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THE LOCATING-CHROMATIC NUMBER OF NON-HOMOGENEOUS AMALGAMATION OF STARS

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Abstract

Let G be a connected graph and c be a proper coloring of G. The color code $c_{\Pi}(v)$ of vertex v in G is the ordered k-tuple distance v to every color classes C_i for $i \in [1, k]$. If all distinct vertices of G have distinct color codes, then c is called a locating-coloring of G. The locating-chromatic number of graph G, denoted by $\chi_L(G)$ is the smallest k such that G has a locating-coloring with k colors. In this paper, we discuss the locating-chromatic number of non-homogeneous amalgamation of stars.

1. Introduction

Research topics about the locating-chromatic number of a graph have grown rapidly since its introduction in 2002 by Chartrand et al. [8]. This concept is combined from the graph partition dimension and graph coloring.

Specially for tree, Chartrand et al. [9] determined the locating-chromatic

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numbers of some particular trees. Asmiati et al. [2, 3] obtained the locating-chromatic number of firecracker graphs and amalgamation of stars, respectively. Next, Welyyanti et al. [10] discussed the locating-chromatic number of complete *n*-arry tree.

In general graphs, Asmiati and Baskoro [1] characterized graph containing cycle with locating-chromatic number three. Behtoei and Omoomi [4-6] determined the locating-chromatic number of Kneser graphs, join of the graphs, and Cartesian product of graphs, respectively. Next, Baskoro and Purwasih [7] discussed the locating-chromatic number of corona product of graphs.

Let $G = (V, E)$ be a connected graph and c be a proper k-coloring of G using the colors $1, 2, ..., k$ for some positive integer k. Let $\Pi = \{C_1, C_2, ..., C_k\}$ be a partition of $V(G)$ which is induced by coloring c. The *color code* $c_\Pi(v)$ of vertex v in G is the ordered k-tuple $(d(v, C_1), ..., d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x) | x \in C_i\}$ for $i \in [1, k]$. If all distinct vertices of G have distinct color codes, then c is called a *locating k-coloring* of G. The locating-chromatic number, denoted by $\chi_L(G)$ is the smallest k such that G has a locating k-coloring.

Let $G_i = K_{1, n_i}$ be a star graph for every $i \in [1, k]$ and $n_i \geq 1$. Non-homogeneous amalgamation of stars, denoted by $S_{k, (n_1, n_2, ..., n_k)}$, $k \geq 2$ is a graph formed by identifying a leaf of G_i for every $i \in [1, k]$. We call the identified vertex as the *center* of $S_{k, (n_1, n_2, ..., n_k)}$, denoted by x. The vertices of distance 1 from the center as the *intermediate vertices*, are denoted by l_i, for $i \in [1, k]$. We denote the *j*th leaf of the intermediate vertex l_i by l_{ij} for $j \in [1, n_j - 1]$. In particular, when $n_i = m$, $m \geq 1$ for all i, we denote the homogeneous amalgamation of k isomorphic stars $K_{1, m}$ by $S_{k, m}$.

The locating-chromatic number of homogeneous amalgamation of stars and its monotonicity property have been studied by Asmiati et al. [3].

Motivated by this, in this paper, we determine the locating-chromatic number of non-homogeneous amalgamation of stars. These results are generalization of results in [3].

The following basic theorem was proved by Chartrand et al. in [8]. The set of neighbours of a vertex v in G is denoted by $N(v)$.

Theorem 1.1. *Let c be a locating-coloring in a connected graph G. If u and v are distinct vertices of G such that $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then $c(u) \neq c(v)$. In particular, if u and v are non-adjacent vertices of G such that $N(u) = N(v)$, then $c(u) \neq c(v)$.*

Corollary 1.1. *If G is a connected graph containing a vertex adjacent to k leaves of G, then $\chi_L(G) \geq k + 1$.*

2. Main Results

In this section, first we give some basic lemmas about the locating-chromatic number of non-homogeneous amalgamation of stars. Let $n_{\max} = \max\{n_1, n_2, ..., n_k\}$.

Lemma 2.1. *Let c be a proper $n_{\max}$-coloring of $S_{k,(n_1, n_2, ..., n_k)}$, where $k \geq 2$, $n_i \geq 1$, and $i \in [1, k]$. The coloring c is a locating-coloring if and only if $c(l_i) = c(l_j)$, $i \neq j$ implies $\{c(l_{is}) | s = 1, 2, ..., n_i - 1\}$ and $\{c(l_{js}) | s = 1, 2, ..., n_j - 1\}$ are distinct.*

Proof. Let $P = \{c(l_{is}) | s = 1, 2, ..., n_i - 1\}$ and $Q = \{c(l_{js}) | s = 1, 2, ..., n_j - 1\}$. Let c be a proper $n_{\max}$-coloring of $S_{k,(n_1, n_2, ..., n_k)}$, where $k \geq 2$, $n_i \geq 1$, and $c(l_i) = c(l_j)$ for some $i \neq n$. Suppose that $P = Q$. Because $d(l_i, u) = d(l_j, u)$ for every $u \in V \backslash \{\{l_{is} | s = 1, 2, ..., n_i - 1\} \cup \{l_{js} | s = 1, 2, ..., n_j - 1\}\}$, then the color codes of l_i and l_j will be the same. So c is not a locating-coloring, a contradiction. Therefore, $P \neq Q$.

Let Π be a partition of $V(G)$ into color classes with $|\Pi| \geq n_{\max}$. Consider $c(l_i) = c(l_j)$, $i \neq j$. Since $P \neq Q$, there are color x and color y such that $(x \in P,\ x \notin Q)$ or $(y \in P,\ y \notin Q)$. We will show that color codes for every $v \in V(S_{k,(n_1,n_2,\ldots,n_k)})$ are unique.

- Clearly, $c_\Pi(l_i) \neq c_\Pi(l_j)$ because their color codes differ in the xth-ordinate and yth-ordinate.

- If $c(l_{kt}) = c(l_{ls})$ for some $l_k \neq l_l$.

We divide into two cases:

Case 1. If $c(l_k) = c(l_l)$, then by the premise of this theorem, $P \neq Q$. So $c_\Pi(l_{kt}) \neq c_\Pi(l_{ls})$.

Case 2. Let $c(l_k) = r_1$ and $c(l_l) = r_2$ with $r_1 \neq r_2$. Then $c_\Pi(l_{kt}) \neq c_\Pi(l_{ls})$ because their color codes are different at least in the r_1 th-ordinate and r_2 th-ordinate.

- If $c(x) = c(l_{is})$, then color code of $c_\Pi(x)$ contains at least two components of value 1, whereas $c_\Pi(l_{is})$ contains exactly one component of value 1. Thus, $c_\Pi(x) \neq c_\Pi(l_{is})$.

From all the above cases, we see that the color code for each vertex in $S_{k,(n_1,n_2,\ldots,n_k)}$ is unique, thus c is a locating-coloring. □

Next, we will determine locating-chromatic number of non-homogeneous amalgamation of stars $S_{k,(n_1,n_2,\ldots,n_k)}$. Recall definition of $S_{k,(n_1,n_2,\ldots,n_k)}$, $\bigcup_{i=1}^{k} K_{1,n_i} = \bigcup_{j=0}^{n_{\max}-1} G_j$, where $G_j = t_j K_{1,n_{\max}-j}$ for some t (value t may be 0). Therefore, G_j induce subgraph of homogeneous amalgamation of stars $S_{t,n_{\max}-j}$, $t \neq 0$.

Lemma 2.2. *If c is a locating $(n_{\max} + a)$-coloring of $S_{k,(n_1, n_2, \ldots, n_k)}$ for $k \geq 2$, $n_i \geq 1$, and* $I(a) = \sum_{i=1}^{n_{\max}} (n_{\max} + a - 1)\binom{n_{\max} + a - 1}{n_{\max} - i}$, $a \geq 0$, $a \in Z$, *then $k \leq I(a)$.*

Proof. Let c be a locating $(n_{\max} + a)$-coloring of $S_{k,(n_1, n_2, \ldots, n_k)}$. By Lemma 2.1, maximum value t for subgraph of homogeneous amalgamation of stars $s_{t, n_{\max} - j}$ for some j, is $(n_{\max} + a - 1)\binom{n_{\max} + a - 1}{n_{\max} - j - 1}$. Because $j = 0, 1, 2, \ldots, n_{\max} - 1$ and $i = j + 1$, then the maximum number of k is $\sum_{i=1}^{n_{\max}} (n_{\max} + a - 1)\binom{n_{\max} + a - 1}{n_{\max} - i} = I(a)$. As a result, $k \leq I(a)$. □

Before we discuss about the locating-chromatic number of non-homogeneous amalgamation of stars $S_{k,(n_1, n_2, \ldots, n_k)}$, we recall to construct the upper bound of the locating-chromatic number of homogeneous amalgamation of stars was determined by Asmiati et al. [3].

The first result. We got that $\chi_L(S_{k,m}) = m$ for $2 \leq k \leq m - 1$, $m \geq 3$. To construct the upper bound of $\chi_L(S_{k,m})$ for $2 \leq k \leq m - 1$, we can assign $c(x) = 1$ and $c(l_i) = i + 1$ for $i = 1, 2, \ldots, k$. To make sure that the leaves will have distinct color code, we assign $\{l_{ij} \mid j = 1, 2, \ldots, m - 1\}$ by $\{1, 2, \ldots, m\} \backslash \{i + 1\}$ for any i. By Lemma 2.1, these colors are a locating-coloring.

The second result. Let $H(a) = (m + a - 1)\binom{m + a - 1}{m - 1}$, $m \geq 2$. Then $\chi_L(S_{k,m}) = m + a$ for $H(a - 1) < k \leq H(a)$, $a \geq 1$. To determine the upper bound of $\chi_L(S_{k,m})$ for $H(a - 1) < k \leq H(a)$, $a \geq 1$. We assign $c(x) = 1$; $c(l_i)$ by $2, 3, \ldots, m + a$ in such a way that the number of the intermediate vertices receiving the same color t does not exceed $\binom{m+a-1}{m-1}$. If $c(l_i) = c(l_n)$, $i \neq n$, then $\{c(l_{ij}) \mid j = 1, 2, ..., m-1\} \neq \{c(l_{nj}) \mid j = 1, 2, ..., m-1\}$. So, by Lemma 2.1, c is a locating-coloring.

Now, we will discuss about the locating-chromatic number of non-homogeneous of amalgamation of stars.

Theorem 2.3. *Let* $I(a) = \sum_{i=1}^{n_{\max}} (n_{\max} + a - 1)\binom{n_{\max} + a - 1}{n_{\max} - i}$, $a \geq 0$, $a \in Z$. *The locating-chromatic number of non-homogeneous amalgamation of stars* $S_{k,(n_1, n_2, ..., n_k)}$, *where* $k \geq 2$, $n_i \geq 1$, *and* $|A| \leq n_{\max} + a - 1$ *is*

$$\chi_L(S_{k,(n_1, n_2, ..., n_k)}) = \begin{cases} n_{\max} & \text{for } 2 \leq k \leq I(0); \\ n_{\max} + a & \text{for } I(a-1) < k \leq I(a),\ a \geq 1; \end{cases}$$

whereas, if $|A| > n_{\max} + a - 1$, *then* $\chi_L(S_{k,(n_1, n_2, ..., n_k)}) = |A| + 1$.

Proof. First, we determine the trivial lower bound for $2 \leq k \leq I(0)$. Because each vertex l_i is adjacent to $(n_{\max} - 1)$ leaves for $i \in [1, k]$, then by Corollary 1.1, $\chi_L(S_{k,(n_1, n_2, ..., n_k)}) \geq n_{\max}$.

Next, we determine the upper bound of $S_{k,(n_1, n_2, ..., n_k)}$ for $2 \leq k \leq I(0)$. Let c be a $n_{\max}$-coloring of $S_{k,(n_1, n_2, ..., n_k)}$. Because non-homogeneous amalgamation of stars $S_{k,(n_1, n_2, ..., n_k)}$ contains subgraph of homogeneous amalgamation of stars $S_{t, n_{\max} - j}$ for some $t \neq 0$, then color vertices of $S_{t, n_{\max} - j}$ for every $j \in [0, n_{\max} - 2]$ follow construction the upper bound of the first results in [3]. Next, for $j = n_{\max} - 1$, color the intermediate vertices by $2, 3, ..., n_{\max} - 1$, respectively. Therefore, c is a locating-coloring. So, $\chi_L(S_{k,(n_1, n_2, ..., n_k)}) \leq n_{\max}$ for $2 \leq k \leq I(0)$.

We improve the upper bound for $I(a-1) \leq k \leq I(a)$, $a \geq 1$. Because $k > I(a-1)$, then by Lemma 2.2, $\chi_L(S_{k,(n_1, n_2, ..., n_k)}) \geq n_{\max} + a$. On the other hand, if $k > I(a)$, then by Lemma 2.2, $\chi_L(S_{k,(n_1, n_2, ..., n_k)}) \geq n_{\max} + a + 1$. As a result, $\chi_L(S_{k,(n_1, n_2, ..., n_k)}) \geq n_{\max} + a$, if $I(a-1) \leq k \leq I(a)$, $a \geq 1$.

Next, we shall show the upper bound of $S_{k,(n_1, n_2, ..., n_k)}$ for $I(a-1) \leq k \leq I(a)$, $a \geq 1$. The vertices $S_{t, n_{\max} - j}$ for every $j \in [0, n_{\max} - 2]$, we give color follow construction the upper bound of the second result in [3] using $(n_{\max} + a)$ colors. Otherwise, for $j = n_{\max} - 1$, the vertices l_i are colored by $2, 3, ..., n_{\max} + a - 1$, respectively. Thus, $\chi_L(S_{k,(n_1, n_2, ..., n_k)}) \leq n_{\max} + a$ for $I(a-1) \leq k \leq I(a)$, $a \geq 1$.

Let $|A| > n_{\max} + a - 1$. By Corollary 1.1, we obtain $\chi_L(S_{k,(n_1, n_2, ..., n_k)}) \geq |A| + 1$, since x is adjacent to $|A|$ leaves. To determine the upper bound, the vertices in A are colored by $2, 3, ..., |A| + 1$, respectively, and the colors for the other vertices alike two cases before it. So, $\chi_L(S_{k,(n_1, n_2, ..., n_k)}) \leq |A| + 1$. □

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