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SOME ANALYSIS AND NUMERICAL STUDY OF RICKER MODEL

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Abstract

In this paper, we investigate the Ricker model describing the density of an isolated population. Our focus is on the long term behavior of the population for various values of parameters and initial conditions. We determine under what conditions, the population behaves chaotically. Some conjectures are made based on numerical computation.

1. Introduction

Different population models have been an area of interest for probability and researchers of stochastic process for a long time. The complex behavior of the deterministic Ricker model is well known in the area of dynamical systems. The Ricker model, a scalar difference equation with density-dependent growth, is one of the most widely known models for modeling fish populations. This model is introduced by Bill Ricker in 1954. Bill Ricker discovered this model when he developed in his studies of stock and recruitment in fisheries [1].

In Section 2, we present the background of Ricker model. In Section 3, some theoretical results are obtained. In Section 4, we carry out some

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numerical experiments to find out under what conditions on the parameters the population will exhibit certain behaviors such as extinct and chaotic. We also make some conjectures based on the analysis results. Finally, we state our findings in Section 5.

2. Background

The biological interpretation of the Ricker model is that of an isolated single-species population that only produces offspring at a specific time each year [2]. Now let us investigate the behavior of the Ricker model. We work with a time increment (Δt) of 1 year, and denote the stock of mature individuals at the census date in year *t* by X_t. Juveniles mature the year after their birth. Adult fish spawn once before dying and produce a maximum of *b* viable recruits to the following year's stock. Due to cannibalism on eggs by adults, the juvenile survivorship in a year when there are X_t adults is e^{-cX_t}, where *c* is a parameter related to the intensity of cannibalism [3].

Thus the population of the next period can be expressed as a function of the population of the previous period as in the equation

$$X_{t+1} = F(X_t),$$

where $F(X_t) = bX_t e^{-cX_t}$, $t = 1, 2, 3, \ldots.$

In the Ecological Dynamics [4] by Gurney and Nisbet, they had found that the Ricker model orbits had different distributions with different *b*. They also found that there is a b_c between 6.5 and 9, and if $b < b_c$, the viable equilibrium is stable, if $b \geq b_c$, it is unstable. They also hypothesized that the cycles displayed when the viable equilibrium is unstable are limit cycles.

3. Some Theoretical Results

For initial population densities larger than zero, the orbits of Ricker population model are quite different at different times. Then we may ask for a fixed small value of c, what is the long term behavior of the population for various values of b?

We fixed the value of parameter c and X_0.

Theorem 1. *When* $b \leq 1$, $\lim_{n \to \infty} X_n = 0$.

We start with a value of b which is $b \leq 1$, the system has no biologically sensible 'viable' equilibrium, and we can find that it tends towards $X_n = 0$ in Figure 2.1, as time progresses.

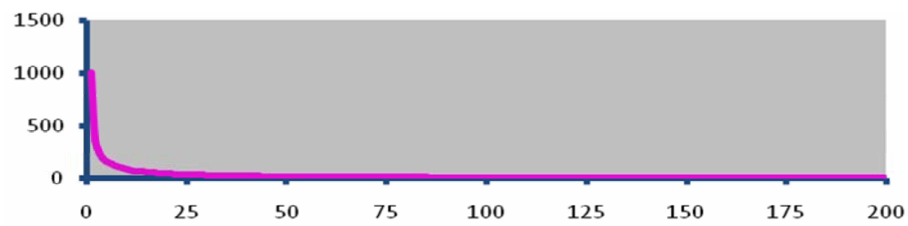

Figure 2.1. Numerical solutions for the Ricker model with $b \leq 1$.

Proof. For the equation $X_{n+1} = bX_n / e^{cX_n}$, it is because that $e^{cX_n} > b$, so $X_{n+1} < X_n$.

It is also easy to discern the mechanism underlying this behavior, the maximum possible juvenile survival is unity and each adult produces an average of only less than 1 egg. So each adult contributes only less than 1 individual to the next generation, and the population turns steadily to extinction.

Theorem 2. *When* $1 < b < e^2$, $\lim_{n \to \infty} X_n = b/ce$.

Proof. When we turn to the condition that $1 < b < e^2$. We may discuss how we find the e^2 first.

For the equation $X = bXe^{-cX}$, the two sides of this equation are equal if $X = 0$ or if $be^{-cX} = 1$. We can also note that if $b = 0$, then the value of the equilibrium is negative, so the only biologically sensible equilibrium is $be^{-cX} = 1$, which $X^* = \ln b/c$.

As we know, $F(X_n) - F(X_{n-1}) = f'(X)(X_n - X_{n-1})$.

So, when $f'(X^*) < 1$, the population distribution will tend to stability [5].

$$f'(x) = be^{-cx} - bcxe^{-cx} = be^{-cx}(1 - cx).$$

Then we can get $b < e^2$.

Under the condition of $b < e^2$, we can get $\lim_{n\to\infty} X_n = b/ce$.

To find the critical number, we should make the $f'(x) = 0$.

And, $f'(x) = be^{-cx} - bcxe^{-cx} = be^{-cx}(1 - cx)$.

So, when $f'(x) = 0$, the only answer is $x = 1/c$.

Then we put this x into the original equation $F(X_t) = bX_te^{-cXt}$.

We can get $f(x)_{\max} = b/ce$.

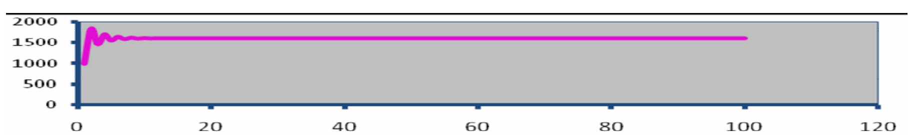

Figure 2.2. Numerical solutions for the Ricker model with $1 < b < e^2$.

4. Preliminary Numerical Investigation

Under the condition of $b > e^2$, the population are not stable. It is difficult to get the initial rules in different conditions. We have made some numerical researches and get some results with certain c. The orbits we get are having frequently cycles in some conditions and sometimes are chaos in some conditions. But the maximum of the equation in all the conditions is $b/ce.$ We have drawn some orbits when $c = 0.001,$ $X_0 = 1000.$

(a) When $e^2 < b \leq 12.5,$ the orbits have 2 *cycles* which can be seen in Figure 2.3.

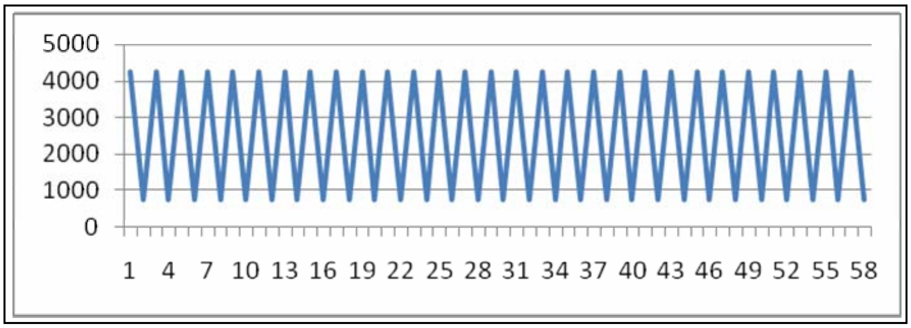

Figure 2.3

(b) When $12.5 < b \leq 14.24,$ the orbits have 4 cycles which is shown in Figure 2.4.

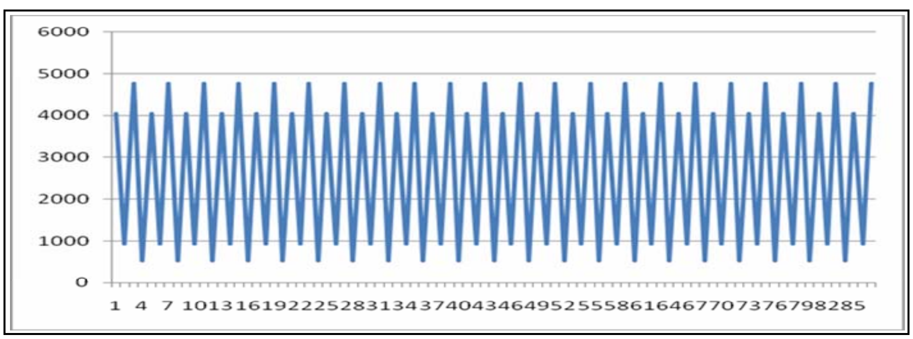

Figure 2.4

(c) When $14.24 < b \leq 14.65$, we can see that the orbits have 8 *cycles* from Figure 2.5.

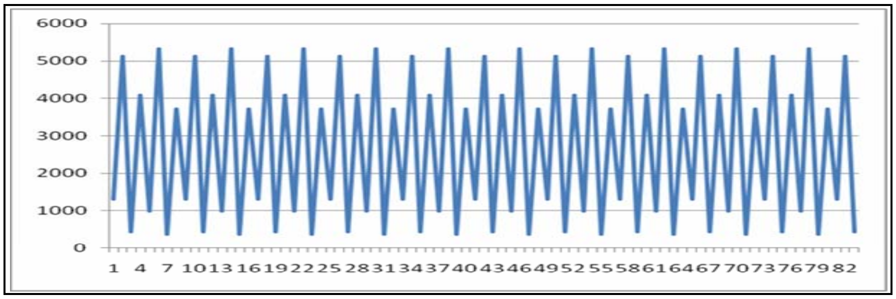

Figure 2.5

(d) When $14.65 < b \leq 14.74$, the orbits have 16 *cycles* which can be shown in Figure 2.6.

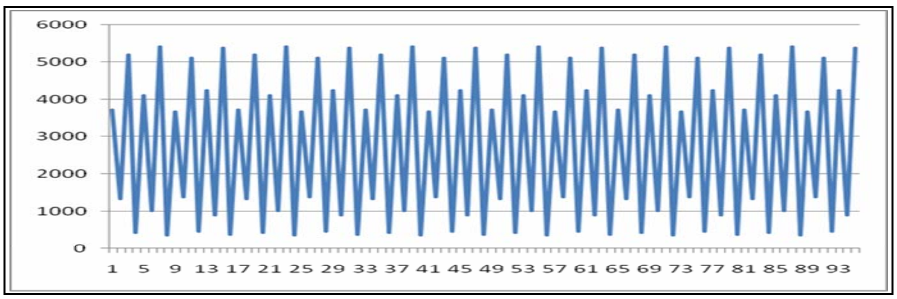

Figure 2.6

(e) When $14.74 < b < 16$, the orbits are *chaos* which can be seen in Figure 2.7.

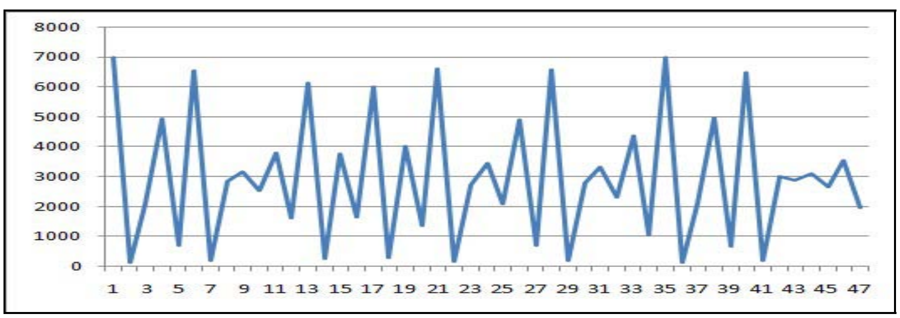

Figure 2.7

(f) When $b = 16$, we can see from Figure 2.8 that the orbits have 6 *cycles*.

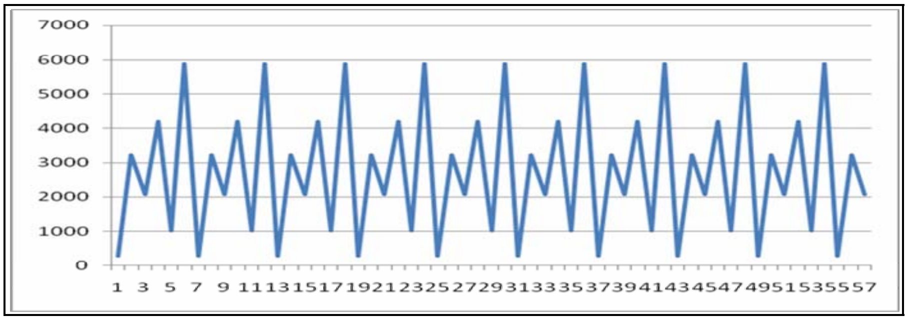

Figure 2.8

(g) When $16 < b$, the orbits are *chaos* which is shown in Figure 2.9.

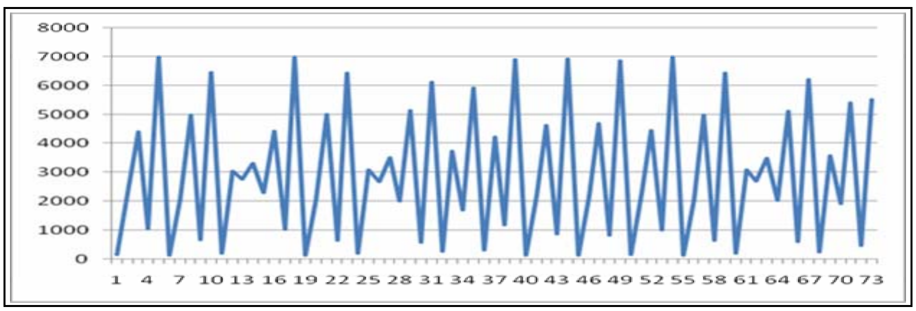

Figure 2.9

5. Conclusions

The orbits of Ricker model have significant disciplinary distributions with different factors b.

- If $b < 1$, then the distribution tends to extinct.
- If $1 < b < e^2$, then the distribution tends to a certain number b/ce.
- If $b > e^2$, then the distributions are not stable.

- If $e^2 < b \leq 12.5,$ then the orbits have 2 cycles.
- If $12.5 < b \leq 14.24,$ then the orbits have 4 cycles.
- If $14.24 < b \leq 14.65,$ then the orbits have 8 cycles.
- If $14.65 < b \leq 14.74,$ then the orbits have 16 cycles.
- If $14.74 < b < 16,$ then the orbits are chaos.
- If $b = 16,$ then the orbits have 6 cycles.
- If $b > 16,$ then the orbits are chaos.

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