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AMPLITUDE INCREASING FORMULA OF BICHROMATIC WAVE PROPAGATION BASED ON FIFTH ORDER SIDE BAND SOLUTION OF KORTEWEG de VRIES EQUATION

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Abstract

Previous analytical, numerical and experimental results show that surface waves which initially bichromatic signals deform during

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their propagation in hydrodynamic laboratories. The deformation is followed by the increase of the bichromatic wave amplitude. It has been investigated that the position where the bichromatic wave generates the highest amplitude is of order $O\left(\frac{1}{q^2}, \frac{1}{\kappa_{lin}^2}\right)$, where q and κ_{lin} are the amplitude and frequency of the bichromatic wave envelope, respectively. The result is based on the third order Korteweg de Vries (KdV) equation and the quantity obtained is called maximal temporal amplitude (MTA). However, the dependence of the increase in the maximum amplitude of the bichromatic wave over wave parameters is unknown. This paper describes the derivation of a formula that can be used to predict the increase of the maximum amplitude of bichromatic wave. Here, the bichromatic wave is defined as a superposition of two monochromatic waves with the same amplitude but slightly different of frequencies. The formula is derived using the approximation of fifth order KdV equation and MTA. It has been found that the increase of a bichromatic wave maximum amplitude is of order $O\left(\frac{q}{\kappa_{lin}}\right)^2$. The result of the increase of maximum amplitude for bichromatic wave found using this formula is in accordance with the experimental results as well as the result of a numerical software HUBRIS.

1. Introduction

This paper focuses on the phenomena of splitting and peaking of surface water waves during their propagation in hydrodynamic laboratories. It is motivated by the needs of the laboratory to generate a wave with high and steep amplitude but do not break during their propagation in the wave tank. Such waves will be used to test a floating body before operated in real. Noted in [1, 2], extreme waves, which is often also called as *giant waves* or *rouge waves* or *freak waves*, are very large water waves whose heights exceed the significant wave height of a measured wave train by a factor more than 2.2. Wave of this type is extremely rare and its appearance is not easily predicted, but its impact can cause severe damage to floating body, offshore structures and other objects that are around this wave (see Earle [3], Mori et al. [4], Divinsky et al. [5], Trulsen and Dysthe [6], Smith [7] and Toffoli and Bitner [8]). Therefore, many researchers seriously conduct researches on extreme wave. Various studies conducted to understand the emerging and propagation phenomena of the extreme wave.

In the real conditions involving sufficiently large spatial and temporal intervals, the generation of extreme waves cannot be easily done. This is due to the physical limitations of the wave generator. It has been understood that as effect of nonlinearity leads to the deformation of wave during its propagation away from the wave maker. Effects of nonlinearity has been described theoretically in van Groesen et al. [9], Groesen and Westhuis [10], Marwan and Andonowati [11] and Andonowati and Groesen [12], numerically by Westhuis et al. [13] as well as experimental investigations on bichromatic wave deformation as in [14-16]. Depending on wave parameters such as wave amplitude and frequency, the position where the wave amplitude reaches the maximum and the increase of the wave amplitude can be determined. Location of the maximum amplitude wave increase in the wave tank also depends on these parameters (see Marwan [17, 18] and Ramli [19]).

Using the third order side band of the KdV equation and MTA was defined in [11, 12], Marwan [17] managed to find a formula which is used to predict the extreme positions of bichromatic wave in their propagation in the wave tank. Extreme position found is compared with the experimental results [15, 16] and the results of numerical calculations of HUBRIS software [16]. However, the wave height at the extreme position calculated using this approach is lower than the experimental results and the results of numerical calculation using HUBRIS software. For the bichromatic wave with the initial amplitude 0.16m, the carrier wave frequency 3.145rad/s and the envelope frequency 0.155rad/s, an increase of amplitude calculated using third order side band KdV equation is 0.22m (1.4 times from the initial amplitude of bichromatic wave) [14]. Meanwhile, the increase in wave amplitude both from experimental results and numerical calculations using software HUBRIS is 0.32m (2 times from the initial amplitude of bichromatic wave) [15, 16]. The amplitude increase is defined as the ratio between the amplitude of the bichromatic wave at the extreme position and the amplitude of the bichromatic wave at the wave maker. Discrepancy is presumably caused by cutting of the solution of KdV equation only up to the third order. Therefore, this research of the amplitude increase of the wave that initially in bichromatic wave needs to be discussed in detail using higher order approximation of KdV equation.

This paper aims to derive the formula of the maximum amplitude increase of waves that initially in bichromatic signals form. This signal is a superposition of two monochromatic signals with the same amplitudes but slightly different of frequencies. The waveform is distorted due to nonlinearity in water medium characterized by an increase in amplitude, as reported in [15-17]. This type of wave is quite attractive when associated with extreme waves, as described in Longuet and Higgins [20], Phillips et al. [21] and Donelan and Hui [22]. Besides, Lo and Mei in 1985 [23], explained that the bichromatic waves unstable in the evolution for a sufficiently long period of evolution. Using the numerical scheme of Dysthe equation, Lo and Mei also observed the separation of a bichromatic wave into two groups of different waves with different of velocity. In addition, they also showed that the bichromatic wave that originally symmetric become unsymmetric and increased its steepness in the front and flattened in the real part. Similar results with Lo and Mei were also obtained by Stansberg in [16] in his experiments. Stansberg [16] showed that the surface waves which in the generator is a bichromatic signal experience deformation of its envelope in the form of peaking and separating during its propagation. As in the previous studies, here KdV equation with exact dispersion relation is used, which models the propagation of surface waves. Different from Marwan and Andonowati [11], Andonowati and Groesen [12] and Marwan [17, 18], here the solution of the KdV equation will be approximated using fifth order asymptotic expansion. Using the fifth order side band of KdV equation and MTA, a formula for amplitude maximum increase of bichromatic wave will be derived.

Outlines of this paper are described as follows. Second section presents briefly the surface wave equation models employed and the construction of its solution. Some interesting parts such as resonant term and side band term will be presented in this session as well. Third section presents the formula of extreme position and the behavior of the amplitude maximum increase of bichromatic wave. It will also present several graphs of the results and its comparison with experimental results and numerical results from HUBRIS software. The last section presents important conclusions of the results.

2. Mathematical Model of Surface Wave

Korteweg de Vries (KdV) equation is known as asymptotic models for unidirectional rather long and small wave that propagates on the surface. In a dimensionless form, it can be written as

$$\partial_t \eta + i\Omega(-i\partial_x)\eta + \frac{3}{4}\partial_x \eta^2 = 0 \tag{1}$$

with η denotes the wave elevation at the surface, x and t denote spatial and time variables, respectively. The symbol Ω is a differential operator which describes the exact dispersion relation between the frequency ω and wave number $k, \omega = \Omega(k) = \sqrt{k \tanh k}$ [24]. Laboratory variables for the wave elevation, spatial and time variables η_{lab}, x_{lab}, t_{lab} are related to the dimensionless variables $\eta_{lab} = h\eta$, $x_{lab} = hx$ and $t_{lab} = \sqrt{h/g}$, where h denotes the water depth and g is the gravity acceleration.

As has been explained in the previous section, in this paper, the approximate solution of equation (1) is determined using the asymptotic expansion up to the fifth order in the form of power series of the amplitude of the wave elevation. Here, the wave elevation is written as

$$\eta \approx \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \varepsilon^3\eta^{(3)} + \varepsilon^4\eta^{(4)} + \varepsilon^5\eta^{(5)} \tag{2}$$

with ε is a small positive number that represents the order of the amplitude of the wave. The symbols $\eta^{(1)}$, $\eta^{(2)}$, $\eta^{(3)}$, $\eta^{(4)}$ and $\eta^{(5)}$ successively describe the first order, second order, third order, fourth order and fifth order nonlinear terms. Using the assumption that the linear form $\eta^{(1)}$ consists of five frequencies close to each other, the third order and fifth order terms only focused on the term that has the largest contribution, namely $\eta_{sb}^{(3)}$ and $\eta_{sb}^{(5)}$. It can be shown that the side band frequency term is quite close to the frequency of the linear term. It has been known that the direct expansion produces resonance in the third order (see [25]). To avoid this, a modification is introduced which is the development Linstead-Poincare [26] techniques, which produces nonlinear dispersion relation [25]

$$k = k^{(0)} + \varepsilon^1 k^{(1)} + \varepsilon^2 k^{(2)} + \varepsilon^3 k^{(3)} + \varepsilon^4 k^{(4)} \tag{3}$$

with $k^{(0)} = \Omega^{-1}(\omega)$.

Since this study focuses on bichromatic waves, here $\eta^{(1)}$ is chosen as

$$\eta^{(1)} = 4q \cos \overline{\theta} \cos \Delta\theta \tag{4}$$

with

$$\overline{\theta} = \frac{\theta_+ + \theta_-}{2} \quad \text{and} \quad \Delta\theta = \frac{\theta_+ - \theta_-}{2},$$

$\theta_\pm = k_\pm x - \omega_\pm t$, $(k_\pm, \omega_\pm)$ satisfy a nonlinear dispersion relation. The following procedure has been adopted in [14, 17, 18], however, it was only up to the third order. Furthermore, substitution of (2) and (3) into (1), for $\eta^{(1)}$ in (4) as a linear solution, at the second order approximation it gives

$$k_\pm^{(1)} = 0,$$

$$\eta^{(2)} = 3q^2[s_+ \cos 2\theta_+ + s_- \cos 2\theta_- + 2s \cos 2\overline{\theta} + s_0 \cos 2\Delta\theta] \tag{5}$$

and

$$\eta_{fw}^2 = 3q^2 \begin{bmatrix} s_+ \cos(\Omega^{-1}(2\omega_+)x - 2\omega_+ t) + s_- \cos(\Omega^{-1}(2\omega_-)x - 2\omega_- t) \\ + 2s \cos(\Omega^{-1}(2\overline{\omega})x - 2\overline{\omega} t) + 2s \cos(\Omega^{-1}(2v)x - 2vt) \end{bmatrix} \tag{6}$$

with

$$\overline{\omega} = \frac{\omega_+ + \omega_-}{2}, \quad v = \frac{\omega_+ - \omega_-}{2}.$$

The coefficients $s_\pm$, s_-, s and s_0 can be found in Marwan [14]. The form of η^2_{fw} is the second order term of *free wave* in order to satisfy boundary conditions at the generator position of the wave in the form of a bichromatic signal (see [11, 14, 17, 18]). Then, at the third order, it successively produces

$$k_\pm^{(2)} = -\frac{9}{4} q^2 k_\pm^{(0)} \frac{(s_\pm + 2s + s_0)}{\Omega'(k_\pm^{(0)})}, \tag{7}$$

$$\eta_{sb}^{(3)} = \frac{9}{2} q^3 [A_{sb+} \cos\overline{\theta} \cos 3\Delta\theta - A_{sb-} \sin\overline{\theta} \sin 3\Delta\theta] \tag{8}$$

and

$$\eta_{sb,\, fw}^{(3)} = \frac{9}{2} q^3 \begin{bmatrix} A_{sb+} \cos(K_+ x - \overline{\omega} t) \cos(K_- x - 3\nu t) \\ -A_{sb-} \sin(K_+ x - \overline{\omega} t) \sin(K_- x - 3\nu t) \end{bmatrix}. \tag{9}$$

In equations (8) and (9) $A_{sb\pm} = a_+ \pm a_-$ with

$$a_\pm = (s_\pm + s_0) \frac{\overline{k}_{lin} \pm 3\kappa_{lin}}{\overline{\omega} \pm 3\nu - \Omega(\overline{k}_{lin} \pm 3\kappa_{lin})}, \tag{10}$$

$$k_\pm = \frac{\Omega^{-1}(\overline{\omega} + 3\nu) \pm \Omega^{-1}(\overline{\omega} - 3\nu)}{2}, \tag{11}$$

$$\overline{k}_{lin} = \frac{k_+^{(0)} + k_-^{(0)}}{2} \quad \text{and} \quad \kappa_{lin} = \frac{k_+^{(0)} - k_-^{(0)}}{2}.$$

As showed in Marwan [14], the quantity $A_{sb\pm}$ in (9) can be written as

$$A_{sb\pm} \approx -\frac{(\sigma_0 + \sigma_2)}{2\kappa_{lin}^2 \Omega''(k_\pm^{(0)}) \pm \frac{4}{3} \kappa_{lin}^3 \Omega'''(k_\pm^{(0)})}$$

with

$$\sigma_0 = \frac{1}{\Omega'(\Omega^{-1}(\overline{\omega})) - 1} \quad \text{and} \quad \sigma_2 = \frac{\Omega^{-1}(\overline{\omega})}{2\overline{\omega} - \Omega(2\Omega^{-1}(\overline{\omega}))} \ [24, 25].$$

Consequently, third order *side band* term can also be written as

$$\eta_{sb}^{(3)} = \frac{9}{2} q \left[\frac{q}{\kappa_{lin}}\right]^2 [\alpha_+ \cos\bar{\theta} \cos 3\Delta\theta - \alpha_- \sin\bar{\theta} \sin 3\Delta\theta], \tag{12}$$

$$\eta_{sb,\,fw}^{(3)} = \frac{9}{2} q \left[\frac{q}{\kappa_{lin}}\right]^2 \left[\begin{array}{l} \alpha_+ \cos(K_+ x - \bar{\omega} t) \cos(K_- x - 3vt) \\ -\alpha_- \sin(K_+ x - \bar{\omega} t) \sin(K_- x - 3vt) \end{array}\right] \tag{13}$$

with

$$\alpha_+ = -\frac{\sigma_0 + \sigma_2}{\Omega''(k_\pm^{(0)})} \quad \text{and} \quad \alpha_- = -\frac{\alpha_+ \Omega'''(k_\pm^{(0)})}{\Omega''(k_\pm^{(0)})} \kappa_{lin}.$$

It shows that the magnitude of the third order side band amplitude is of order $O(q[q/\kappa_{lin}]^2)$ as presented in Marwan [14] and Cahyono [25]. For sufficiently small κ_{lin}, third order side band term dominate the existence of other second and third order terms. It should be noted that the expression in (7)-(13) has also been found in Marwan [14, 17, 18]. Furthermore, in the fourth order, the following expression is obtained:

$$k_\pm^{(3)} = 0,$$

$$\eta^{(4)} = \frac{27}{4} q^4 \left[\begin{array}{l} T_+ \cos 4\theta_+ + T_- \cos 4\theta_- + P_+ \cos(3\theta_+ + \theta_-) + P_- \cos(3\theta_- + \theta_+) \\ + Q_+ \cos(3\theta_+ - \theta_-) + Q_- \cos(3\theta_- - \theta_+) \end{array}\right] \tag{14}$$

and

$$\eta_{fw}^4 = \frac{27}{4} q^4 \left[\begin{array}{l} T_+ \cos(\Omega^{-1}(4\omega_+) x - 4\omega_+ t) + T_- \cos(\Omega^{-1}(4\omega_-) x - 4\omega_- t) \\ + P_+ \cos(\Omega^{-1}(3\omega_+ + \omega_-) x - (3\omega_+ + \omega_-) t) \\ + P_- \cos(\Omega^{-1}(3\omega_- + \omega_+) x - (3\omega_- + \omega_+) t) \\ + Q_+ \cos(\Omega^{-1}(3\omega_+ - \omega_-) x - (3\omega_+ - \omega_-) t) \\ + Q_- \cos(\Omega^{-1}(3\omega_- - \omega_+) x - (3\omega_- - \omega_+) t) \end{array}\right] \tag{15}$$

with

$$T_{\pm} = \frac{4k_{\pm}^{(0)}(A_{\pm} + s_{\pm})}{4\omega_{\pm} - \Omega(4k_{\pm}^{(0)})}, \quad P_{\pm} = \frac{(3k_{\pm}^{(0)} + k_{\mp}^{(0)})(A_{\pm} + 2ss_{\pm})}{3\omega_{\pm} + \omega_{\pm} - \Omega(3k_{\pm}^{(0)} + k_{\mp}^{(0)})},$$

$$Q_{\pm} = \frac{(3k_{\pm}^{(0)} - k_{\mp}^{(0)})(A_{\pm} + s_0 s_{\pm})}{3\omega_{\pm} - \omega_{\pm} - \Omega(3k_{\pm}^{(0)} - k_{\mp}^{(0)})} \quad \text{and} \quad A_{\pm} = \frac{3k_{\pm}^{(0)} s_{\pm}}{3\omega_{\pm} - \Omega(3k_{\pm}^{(0)})}.$$

As the focus of this paper only to the term that gives the highest contribution in each order, then in the fifth order gives

$$k_{\pm}^{(4)} = -\frac{81}{16} q^4 k_{\pm}^{(0)} \frac{(s_{\pm}A_{\pm} + s_{\mp}C_{\mp} + 2sC_{\pm} + s_0 a_{\mp} + s_{\mp}a_{\mp})}{\Omega'(k_{\pm}^{(0)})} \tag{16}$$

with

$$C_{\pm} = \frac{(s_{+} + 2s)(2k_{\pm} + k_{\mp})}{(2\omega_{\pm} + \omega_{\mp}) - \Omega(2k_{\pm} + k_{\mp})}.$$

After that, fifth order side band term can be written as

$$\eta_{sb}^{(5)} = \frac{81}{8} q^5 [D_{+} \cos\overline{\theta} \cos 5\Delta\theta - D_{-} \sin\overline{\theta} \sin 5\Delta\theta$$
$$+ E_{+} \cos\overline{\theta} \cos 3\Delta\theta - E_{-} \sin\overline{\theta} \sin 3\Delta\theta], \tag{17}$$

$$\eta_{sb,\,fw}^{(5)} = \frac{81}{8} q^5 \begin{bmatrix} D_{+} \cos(H_{+}x - \overline{\omega}t)\cos(H_{-}x - 5vt) \\ -D_{-} \sin(H_{+}x - \overline{\omega}t)\sin(H_{-}x - 5vt) \\ + E_{+} \cos(K_{+}x - \overline{\omega}t)\cos(K_{-}x - 3vt) \\ -E_{-} \sin(K_{+}x - \overline{\omega}t)\sin(K_{-}x - 3vt) \end{bmatrix} \tag{18}$$

with $D_{\pm} = d_{+} \pm d_{-}$, $E_{\pm} = e_{+} \pm e_{-}$,

$$d_{\pm} = (s_{\mp}A_{\pm} + s_0 A_{sb\pm} + s_{\pm}A_{sb\mp}) \frac{\overline{k}_{lin} \pm 5\kappa_{lin}}{\overline{\omega} \pm 5v - \Omega(\overline{k}_{lin} \pm 5\kappa_{lin})},$$

$$e_{\pm} = 2s(A_{\pm} + A_{sb\mp}) \frac{\overline{k}_{lin} \pm 3\kappa_{lin}}{\overline{\omega} \pm 3v - \Omega(\overline{k}_{lin} \pm 3\kappa_{lin})}$$

and

$$H_{\pm} = \frac{\Omega^{-1}(\overline{\omega} + 5\nu) \pm \Omega^{-1}(\overline{\omega} - 5\nu)}{2}.$$

The form in (17) is called as the *fifth order side band term* and has frequencies close enough to the first order term frequency. In the same way as in the third order side band term, it can be shown that this term has an amplitude magnitude of order $O(q[q/\kappa_{lin}]^4)$. As is the case with the third order side band terms, for a sufficiently small value of κ_{lin}, fifth order side band term can dominate the existence of other fourth order and fifth order terms.

3. Formula for Extreme Position and Amplitude Increase

The nature of a surface wave is nonlinear. This causes water surface waves experience the phenomena of peaking and splitting during their propagation in the wave tank in hydrodynamics laboratory, as reported in Groesen et al. [9], Westhuis et al. [13, 15], Stansberg [16] and Zakharov [27].

To observe this phenomena, especially in peaking, Marwan and Andonowati [11] introduced a quantity called the *MTA* and has been used also for optical problems [12]. The scale used to measure the height of the wave at each position, which is defined as

$$m(x) = \max_{t} \eta(x, t), \quad 0 < x < L,$$

L specifies the length of the wave tank.

In the deterministic wave generation in hydrodynamic laboratories, MTA is used to predict the position in which the wave will reach the highest peaking during their propagation in the wave tank [17, 18]. To know the wave elevation changes, it observes the ratio of the highest value of MTA and the value of MTA at the wave maker. The comparison is called the *amplitude amplification factor* (*AAF*), which is defined as

$$AAF = \frac{m(x_{\max})}{m(0)}$$

with $x_{\max}$ represent the position where MTA reach maximum.

Here, we present derivation of formula for extreme position and the increase of the wave amplitude. Here, we review again the first order solution term, the third order side band term and the fifth order side band term presented in (4), (8) and (17). Based on these formulas, it can be seen that the first order term, the third order side band term and the fifth order side band term have the same carrier wave. Assuming that $A_{sb-} \ll A_{sb+}$, $D_{-} \ll D_{+}$ and $E_{-} \ll E_{+}$, superposition the first order term, the third order side band term, the third order side band free waves, the fifth order side band term, and the fifth order side band free waves will generate an envelope that modulate the carrier wave. Using the same procedure as in [18] it can be shown that the modulated length of the carrier wave is

$$\lambda = \frac{4\pi}{|\,\overline{H} - \bar{k}\,|},$$

with $\overline{H} = (H_{+} + K_{+})/2$. Consequently, the position where bichromatic waves experience the highest peaking can be written as

$$\hat{x}_{\max} = \frac{2\pi}{|\,\overline{H} - \bar{k}\,|}. \tag{19}$$

As explained earlier, this paper focuses only on the term that have the greatest contribution in each order, then the solution of equation (1) that meets the boundary conditions in a bichromatic signal wave generator can be written as

$$\eta = \eta^{(1)} + \eta^{(3)}_{sb} - \eta^{(3)}_{sb,\,fw} + \eta^{(5)}_{sb} - \eta^{(5)}_{sb,\,fw}. \tag{20}$$

By substitution of the expressions of (4), (8), (9), (17) and (18) into (20), and using the assumption that $A_{sb-} \ll A_{sb+}$, $D_{-} \ll D_{+}$ and $E_{-} \ll E_{+}$, it is found that

$$\begin{aligned}
\eta(x, t) = {} & 4q\cos(\bar{k}x - \bar{\omega}t)\cos(\kappa x - \nu t) \\
& + \frac{9}{2}q^3 A_{sb+}\cos(\bar{k}x - \bar{\omega}t)\cos(3\kappa x - 3\nu t) \\
& - \frac{9}{2}q^3 A_{sb+}\cos(K_+ x - \bar{\omega}t)\cos(K_- x - 3\nu t) \\
& + \frac{81}{8}q^5 D_+\cos(\bar{k}x - \bar{\omega}t)\cos(5\kappa x - 5\nu t) \\
& - \frac{81}{8}q^5 D_+\cos(H_+ x - \bar{\omega}t)\cos(H_- x - 5\nu t) \\
\approx {} & 4q\begin{bmatrix}\cos(kx - \nu t) + \dfrac{9}{4}q^2 A_{sb+}[\cos(3\kappa x - 3\nu t) - \cos(K_- x - 3\nu t)] \\ + \dfrac{81}{16}q^4 D_+[\cos(5\kappa x - 5\nu t) - \cos(H_- x - 5\nu t)]\end{bmatrix} \\
& \cdot \cos(\bar{k}x - \bar{\omega}t). \qquad (21)
\end{aligned}$$

Based on the expression in (21), it can be shown that the increase of the bichromatic wave amplitude can be written as

$$AAF = 1 + \frac{9}{4}q^2 A_{sb+} + \frac{81}{16}q^4[D_+ + E_+]. \qquad (22)$$

Furthermore, by (12) and (18), the magnitude of the amplitude of the third order side band term and the third order side band term consecutively is of order $O(q[q/\kappa_{lin}]^2)$ and $O(q[q/\kappa_{lin}]^4)$, then the AAF in (22) is of order $O([q/\kappa_{lin}]^2)$. It means, the increase of the bichromatic wave amplitude is proportional to the square of the initial amplitude and inversely proportional to the square of the envelope frequency.

4. Results and Discussion

Here, some illustrations are presented in graphical form based on the solution up to the fifth order approximation of the KdV equation. All values in this calculation are given in the laboratory coordinate system units

(m, kg, s). Comparison of calculation results with the experimental results of Stansberg and the results of calculations using numerical software HUBRIS is also presented.

As a case example, it uses a bichromatic wave generation in a laboratory of hydrodynamics with the depth and length of the wave tank is 5m and 250m, respectively.

Table 1. Bichromatic wave parameters in laboratory scale and parameter in the normalized scale

Parameter	Laboratory scale	Normalized scale
$4q$	0.16m	0.0320
$\overline{\omega}$	3.145rad/s	2.2464
ν	0.155rad/s	0.1107
$\bar{k}_{lin}$	1.0185rad/m	5.0923
κ_{lin}	0.1005rad/m	0.5025

At the initial position, $x = 0$, the value of MTA for the bichromatic wave is $m(0) = 4q = 0.16\text{m}$. The graph of this MTA over $x, 0 \leq x \leq 4000\text{m}$, is presented in Figure 1. This figure presents four graphs of MTA, which, respectively, represent, (blue) shows the value of MTA that calculated using all order, $m(x) = \max_t \eta(x, t)$, (red) shows the value of MTA that calculated using first order solution, third order side band solution and fifth order side band solution, $m_{-[2,3,4]}(x) = \max_t[\eta(x, t) - \eta^2(x, t) - \eta^3(x, t) - \eta^4(x, t)]$, (black) shows the value of MTA that calculated using first order solution and third order side band solution, $m_{-[2,3,4,5]}(x) = \max_t[\eta(x, t) - \eta^2(x, t) - \eta^3(x, t) - \eta^4(x, t) - \eta_{sb}^{(5)}(x, t)]$, and (violet) shows the value of MTA that calculated using first order solution, and fifth order side band solution $m_{-[2,3,4]}(x) = \max_t[\eta(x, t) - \eta^2(x, t) - \eta^3(x, t) - \eta_{sb}^{(3)}(x, t) - \eta^4(x, t)]$. It can be observed that $m_{-[2,3,4]}(x)$, $m_{-[2,3,4,5]}(x)$ and $m_{-[2,3,4]}(x)$ are different from $m(x)$ only in the number of MTA oscillation per interval of distance. This difference is due to the influence of second order solutions, third order solutions and fourth order solutions that have a high-frequency carrier wave, successively represent by $2\overline{\omega}$, $3\overline{\omega}$ and $4\overline{\omega}$. Then from the expression in (4), (12) and (17) it shows that the phase of the carrier wave of the first order solution is the same as the phase of the carrier wave of the third order side band solution and the fifth order side band solution, namely $(kx - \omega t)$. As a result, the global structure of the bichromatic wave MTA can be determined just from the interaction of first order solution, third order side band solution and fifth order side band solution, as shown in Figure 1. In addition, there were also indications of the repetition of the MTA value for a certain period.

Furthermore, the MTA graph of bichromatic waves calculated using numerical software HUBRIS, the fifth order approximation of KdV equation and third order approximation of KdV equation is presented in Figure 2. In this figure, it shows the bichromatic wave MTA graph is presented for $q = 0.04\text{m}$, $\overline{\omega} = 3.145\text{rad/s}$ and $\nu = 0.155\text{rad/s}$, which is calculated using a numerical software HUBRIS (black), fifth order approximation of KdV equation (red) and third order approximation of KdV equation (blue). It can be seen that there is a compatibility in the position where MTA of bichromatic wave reaches its maximum when calculated using all those three approaches. In addition, it appears also that the maximum value of the bichromatic wave MTA calculated using the fifth order approximation of the KdV equation is higher than the third order approximation of the KdV equation, although still lower than the maximum value calculated using the numerical software HUBRIS. This gives an indication that there is a high order term effect on the increase in the maximum amplitude, but does not give effect to the position where the maximum value is achieved.

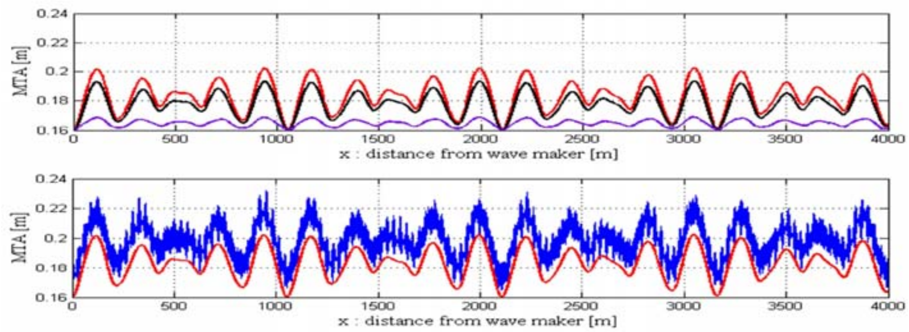

Figure 1. Plot of MTA over the position of a bichromatic wave with $q = 0.04\text{m}$, $\overline{\omega} = 3.145\text{rad/s}$ and $\nu = 0.155\text{rad/s}$ based on KdV solution, (blue) based on all order, (black) based on the first order term and the third order side band term [17], (violet) based on the first order term and the fifth order side band term. It can be seen that the effect of the higher order other than side band only at high oscillation.

It has been shown that the increase of the bichromatic wave amplitude is proportional to the square of the initial amplitude and inversely proportional to the square of the envelope frequency, the $AAF \approx O\left(\frac{q}{\kappa_{lin}}\right)^2$, as seen in Figure 1. Graph of the dependency of the amplitude increase of bichromatic wave over the envelope frequency is presented in Figure 3. Figure 3 presents a graph of the increase in bichromatic wave amplitude with respect to two wave parameters, namely the envelope amplitude and frequency, which is calculated using numerical software HUBRIS (black), the third order approximation of KdV equation (blue) and the fifth order approximation of KdV equation (red). It shows that the increase of bichromatic wave amplitudes calculated using all these three approaches showed closely the same pattern of dependence on the amplitude and frequency of the envelope.

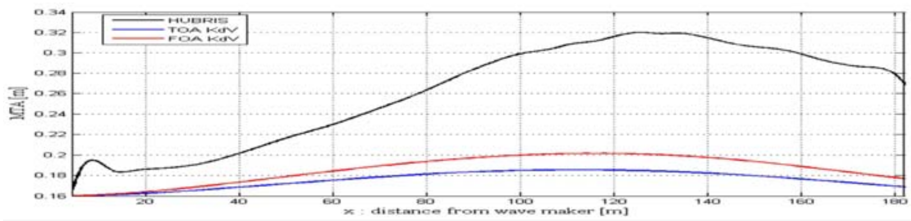

Figure 2. Plot of MTA over the position for bichromatic wave with $q = 004\text{m}$, $\overline{\omega} = 3.145\text{rad/s}$ and $v = 0.155\text{rad/s}$, (black) using numerical software HUBRIS [13, 15], (red) based on fifth order approximation of the KdV equation and (blue) based on third order approximation of the KdV equation [17]. It shows that there is conformity of where MTA reaches a maximum, computed based on all these the three approaches.

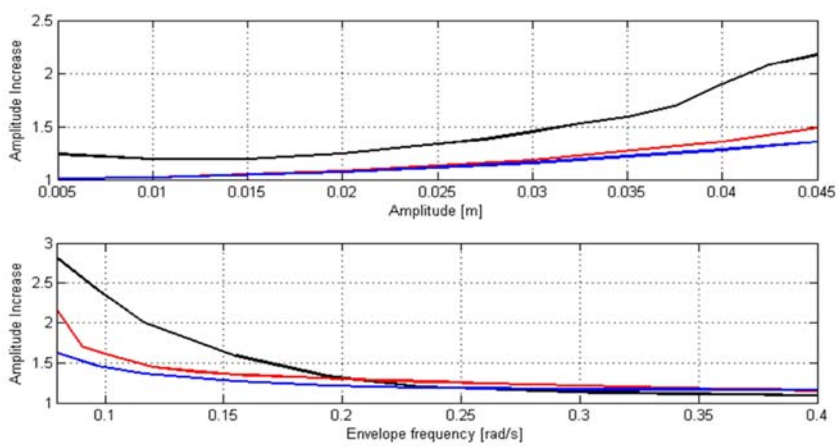

Figure 3. Plot of the increase of bichromatic wave amplitude, (top) the amplitude (q) for $\overline{\omega} = 3.145\text{rad/s}$ and $v = 0.155\text{rad/s}$ and (bottom) the envelope frequency (ν) for $q = 0.04\text{m}$ and $\overline{\omega} = 3.145\text{rad/s}$. (Black) calculations with numerical software HUBRIS [14], (red) based on fifth order approximation of the KdV equation and (blue) based on third order approximation of the KdV equation [17]. It shows that the increase in bichromatic wave amplitudes calculated using the all these three approaches showed nearly the same pattern of dependence on the amplitude and frequency of the envelope.

Furthermore, the bichromatic wave form for $q = 0.04\text{m}$, $\overline{\omega} = 3.145\text{rad/s}$ and $\nu = 0.155\text{rad/s}$ at wave generator position $(x = 0)$ and the extreme position $(x = x_{\max})$ are presented in Figure 4. Based on the graph, it can be seen that the bichromatic wave signal on extreme positions, $x = x_{\max}$, which is calculated using the approach of the fifth order KdV equation has a lower amplitude than the bichromatic wave signal calculated using a numerical software HUBRIS. However, there is a form of conformity for the bichromatic wave signal calculated by both the approaches at the extreme positions.

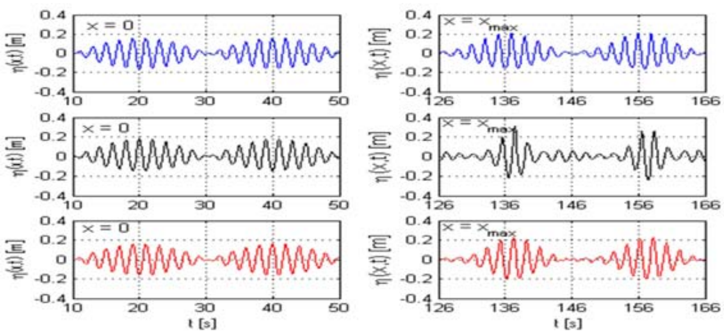

Figure 4. The form of bichromatic wave signal at the initial position and the position where the MTA reached a maximum for $q = 0.04\text{m}$, $\overline{\omega} = 3.145\text{rad/s}$ and $\nu = 0.155\text{rad/s}$. (Blue) approach based on third order KdV equation [14], (black) based on experimental results [15, 16] and (red) based on the fifth order KdV equation.

Prediction of extreme position, $x_{\max}$, and the increase of maximum amplitude, AAF, of the bichromatic wave which is derived based on the fifth order approximation of the KdV equation is compared with the results of numerical calculations using the software HUBRIS [15] and experimental results [15, 16]. Comparison of the results of calculation of $x_{\max}$ and AAF of the bichromatic wave for some parameters of bichromatic wave using all three approaches are presented in Table 2. It shows the calculation result of $x_{\max}$ for the MTA of bichromatic waves calculated using the fifth order

KdV equation conform with the calculation result of $x_{\max}$, which is calculated using numerical software HUBRIS and experimental results. Furthermore, it appears also that the bichromatic wave height in extreme position that is calculated by using the fifth order KdV equation is still lower than that calculated using numerical software HUBRIS and experimental results. Nonetheless there is an increase in the value of the maximum amplitude of the bichromatic wave calculated using the approach of the fifth order KdV equation compared to the value of the maximum amplitude of the bichromatic wave calculated using the third order KdV equation. This gives an indication that the high order side band solutions give effect to the increase in the maximum amplitude of the bichromatic wave.

Table 2. Comparison of the results of the calculation of extreme positions and the increase of the maximum amplitude for the bichromatic wave using fifth order approximation of KdV equation, third order approximation of KdV equation, numerical software HUBRIS and experimental results

Wave parameters	Extreme position $[m]$				Maximum amplitude $[m]$		
	HUBRIS	Experiment	TOA KdV	FOA KdV	Experiment	TOA KdV	FOA KdV
$q = 0.45$, $\omega_+ = 3.264$, $\omega_- = 3.028$	155.0	140 – 160 [15]	157.5 [17]	157.53	0.38 [15]	0.2954	0.3459
$q = 0.04$, $\omega_+ = 3.30$, $\omega_- = 2.99$	127.0	100 – 120 [15, 16]	118.0 [17]	118.22	0.31 [15, 16]	0.2189	0.2285
$q = 0.045$, $\omega_+ = 3.30$, $\omega_- = 2.99$	109.8	100 – 120 [15]	110.9 [17]	110.91	0.35 [15]	0.2579	0.2745
$q = 0.1$, $\omega_+ = 3.491$, $\omega_- = 2.856$	47.0	40 – 50 [15]	38.5 [17]	38.5	0.28 [15]	0.2541	0.2603

5. Concluding Remarks

We have investigated propagation of wave where initially bichromatic signal at the wave maker in hydrodynamics laboratory. Investigation is focused on derivation of a formula that can be used to predict the increase of maximum amplitude of bichromatic wave during their propagation in the wave tank. The formula is derived by using approach of fifth order of KdV equation and a quantity called as *maximal temporal amplitude* (*MTA*). Results show that the increase of maximum amplitude of bichromatic wave has order $O\left(\frac{q}{\kappa_{lin}}\right)^2$ with q and κ_{lin}, respectively, express amplitude and frequency of the envelope of bichromatic wave. The obtained formula is verified with the experimental results and numerical software HUBRIS. Based on the comparison result it is obtained that the increase of the maximum amplitude of bichromatic wave calculated with approach of fifth order of KdV equation is suitable with the experimental results and also numerical software HUBRIS. For the future research, the investigation carried out for the KdV equation on the solution of higher order.

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