of the MLE (MMLE). Existing literature provides some partial results about the bias and MSE of these estimators, but not much about their sampling distributions. In this investigation we go into details in observing the nature of the sampling distributions of the estimators through a comprehensive simulation study.

1. Introduction

A two parameter gamma distribution with the following pdf:

$$f(x|\alpha, \theta) = (\Gamma(\alpha)\theta^{\alpha})^{-1} e^{-x/\theta} x^{\alpha-1}, \quad x > 0, \alpha > 0, \theta > 0, \tag{1.1}$$

is a well-known positively skewed distribution widely used in modeling lifetime data. We refer to (1.1) as the $G(\alpha, \theta)$, model.

Given a complete random sample of size *n*, say $X_1, X_2, ..., X_n$, from $G(\alpha, \theta)$, the maximum likelihood estimates (MLE) of α and θ, henceforth denoted by $\hat{\alpha}$ and $\hat{\theta}$, respectively, are obtained as follows. Get $\hat{\alpha}$ by solving

$$h(\hat{\alpha}) := \ln(\hat{\alpha}) - \psi(\hat{\alpha}) = \ln R, \tag{1.2}$$

where $\psi(x) = \partial \ln \Gamma(x)/\partial x$ is the digamma function and *R* is the ratio of the arithmetic mean (AM) and the geometric mean (GM) of the sampled observations, i.e.,

$$R = \left(\sum_{i=1}^{n} X_i / n \right) \Bigg/ \left(\prod_{i=1}^{n} X_i^{1/n} \right). \tag{1.3}$$

Note that $R > 1$ with probability 1, i.e., $\ln R > 0$ with probability 1. After obtaining $\hat{\alpha}$ get $\hat{\theta}$ by the expression

$$\hat{\theta} = \overline{X}/\hat{\alpha}. \tag{1.4}$$

Since $\hat{\alpha}$ (and hence $\hat{\theta}$ too) does not have any closed expression, studying the exact sampling distribution of the above two parameter estimates analytically is nearly impossible. However, it can be observed that the sampling distribution of $\hat{\alpha}$ depends analytically solely on (α, n) and it is free from θ. This can be seen from the fact that if we transform each X_i to $Y_i = cX_i$ for any $c > 0$, then

$$R = \left(\sum_{i=1}^{n} Y_i/n\right)\Bigg/\left(\prod_{i=1}^{n} Y_i^{1/n}\right) = \left(\sum_{i=1}^{n} X_i/n\right)\Bigg/\left(\prod_{i=1}^{n} X_i^{1/n}\right). \tag{1.5}$$

Thus, the value of $\hat{\alpha}$ depends solely on R (from (1.2)) which remains unaffected by any scale transformation of the data, and thus without loss of generality one can use $c = 1/\theta$. Using the same argument one can see that the probability distribution of $(\hat{\theta}/\theta)$ depends only on (α, n).

It is possible to estimate α and θ by the method of moments estimation, but such estimates do not utilize the full sample information (i.e., they are not functions of the sufficient statistics) and hence such estimates are not considered in this paper.

Berman [1] proved that for $n \geq 2$ the estimator $\hat{\alpha}$ is always positively biased. Using the reparameterization $\lambda = 1/\theta$, it was also shown that $\hat{\lambda} = 1/\hat{\theta}$ is positively biased for λ; i.e. $\hat{\theta}$ is negatively biased for θ. Shenton and Bowman [3] have given the following bias expressions for $\hat{\alpha}$ and $\hat{\lambda}$, respectively as:

$$\begin{aligned} B(\hat{\alpha} \mid \alpha, n) &= E(\hat{\alpha} - \alpha) \\ &= (3\alpha/(n-3))\{1 - 2/(9\alpha) + (n-1)/(27n\alpha^2) \\ &\quad + 7(n^2 - 1)/(162(n+3)n^2\alpha^3) + \cdots\}; \end{aligned} \tag{1.6}$$

and

$$B(\hat{\lambda} \mid \alpha,\ n)/\lambda = E((\hat{\lambda}/\lambda) - 1)$$
$$= [n\alpha\{(n-3)(n\alpha-1)\}]\{3 + 1/(3\alpha) - 3/(n\alpha)$$
$$+ (n-1)/(9n\alpha^2) + \cdots\}. \tag{1.7}$$

Shenton and Bowman [2] have also shown that

$$E((\hat{\alpha}/\alpha)^s) \approx (n^s/((n-3)(n-5)\cdots(n-2s-1)))\{1 - s(s+1)/(3n\alpha)\} \tag{1.8}$$

and

$$E(\hat{\theta}/\theta) = 1 - (1/n) + O(\alpha^{-2}). \tag{1.9}$$

Asymptotically $(\text{i.e., as } n \to \infty)$ it is known that

$$\lim_{n\to\infty} E[\sqrt{n}(\hat{\alpha} - \alpha)] = 0 = \lim_{n\to\infty} E[\sqrt{n}(\hat{\theta} - \theta)], \tag{1.10}$$

$$\lim_{n\to\infty} Var(\sqrt{n}\hat{\alpha}) = \alpha/\{\alpha\psi'(\alpha) - 1\}, \tag{1.11}$$

$$\lim_{n\to\infty} Var(\sqrt{n}\hat{\theta}) = \theta^2\psi'(\alpha)/\{\alpha\psi'(\alpha) - 1\}, \tag{1.12}$$

$$\lim_{n\to\infty} Corr(\hat{\alpha},\ \hat{\theta}) = -1/\sqrt{\alpha\psi'(\alpha)}, \tag{1.13}$$

where $\psi(\cdot)$ is the digamma function as mentioned earlier. In other words, for sufficiently large *n*, both $\hat{\alpha}$ and $\hat{\theta}$ are nearly unbiased, and

$$Var(\hat{\alpha}) = \alpha/\{n(\alpha\psi'(\alpha) - 1)\} + O(n^{-2}), \tag{1.14}$$

$$Var(\hat{\theta}) = \theta^2\psi'(\alpha)/\{n(\alpha\psi'(\alpha - 1))\} + O(n^{-2}). \tag{1.15}$$

In this study our first goal is to understand the sampling distribution of $\hat{\alpha}$ and $\hat{\theta}$ in more details since the above bias and/or moment expressions (which can give rise to mean squared error (MSE)) do not provide an adequate understanding about the true behavior of $\hat{\alpha}$ and $\hat{\theta}$.

It has been noted in our study that bias corrected versions of $\hat{\alpha}$ and $\hat{\theta}$ as given below tend to perform better. These estimators of α and θ are given as

$$\tilde{\alpha} = \hat{\alpha}((n-3)/n) + (2/(3n)), \tag{1.16}$$

$$\tilde{\theta} = \overline{X}/\tilde{\alpha}. \tag{1.17}$$

Yanagimoto [4] proposed a superior estimator of α as $\hat{\alpha}^*$ which is obtained by solving the following equation:

$$h(\hat{\alpha}^*) - h(n\hat{\alpha}^*) = \ln R \tag{1.18}$$

where the function $h(\cdot)$ is given in (1.2). It was shown by Yanagimoto [4] as well as Zaigraev and Podraza-Karakulska [5] that $\hat{\alpha}^*$ has smaller bias and mean squared error (MSE) than those of $\hat{\alpha}$. The rationale behind obtaining $\hat{\alpha}^*$ by solving (1.18) is that the expectation of $(\ln R)$ happens to be $(h(\alpha) - h(n\alpha))$. Therefore, it makes sense to solve (1.18) to obtain an estimator of α. After obtaining $\hat{\alpha}^*$ as stated above, one can obtain an estimator of θ as

$$\hat{\theta}^* = \overline{X}/\hat{\alpha}^*. \tag{1.19}$$

However, it is not known yet how $\hat{\theta}^*$ compares against $\hat{\theta}$ (in (1.4)) and $\tilde{\theta}$ (in (1.17)) which will be part of our investigation in this paper.

After studying the estimates of α and θ, we will then focus our attention to estimate the reliability of an entire system by using the above three types of estimators. The reliability of the system at time $t > 0$ is defined as

$$\tau(t) = P(X > t) = \int_t^{\infty} f(x|\alpha, \theta)dx, \tag{1.20}$$

where $f(x|\alpha, \theta)$ is given in (1.1). We may drop "t" from $\tau(t)$, and write only τ, when it is obvious from the context.

The MLE of τ is $\hat{\tau}$ given as

$$\hat{\tau} = \int_t^{\infty} f(x|\hat{\alpha}, \hat{\theta})dx \tag{1.21}$$

where $\hat{\alpha}$ and $\hat{\theta}$ are given in (1.2) and (1.3). Similarly, we have two other estimators of τ as $\tilde{\tau}$ and $\hat{\tau}^*$ obtained by replacing $(\hat{\alpha}, \hat{\theta})$ in (1.21) by $(\tilde{\alpha}, \tilde{\theta})$ and $(\hat{\alpha}^*, \hat{\theta}^*)$, respectively.

In this paper, we have undertaken a comprehensive simulation to study the nature of the sampling distributions of the various estimators (as described above) of α, θ and τ. Throughout this study, we have used $\theta = 1$, and varied α as well as n. In Section 2, we first study the behavior of the estimators of α and θ; whereas in Section 3, we study the behavior of the estimators of τ. The paper ends with a concluding remark which summarizes our observations.

2. Sampling Distributions of the Model Parameter Estimators

In this section, we consider a system whose lifetime follows a $G(\alpha, \theta)$ distribution. As stated in the previous section, α and θ can be estimated by their MLEs $\hat{\alpha}$, $\hat{\theta}$ given in (1.2) and (1.4), respectively. These two parameters can also be estimated by $\tilde{\alpha}$, $\tilde{\theta}$ (given in (1.16), (1.17)); as well as by $\hat{\alpha}^*$, $\hat{\theta}^*$ (given in (1.18), (1.19)).

We start this section by studying first the scaled estimators of α, i.e., $(\hat{\alpha}/\alpha)$, $(\tilde{\alpha}/\alpha)$ and $(\hat{\alpha}^*/\alpha)$. Each estimator is scaled because the corresponding bias and MSE provide a better understanding about the behavior of the estimator relative to the parameter value. Given (n, α), we generate $X_1, X_2, \ldots, X_n$, *iid* from $G(\alpha, \theta = 1)$ using the software SAS. Based on the generated data, we compute the value of each estimator, and then this process is replicated $N = 10^5$ times. Then we draw the histogram of the simulated scaled estimator for $\alpha = 0.5, 1.0, 5.0, 10.0, 15.0$ and $n = 5$, 10, 20, 50 (i.e., a total 20 combinations of α and n). For example, Figure 2.1 indicates the histograms of $(\hat{\alpha}/\alpha)$. Similarly, Figures 2.2 and 2.3 are for $(\tilde{\alpha}/\alpha)$ and $(\hat{\alpha}^*/\alpha)$, respectively. Note that in Table 2.1 we are computing: Mean $\approx$ average of $(\tilde{\alpha}/\alpha)$, Variance $\approx$ average of $(\hat{\alpha}/\alpha - \text{mean})^2$ and MSE

$\approx$ average of $(\hat{\alpha}/\alpha - 1)^2$, respectively. The standard error of each table entry is less than 0.001.

Observation 2.1. Note that all the histograms in Figures 2.1-2.3 are positively skewed. The general trends that emerge from these figures are given as follows:

(a) The skewness of each scaled estimator of α goes from extreme to moderate as α increases, and *n* increases.

(b) Even though $\hat{\alpha}^*$ is known to be better than $\hat{\alpha}$ analytically, and evident from their respective histograms, the performance of $\tilde{\alpha}$ came as surprise. For $\alpha \leq 10$, and small n (< 20), or for $n \geq 20$ and all values of α, the distribution of $(\tilde{\alpha}/\alpha)$ is consistently more concentrated near 1 compared to other two estimators, and as a result $\tilde{\alpha}$ seems to be the preferred estimator for α. As seen in Table 2.1, in terms of MSE, $\tilde{\alpha}$ is far better than the other two estimators which has never been reported in the literature before.

(c) It is tempting to guess a suitable positively skewed distribution which can approximate the true distribution of each estimator of α (either scaled or unscaled). Therefore, it may seem natural to fit a new gamma distribution for each estimator of α. For example, assume that $(\hat{\alpha}/\alpha)$ follows $G(\alpha_*, \theta_*)$. One can find α_* and θ_* such that $\alpha_*\theta_* =$ mean of $(\hat{\alpha}/\alpha)$ and $\alpha_*\theta_*^2 =$ variance of $(\hat{\alpha}/\alpha)$. For this purpose, Table 2.1 can be useful to approximate α_* and θ_*. Similarly, one can approximate the distributions of $(\tilde{\alpha}/\alpha)$ and $(\hat{\alpha}^*/\alpha)$ by suitable gamma distributions.

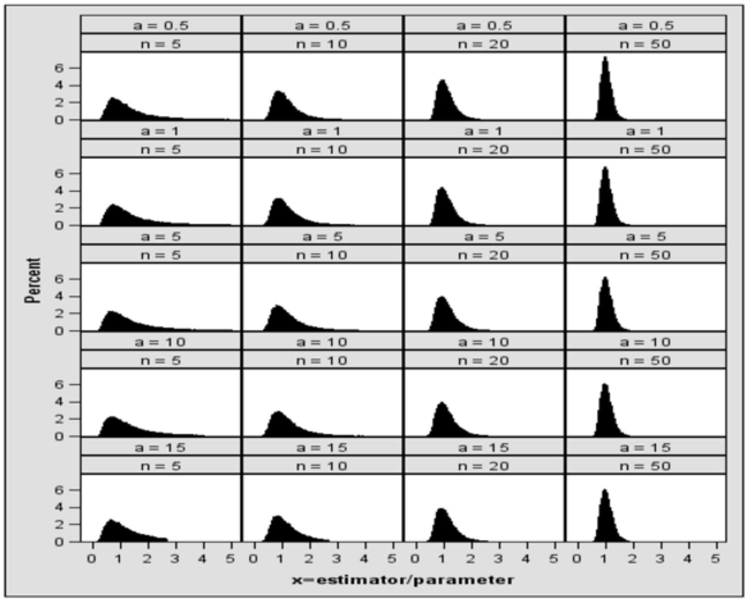

Figure 2.1. Histograms of $(\hat{\alpha}/\alpha)$ for various of α and *n*. (In the figure 'a' stands for α.)

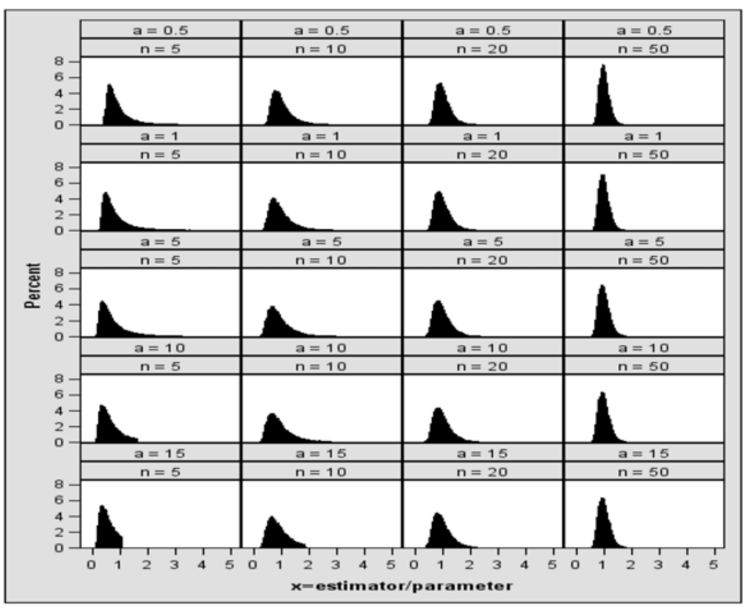

Figure 2.2. Histograms of $(\tilde{\alpha}/\alpha)$ for various of α and *n*. (In the figure 'a' stands for α.)

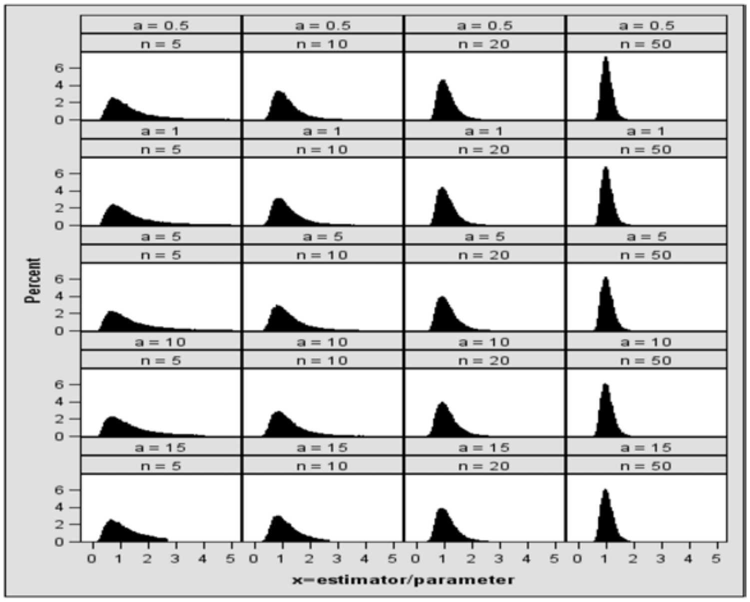

Figure 2.3. Histograms of $(\hat{\alpha}^*/\alpha)$ for various of α and *n*. (In the figure 'a' stands for α.)

Table 2.1. Mean, variance and MSE of the three scaled estimators of α

n	α	$\hat{\alpha}/\alpha$			$\tilde{\alpha}/\alpha$			$\hat{\alpha}^*/\alpha$		
		Mean	Var	MSE	Mean	Var	MSE	Mean	Var	MSE
5	0.5	1.997	6.910	7.904	1.065	1.106	1.110	1.675	5.072	5.528
	1	2.127	6.674	7.943	0.984	1.068	1.068	1.764	4.847	5.430
	5	1.876	1.992	2.758	0.777	0.319	0.368	1.584	1.656	1.997
	10	1.527	0.722	1.000	0.624	0.115	0.257	1.329	0.670	0.779
	15	1.287	0.343	0.425	0.524	0.055	0.282	1.141	0.337	0.357
10	0.5	1.305	0.459	0.552	1.047	0.225	0.227	1.208	0.380	0.424
	1	1.353	0.590	0.715	1.014	0.289	0.289	1.241	0.482	0.540
	5	1.402	0.647	0.808	0.994	0.317	0.317	1.271	0.544	0.618
	10	1.343	0.429	0.547	0.947	0.210	0.213	1.228	0.386	0.438
	15	1.245	0.252	0.312	0.876	0.123	0.139	1.152	0.240	0.263

	0.5	1.126	0.112	0.128	1.024	0.081	0.082	1.086	0.103	0.110
	1	1.145	0.137	0.158	1.006	0.099	0.099	1.099	0.125	0.134
20	5	1.168	0.171	0.200	0.999	0.124	0.124	1.112	0.155	0.168
	10	1.172	0.174	0.203	1.000	0.125	0.125	1.116	0.158	0.171
	15	1.157	0.148	0.172	0.986	0.107	0.107	1.104	0.138	0.149
	0.5	1.046	0.033	0.035	1.010	0.029	0.029	1.031	0.032	0.033
	1	1.052	0.039	0.041	1.002	0.034	0.034	1.036	0.037	0.038
50	5	1.061	0.047	0.051	1.000	0.042	0.042	1.041	0.045	0.047
	10	1.062	0.048	0.052	1.000	0.043	0.043	1.042	0.046	0.048
	15	1.064	0.049	0.053	1.001	0.043	0.043	1.043	0.047	0.049

Next, we present the simulated histograms of $(\hat{\theta}/\theta)$, $(\tilde{\theta}/\theta)$ and $(\hat{\theta}^*/\theta)$ in Figures 2.4-2.6 for various combinations of α and n.

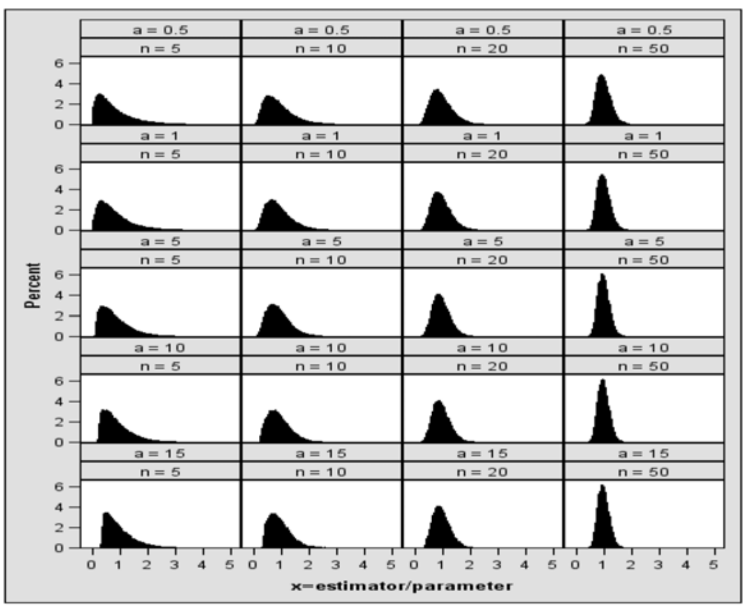

Figure 2.4. Histograms of $(\hat{\theta}/\theta)$ for various values of α and n. (In the figure 'a' stands for α.)

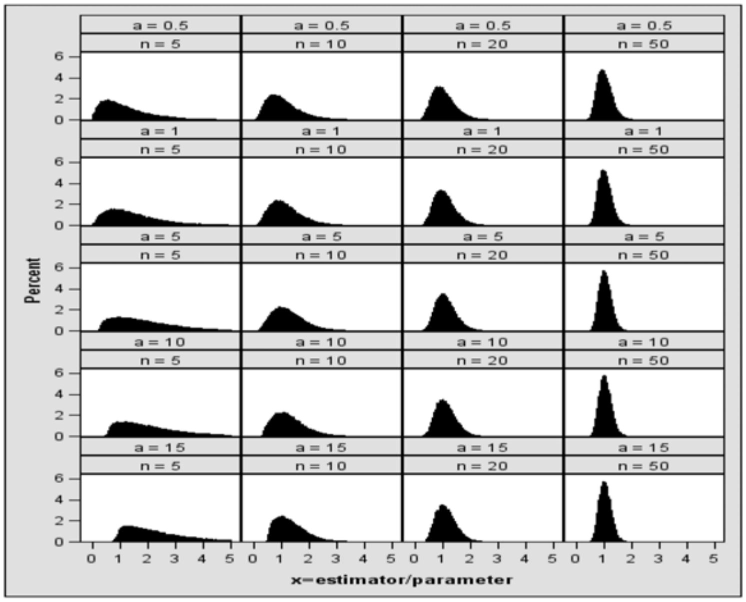

Figure 2.5. Histograms of $(\tilde{\theta}/\theta)$ for various values of α and *n*. (In the figure 'a' stands for α.)

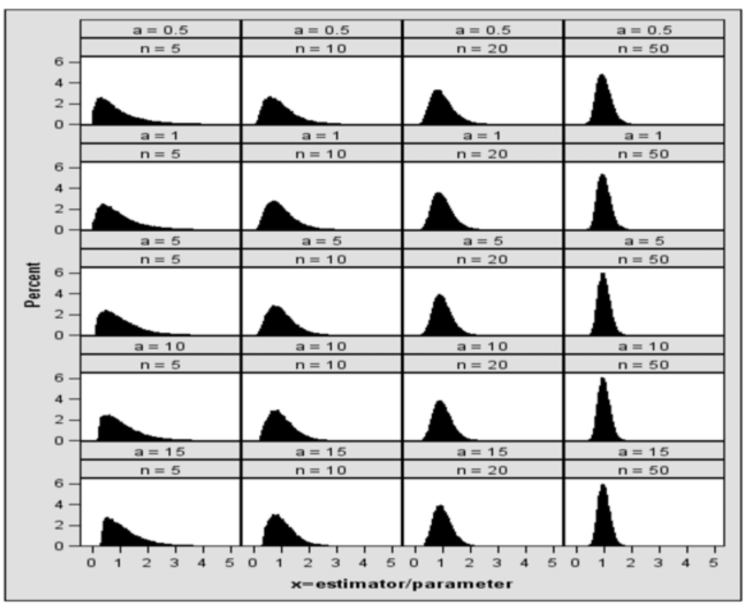

Figure 2.6. Histograms of $(\hat{\theta}^{*}/\theta)$ for various values of α and *n*. (In the figure 'a' stands for α.)

The following Table 2.2 provides mean, variance and MSE values of simulated $(\hat{\theta}/\theta)$, $(\tilde{\theta}/\theta)$ and $(\hat{\theta}^{*}/\theta)$ for various n and α.

Table 2.2. Mean and variance of the three scaled estimators of θ

n	α	$\hat{\theta}$			$\tilde{\theta}$			$\hat{\theta}^{*}$		
		Mean	Var	MSE	Mean	Var	MSE	Mean	Var	MSE
5	0.5	0.826	0.594	0.624	1.236	0.927	1.110	0.968	0.800	0.801
	1	0.816	0.445	0.479	1.508	1.099	1.068	0.970	0.610	0.611
	5	0.832	0.331	0.360	1.930	1.563	0.368	1.011	0.491	0.491
	10	0.896	0.306	0.316	2.155	1.629	0.257	1.074	0.468	0.474
	15	0.984	0.299	0.299	2.394	1.658	0.282	1.158	0.464	0.489
10	0.5	0.913	0.318	0.326	1.090	0.392	0.227	0.982	0.364	0.365
	1	0.906	0.239	0.248	1.177	0.361	0.289	0.983	0.277	0.277
	5	0.900	0.186	0.196	1.260	0.352	0.317	0.993	0.224	0.224
	10	0.915	0.179	0.186	1.294	0.351	0.213	1.008	0.220	0.220
	15	0.945	0.163	0.166	1.341	0.324	0.139	1.034	0.204	0.205
20	0.5	0.958	0.164	0.166	1.044	0.183	0.082	0.993	0.175	0.175
	1	0.952	0.122	0.124	1.077	0.149	0.099	0.991	0.131	0.131
	5	0.951	0.099	0.101	1.109	0.132	0.124	0.997	0.108	0.108
	10	0.950	0.097	0.100	1.113	0.133	0.125	0.998	0.107	0.107
	15	0.956	0.094	0.096	1.121	0.128	0.107	1.003	0.104	0.104
50	0.5	0.984	0.067	0.067	1.018	0.070	0.029	0.998	0.068	0.068
	1	0.982	0.051	0.051	1.029	0.055	0.034	0.997	0.052	0.052
	5	0.980	0.041	0.041	1.040	0.046	0.042	0.999	0.042	0.042
	10	0.980	0.039	0.040	1.041	0.044	0.043	0.999	0.041	0.041
	15	0.979	0.039	0.040	1.040	0.044	0.043	0.998	0.041	0.041

Observation 2.2. Again, similar to α estimation, the estimators of θ are positively skewed. These sampling distributions tend to become less and less skewed as α and/or n increases. Interestingly, for θ estimation, $\hat{\theta}$ tends to have smaller bias and variance (and hence the MSE) compared to the other two estimators. As a result, $\hat{\theta}$ seems to be the best estimator among the three estimators. Further, if an estimator's distribution is approximated by a suitable gamma distribution, say $G(\alpha_{**}, \theta_{**})$, then the parameters α_{**} and θ_{**} can be found by matching the mean and variance from Table 2.2.

3. Sampling Distribution of τ Estimators

Since without loss of generality, we have taken $\theta = 1$, τ_1, τ_2 and τ_3 are roughly (not exactly) the second quartile (i.e. median), the third quartile and the first quartile for large α.

Here we study the sampling distributions of the three estimators of τ, namely - $\hat{\tau}$, $\tilde{\tau}$ and $\hat{\tau}^*$ given in (1.21) and the expressions after (1.21). Here we are going to estimate τ at three specific values of t given as

$$\tau_1 = \tau(\alpha) = P(X > \alpha) = \int_{\alpha}^{\infty} f(x|\alpha, \theta)dx,$$

$$\tau_2 = \tau(\alpha + \sqrt{\alpha}) = P(X > \alpha + \sqrt{\alpha}) = \int_{\alpha+\sqrt{\alpha}}^{\infty} f(x|\alpha, \theta)dx,$$

$$\tau_3 = \begin{cases} \tau(\alpha - \sqrt{\alpha}) = P(X > \alpha - \sqrt{\alpha}) = \int_{\alpha-\sqrt{\alpha}}^{\infty} f(x|\alpha, \theta)dx, & \text{if } \alpha > 1 \\ \tau(\alpha - \sqrt{\alpha}/2) = P(X > \alpha - \sqrt{\alpha}/2) = \int_{\alpha-\sqrt{\alpha}/2}^{\infty} f(x|\alpha, \theta)dx, & \text{if } \alpha \le 1 \end{cases}$$

and the corresponding τ values will be referred to as τ_1, τ_2 and τ_3, respectively.

The following Table 3.1, Table 3.2 and Table 3.3 provide mean, variance and MSE value of τ_1, τ_2 and τ_3 estimators for various n and α.

Table 3.1. Mean and variance of the three scaled estimators of τ_1

n	α	$\hat{\tau}_1$			$\tilde{\tau}_1$			$\hat{\tau}_1^*$		
		Mean	Var	MSE	Mean	Var	MSE	Mean	Var	MSE
5	0.5	0.296	0.034	0.034	0.268	0.020	0.023	0.287	0.030	0.030
	1	0.354	0.038	0.039	0.320	0.021	0.024	0.346	0.033	0.034
	5	0.433	0.040	0.040	0.410	0.020	0.021	0.429	0.035	0.035
	10	0.451	0.035	0.036	0.433	0.017	0.018	0.448	0.032	0.032
	15	0.459	0.031	0.031	0.443	0.015	0.015	0.457	0.029	0.029
10	0.5	0.304	0.015	0.015	0.292	0.012	0.013	0.299	0.014	0.014
	1	0.358	0.017	0.017	0.344	0.013	0.014	0.354	0.016	0.016
	5	0.437	0.019	0.019	0.428	0.014	0.014	0.434	0.017	0.017
	10	0.455	0.018	0.018	0.448	0.013	0.014	0.453	0.017	0.017
	15	0.463	0.017	0.017	0.457	0.013	0.013	0.461	0.016	0.016
20	0.5	0.311	0.007	0.007	0.304	0.006	0.007	0.308	0.007	0.007
	1	0.363	0.008	0.008	0.355	0.007	0.007	0.360	0.007	0.008
	5	0.438	0.009	0.009	0.434	0.007	0.007	0.437	0.008	0.008
	10	0.456	0.009	0.009	0.453	0.008	0.008	0.455	0.008	0.008
	15	0.464	0.009	0.009	0.462	0.007	0.007	0.464	0.008	0.008
50	0.5	0.315	0.003	0.003	0.312	0.003	0.003	0.314	0.003	0.003
	1	0.366	0.003	0.003	0.363	0.003	0.003	0.365	0.003	0.003
	5	0.439	0.003	0.003	0.438	0.003	0.003	0.439	0.003	0.003
	10	0.457	0.003	0.003	0.456	0.003	0.003	0.457	0.003	0.003
	15	0.465	0.003	0.003	0.464	0.003	0.003	0.465	0.003	0.003

Table 3.2. Mean and variance of the three scaled estimators of τ_2

n	α	$\hat{\tau}_2$			$\tilde{\tau}_2$			$\hat{\tau}_2^*$		
		Mean	Var	MSE	Mean	Var	MSE	Mean	Var	MSE
5	0.5	0.601	0.041	0.041	0.512	0.027	0.033	0.571	0.038	0.038
	1	0.621	0.040	0.040	0.520	0.027	0.034	0.595	0.037	0.037
	5	0.662	0.035	0.071	0.574	0.022	0.100	0.643	0.032	0.076
	10	0.663	0.031	0.064	0.582	0.017	0.087	0.648	0.028	0.068
	15	0.653	0.024	0.061	0.579	0.013	0.083	0.643	0.023	0.063
10	0.5	0.597	0.017	0.017	0.556	0.014	0.015	0.581	0.017	0.017
	1	0.615	0.018	0.018	0.569	0.015	0.016	0.601	0.017	0.017
	5	0.662	0.017	0.053	0.623	0.014	0.067	0.651	0.016	0.057
	10	0.671	0.016	0.047	0.635	0.013	0.058	0.661	0.015	0.050
	15	0.667	0.014	0.045	0.634	0.011	0.055	0.660	0.013	0.047
20	0.5	0.593	0.008	0.008	0.573	0.007	0.007	0.585	0.008	0.008
	1	0.611	0.008	0.008	0.588	0.007	0.008	0.603	0.008	0.008
	5	0.657	0.008	0.046	0.638	0.007	0.053	0.651	0.008	0.049
	10	0.669	0.008	0.039	0.651	0.007	0.045	0.663	0.008	0.041
	15	0.673	0.007	0.036	0.657	0.006	0.041	0.669	0.007	0.038
50	0.5	0.590	0.003	0.003	0.583	0.003	0.003	0.587	0.003	0.003
	1	0.608	0.003	0.003	0.599	0.003	0.003	0.605	0.003	0.003
	5	0.654	0.003	0.043	0.646	0.003	0.046	0.651	0.003	0.044
	10	0.666	0.003	0.036	0.659	0.003	0.038	0.663	0.003	0.036
	15	0.674	0.003	0.032	0.667	0.003	0.034	0.672	0.003	0.032

Table 3.3. Mean and variance of the three scaled estimators of τ_3

n	α	$\hat{\tau}_3$			$\tilde{\tau}_3$			$\hat{\tau}_3^*$		
		Mean	Var	MSE	Mean	Var	MSE	Mean	Var	MSE
5	0.5	0.038	0.106	0.011	0.011	0.113	0.008	0.008	0.108	0.010
	1	0.037	0.121	0.013	0.013	0.139	0.009	0.009	0.127	0.012
	5	0.076	0.141	0.015	0.015	0.189	0.011	0.012	0.152	0.014
	10	0.068	0.152	0.015	0.015	0.210	0.011	0.014	0.163	0.014
	15	0.065	0.163	0.014	0.014	0.226	0.010	0.015	0.174	0.014
10	0.5	0.017	0.110	0.006	0.006	0.115	0.005	0.005	0.112	0.006
	1	0.017	0.126	0.007	0.007	0.136	0.006	0.006	0.129	0.007
	5	0.057	0.143	0.008	0.008	0.167	0.007	0.007	0.150	0.008
	10	0.050	0.148	0.008	0.008	0.175	0.007	0.008	0.155	0.008
	15	0.047	0.153	0.008	0.008	0.183	0.007	0.008	0.160	0.008
20	0.5	0.008	0.115	0.003	0.003	0.117	0.003	0.003	0.116	0.003
	1	0.008	0.130	0.004	0.004	0.135	0.003	0.003	0.131	0.004
	5	0.049	0.147	0.004	0.004	0.159	0.004	0.004	0.151	0.004
	10	0.041	0.150	0.004	0.004	0.164	0.004	0.004	0.154	0.004
	15	0.039	0.152	0.004	0.004	0.166	0.004	0.004	0.156	0.004
50	0.5	0.003	0.118	0.001	0.001	0.119	0.001	0.001	0.118	0.001
	1	0.003	0.133	0.002	0.002	0.135	0.001	0.001	0.134	0.001
	5	0.044	0.150	0.002	0.002	0.155	0.002	0.002	0.152	0.002
	10	0.036	0.153	0.002	0.002	0.159	0.002	0.002	0.155	0.002
	15	0.034	0.154	0.002	0.002	0.160	0.002	0.002	0.156	0.002

Observation 2.3. For τ_1 estimation, $\tilde{\tau}_1$ seems to be the best estimator among the three. For τ_2 estimation, though there is no uniformly best estimator, $\hat{\tau}_2$ seems to have the best overall performance. For τ_3 estimation, $\tilde{\tau}_3$ seems to have the best performance.

Concluding Remark. This work deals with three estimators of a gamma model parameters as well as the reliability function. Based on a comprehensive simulation study it has been observed that the BMLE is the best estimator of the shape parameter, whereas the MLE is the best estimator of the scale parameter. For reliability estimation, if we plug in the BMLEs (of shape and scale) then we get the overall best estimator. Hopefully this study will help the applied researchers in modelling their data by a gamma distribution.

References

[1] M. Berman, The maximum likelihood estimators of the parameters of the gamma distribution are always positively biased, Communications in Statistics-Theory and Methods 10 (1981), 693-697.

[2] L. R. Shenton and K. O. Bowman, Maximum likelihood estimator moments for the two-parameter gamma distribution, Sankhyā Series B 31 (1969), 379-396.

[3] L. R. Shenton and K. O. Bowman, Further remarks on maximum likelihood estimators for the gamma distribution, Technometrics 14 (1972), 725-733.

[4] T. Yanagimoto, The conditional maximum likelihood estimator of the shape parameter in the gamma distribution, Metrika 35 (1988), 161-175.

[5] A. Zaigraev and A. Podraza-Karakulska, On estimation of the shape parameter of the gamma distribution, Statistics & Probability Letters 78 (2008), 286-295.