Far East Journal of Mathematical Sciences (FJMS)

Published Online: November 2014
Available online at http://pphmj.com/journals/fjms.htm
Volume 93, Number 1, 2014, Pages 1-21

FIBONACCI NUMBERS OF χ^2 OVER *p*-METRIC SPACES DEFINED BY SEQUENCE OF MODULUS

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Abstract

In this paper, we introduce moduli Fibonacci numbers of χ_M^{2F} and Λ_M^{2F} sequence spaces over *p*-metric spaces defined by sequences of modulus functions and also discuss some of the general properties of these spaces.

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

Received: June 26, 2014; Revised: August 19, 2014; Accepted: August 25, 2014
2010 Mathematics Subject Classification: 40A05, 40C05, 40D05.
Keywords and phrases: analytic sequence, double sequences, χ^2 space, Fibonacci number, Musielak-modulus function, *p*-metric space.
Communicated by B. C. Tripathy

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then w^2 is a linear space under the coordinate-wise addition and scalar multiplication.

Some initial works on double sequence spaces are found in Bromwich [2]. Later on, they were investigated by Hardy [3], Moricz [7], Moricz and Rhoades [8], Basarir and Solancan [1], Tripathy [11], Turkmenoglu [12], and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \{(x_{mn}) \in w^2 : \sup_{m, n \in N} | x_{mn} |^{t_{mn}} < \infty\},$$

$$\mathcal{C}_p(t) := \{(x_{mn}) \in w^2 : p - \lim_{m, n \to \infty} | x_{mn} - l |^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C}\},$$

$$\mathcal{C}_{0p}(t) := \{(x_{mn}) \in w^2 : p - \lim_{m, n \to \infty} | x_{mn} |^{t_{mn}} = 1\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} | x_{mn} |^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m, n \to \infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$, the above classes of sequences are denoted by $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Çolak [14, 15] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α-, β-, γ-duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Zeltser [16] in her Ph.D.

Thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [17] and Tripathy [11] have independently introduced the statistical convergence and statistical Cauchy for double sequences and established the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Başar [20] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums is in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(\vartheta)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Başar and Sever [21] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Subramanian and Misra [22] have studied the space $\chi_M^2(p, q, u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [6] as an extension of the definition of strongly Cesàro summable sequences. Cannor [23] further extended this definition to a definition of strong A-summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A-summability, strong A-summability with respect to a modulus, and A-statistical convergence. In [24], the notion of convergence of double sequences was presented by Pringsheim. Also, in [25-27], the four dimensional matrix transformation

$$(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$$

was studied extensively by Robison and Hamilton.

We shall use the following inequality in the sequel of the paper. For $a, b \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called *convergent* if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}$ $(m, n \in \mathbb{N})$.

A sequence $x = (x_{mn})$ is said to be *double analytic* if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called *double gai sequence* if

$$((m+n)!|x_{mn}|)^{1/m+n} \to 0 \text{ as } m, n \to \infty.$$

The double gai sequences will be denoted by χ^2. Let $\phi = \{$all finite sequences$\}$.

Consider a double sequence $x = (x_{ij})$. The (m, n)th section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only nonzero term is a $\frac{1}{(i+j)!}$ in the (i, j)th place for each $i, j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have *AK property* if $(\Im_{mn})$ is a Schauder basis for X or equivalently, $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

By Kamthan and Gupta [13], consider the kernel $p(t)$ associated with an Orlicz function $M(t)$ and let

$$q(s) = \sup\{t;\ p(t) \leq s\}.$$

The q possesses the same properties as the function p. Suppose now

$$N(x) = \int_0^x q(s)ds.$$

Then N is an Orlicz function. Then functions M and N are called *mutually complementary Orlicz functions*.

Let M and Φ are mutually complementary modulus functions. Then we have:

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y) \text{ (Young's inequality) (one may refer to [13]).} \quad (1.2)$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \quad (1.3)$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u). \quad (1.4)$$

Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}.$$

The space ℓ_M with the norm

$$\| x \| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an *Orlicz sequence space*. For $M(t) = t^p$ $(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a *Musielak-modulus function*. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = \sup\{| v | u - (f_{mn})(u) : u \geq 0\}, \; m, \; n = 1, 2, \ldots$$

is called the *complementary function* of a Musielak-modulus function *f*. For a given Musielak-modulus function *f*, the Musielak-modulus sequence space t_f is defined as follows:

$$t_f = \{x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \to 0 \text{ as } m, n \to \infty\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, \; x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sup_{mn}\left\{\inf\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)\right) \leq 1\right\}.$$

2. Definition and Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on X satisfying the following conditions:

(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ if and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly dependent,

(ii) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ is invariant under permutation,

(iii) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = \|(d_1(x_n, 0), \ldots, d_n(x_1, 0))\|_p$,

(iv) $\|(\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = |\alpha| \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R}$,

(v) $\|(d_1(x_1 + x_1', 0), d_2(x_2, 0), \ldots, d_n(x_n, 0))\|_p = \|(d_1(x_1, 0), d_2(x_2, 0), \ldots, d_n(x_n, 0))\|_p + \|(d_1(x_1', 0), d_2(x_2, 0), \ldots, d_n(x_n, 0))\|_p$,

(vi) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \ldots, x_n)^p + d_Y(y_1, y_2, \ldots, y_n)^p)^{1/p}$ for $1 \leq p < \infty$; (or)

(vii) $d((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)) := \sup\{d_X(x_1, x_2, ..., x_n), d_Y(y_1, y_2, ..., y_n)\}$,

for $x_1, x_2, ..., x_n \in X$, $y_1, y_2, ..., y_n \in Y$ is called the *p product metric* of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

The pair $(X, \|.,.\|)$ is called 2-*metric space*. Standard examples of 2-metric space are $\mathbb{R}^2$ equipped with the following conditions:

(1) $\| d(x, 0), d(y, 0) \| = | d(x_1, 0), d(y_2, 0) - d(x_2, 0), d(y_1, 0) |$, where $d(x, 0) = (d(x_1, 0), d(x_2, 0)), d(y, 0) = (d(y_1, 0), d(y_2, 0))$,

(2) $\| d(x, 0), d(y, 0) \| =$ the area of the triangle having vertices 0, $d(x, 0)$ and $d(y, 0)$.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\| (d_1(x_1, 0), ..., d_n(x_n, 0)) \|_E$$

$$= \sup(| \det(d_{mn}(x_{mn}, 0)) |)$$

$$= \sup\left(\begin{vmatrix} d_{11}(x_{11}, 0) & d_{12}(x_{12}, 0) & \cdots & d_{1n}(x_{1n}, 0) \\ d_{21}(x_{21}, 0) & d_{22}(x_{22}, 0) & \cdots & d_{2n}(x_{1n}, 0) \\ \vdots & & & \\ d_{n1}(x_{n1}, 0) & d_{n2}(x_{n2}, 0) & \cdots & d_{nn}(x_{nn}, 0) \end{vmatrix}\right),$$

where $x_i = (x_{i1}, ..., x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, ..., n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be *complete* with respect to the p-metric. Any complete p-metric space is said to be *p-complete metric space*.

Definition 2.1. Let λ be a sequence space. Then λ is called

(i) *Solid* (or *normal*) of $(\alpha_{mn}x_{mn}) \in \lambda$ whenever $(x_{mn}) \in \lambda$ for all sequences (α_{mn}) of scalars with $|\alpha_{mn}| \leq 1$.

(ii) *Monotone* if provided λ contains the canonical preimages of all its step spaces.

(iii) *Perfect* if $\lambda = \lambda^{\alpha\alpha}$, see [13].

Definition 2.2. A sequence space E is said to be *convergence free* if $(y_{mn}) \in E$ whenever $(x_{mn}) \in E$ and $x_{mn} = 0$ implies $(y_{mn}) = 0$.

Definition 2.3. A sequence space E is said to be a *sequence algebra* if $(x_{mn} \circ y_{mn}) \in E$ whenever $(x_{mn}) \in E$, $(y_{mn}) \in E$.

Definition 2.4. A sequence space E is said to be *symmetric* if $(x_{\pi(mn)}) \in E$ whenever $(x_{mn}) \in E$, where π is a permutation on $\mathbb{N} \times \mathbb{N}$.

Definition 2.5. A map h defined on a domain $D \subset X$, i.e., $h : D \subset X \to \mathbb{R}$ is said to satisfy *Lipschitz condition* if $|h(x) - h(y)| \leq K|x - y|$, where K is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $h \in (D, K)$.

Definition 2.6. A convergence field of convergence is a set

$$F = \{x = (x_{mn}) \in \Lambda^2 : \text{there exists} \lim x_{mn} \in \mathbb{R}\}.$$

The convergence field F is a closed linear subspace of Λ^2 with respect to the supremum metric $F = \Lambda^2 \bigcap c^2$.

Define a function $h : F \to \mathbb{R}$ is a Lipschitz function.

Definition 2.7. Let $A = (a_{k,\ell}^{mn})$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax, where the k, ℓth term to Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$$

such transformation is said to be *nonnegative* if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations are the following four dimensional analog of regularity for double sequences in which they both added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded.

Let λ and μ be two sequence spaces and $A = (a_{k,\ell}^{mn})$ be a four dimensional infinite matrix of real numbers $(a_{k,\ell}^{mn})$, where $m, n, k, \ell \in \mathbb{N}$. Then we say A defines a matrix mapping from λ into μ and we denote it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_{mn}) \in \lambda$, the sequence $Ax = \{(Ax)_{k\ell}\}$, the A-transform of x, is in μ, where

$$(Ax)_{k\ell} = A = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k,\ell}^{mn} x_{mn} \ (k, \ell \in \mathbb{N}). \tag{2.1}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series converges for each $k, \ell \in \lambda$. A sequence x is said to be *A-summable* to α if Ax converges to α which is called as the *A-limit* of x.

Lemma 2.8 [See 32]. *Matrix $A = (a_{k,\ell}^{mn})$ is regular if and only if the following three conditions hold*:

(1) *There exists $M > 0$ such that for every $k, \ell = 1, 2, \ldots,$ the following inequality holds*: $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} | a_{k\ell}^{mn} | \leq M$,

(2) $\lim_{k,\ell \to \infty} a_{k\ell}^{mn} = 0$ *for every* $k, \ell = 1, 2, \ldots,$

(3) $\lim_{k,\ell \to \infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} = 1.$

Let (q_{mn}) be a sequence of positive numbers and

$$Q_{k\ell} = \sum_{m=0}^{k} \sum_{n=0}^{\ell} q_{mn} \ (k, \ell \in \mathbb{N}). \tag{2.2}$$

Then the matrix $R^q = (r_{k\ell}^{mn})^q$ of the Riesz mean is given by

$$(r_{k\ell}^{mn})^q = \begin{cases} \frac{q_{mn}}{Q_{k\ell}}, & \text{if } 0 \le m, n \le k, \ell, \\ 0, & \text{if } (m, n) > k\ell. \end{cases} \tag{2.3}$$

The Fibonacci numbers are the sequence of numbers $f_{k\ell}^{mn}$ $(k, \ell, m, n \in \mathbb{N})$ defined by the linear recurrence equations $f_{00} = 1$ and $f_{11} = 1$, $f_{mn} = f_{m-1n-1} + f_{m-2n-2}$; $m, n \ge 2$. Fibonacci numbers have many interesting properties and applications in sciences and technology. Also, some basic properties of Fibonacci numbers are given as follows:

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} f_{mn} = f_{m+2n+2} - 1; \; m, n \ge 1,$$

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} f_{mn}^2 = f_{mn} f_{m+1n+1}; \; m, n \ge 1,$$

$$\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{1}{f_{k\ell}^{mn}} \text{ converges.}$$

In this paper, we define the Fibonacci matrix $F = (f_{k\ell}^{mn})_{m,n=1}^{\infty}$, which differs from existing Fibonacci matrix by using Fibonacci numbers f_{mn} and introduce some new sequence spaces χ^2 and Λ^2. Now, we define the Fibonacci matrix $F = (f_{k\ell}^{mn})_{m,n=1}^{\infty}$, by

$$(f_{k\ell}^{mn}) = \begin{cases} \frac{f_{k\ell}}{f_{(m+2)(n+2)} - 1}, & \text{if } 0 \le k \le m; 0 \le \ell \le n, \\ 0, & \text{if } (m, n) > k\ell, \end{cases}$$

that is,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} & 0 & 0 & \cdots \\ \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is obvious that the four dimensional infinite matrix F is a triangular matrix. Also, it follows from Lemma 2.8 that the method F is regular.

Let M be a Musielak-modulus function. Now, we introduce the following sequence spaces based on the four dimensional infinite matrix F:

$$[\Lambda_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$$

$$= F_\eta(x) = \sup\left\{\sum_{m=1}^\infty \sum_{n=1}^\infty\right.$$

$$\left.\cdot [M(f_{k\ell}^{mn} | x_{mn} |^{1/m+n}, \| (d(x_1), d(x_2), ..., d(x_{n-1}) \|_p))] < \infty\right\}$$

$$= \sup\left\{\frac{1}{f_{(m+2)(n+2)} - 1} \sum_{m=1}^\infty \sum_{n=1}^\infty\right.$$

$$\left.\cdot [M(f_{k\ell}^{mn} | x_{mn} |^{1/m+n}, \| (d(x_1), d(x_2), ..., d(x_{n-1}) \|_p))] < \infty\right\}$$

$$(k, \ell \in \mathbb{N}).$$

Let us consider $[\Lambda_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$ is a metric space with the metric

$$d(x, y) = \sup\{M(F_\eta(x) - F_\eta(y)) : m, n = 1, 2, 3, ...\}, \tag{2.4}$$

$$[\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$$

$$= F_\mu(x) = \lim_{m, n \to \infty}\left\{\sum_{m=1}^\infty \sum_{n=1}^\infty\right.$$

$$\cdot \left[M\left(f_{k\ell}^{mn}((m+n)!|x_{mn}|)^{1/m+n}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p\right)\right] = 0\Bigg\}$$

$$= \lim_{m, n \to \infty}\left\{\frac{1}{f_{(m+2)(n+2)} - 1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\right.$$

$$\left.\cdot \left[M\left(f_{k\ell}^{mn}((m+n)!|x_{mn}|)^{1/m+n}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p\right)\right] = 0\right\}$$

$(k, \ell \in \mathbb{N})$.

Let us consider $[\chi_M^{2F}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p]$ be the metric space with the metric

$$d(x, y) = \sup\{M(F_\mu(x) - F_\mu(y)) : m, n = 1, 2, 3, \ldots\}. \tag{2.5}$$

3. Main Results

Theorem 3.1. *The classes of sequences*

$$[\chi_M^{2F}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p] \text{ and}$$

$$[\Lambda_M^{2F}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p]$$

of moduli Fibonacci $F = (f_{k\ell}^{mn})$, *are linear spaces.*

Proof. It can be established using standard technique.

Theorem 3.2. *The spaces* $[\chi_M^{2F}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p]$ *and* $[\Lambda_M^{2F}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p]$ *of moduli Fibonacci* $F = (f_{k\ell}^{mn})$, *are solid and monotone.*

Proof. We shall prove the result for

$$[\chi_M^{2F}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p].$$

Let $x_{mn} \in [\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$. Then

$$\{\lim_{m,n\to\infty}[M(f_{k\ell}^{mn}((m+n)!|x_{mn}|)^{1/m+n},$$

$$\| (d(x_1), d(x_2), ..., d(x_{n-1}) \|_p))] = 0\}. \tag{3.1}$$

Let (α_{mn}) be a sequence of scalars with $|\alpha_{mn}|^{1/m+n} \leq 1$ for all $m, n \in \mathbb{N}$. Therefore, the equation from (3.1) and the following inequality:

$$\{[M(f_{k\ell}^{mn}((m+n)!|\alpha x_{mn}|)^{1/m+n}, \| (d(x_1), d(x_2), ..., d(x_{n-1}) \|_p))]\}$$

$$\leq \{|\alpha_{mn}|^{1/m+n}[M(f_{k\ell}^{mn}((m+n)!|x_{mn}|)^{1/m+n},$$

$$\| (d(x_1), d(x_2), ..., d(x_{n-1}) \|_p))]\}$$

$$\leq \{[M(f_{k\ell}^{mn}((m+n)!|x_{mn}|)^{1/m+n}, \| (d(x_1), d(x_2), ..., d(x_{n-1}) \|_p))]\}$$

for all $m, n \in \mathbb{N}$. Therefore, $[\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$ is a sequence space. If $[\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$ is solid, then

$$[\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$$

is monotone. Hence, the space $[\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$ is monotone.

Similarly, the result is true for $[\Lambda_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$.

Theorem 3.3. *The space* $[\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$ *is a sequence algebra.*

Proof. Let $(x_{mn}), (y_{mn}) \in [\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$. Then

$$\lim_{m,n\to\infty}\left\{\frac{1}{f_{(m+2)(n+2)}-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\right.$$

$$\left.\cdot\left[M\left(f_{k\ell}^{mn}((m+n)!\,|\,x_{mn}\,|)^{1/m+n},\ \|\,(d(x_1),\ d(x_2),\ \ldots,\ d(x_{n-1})\,\|_p)\right)\right]=0\right\},$$

$$\lim_{m,n\to\infty}\left\{\frac{1}{f_{(m+2)(n+2)}-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\right.$$

$$\left.\cdot\left[M\left(f_{k\ell}^{mn}((m+n)!\,|\,y_{mn}\,|)^{1/m+n},\ \|\,(d(x_1),\ d(x_2),\ \ldots,\ d(x_{n-1})\,\|_p)\right)\right]=0\right\}.$$

Then we have

$$\lim_{m,n\to\infty}\left\{\frac{1}{f_{(m+2)(n+2)}-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\right.$$

$$\left.\cdot\left[M\left(f_{k\ell}^{mn}((m+n)!\,|\,x_{mn}y_{mn}\,|)^{1/m+n},\ \|\,(d(x_1),\ d(x_2),\ \ldots,\ d(x_{n-1})\,\|_p)\right)\right]=0\right\}.$$

Thus, $(x_{mn}y_{mn})\in[\chi_M^{2F},\ \|\,(d(x_1),\ d(x_2),\ \ldots,\ d(x_{n-1}))\,\|_p]$ is a sequence algebra.

Theorem 3.4. *The space* $[\chi_M^{2F},\ \|\,(d(x_1),\ d(x_2),\ \ldots,\ d(x_{n-1}))\,\|_p]$ *is not convergence free in general.*

Proof. Here we give a counterexample. Let $M(x)=x^3$ for all $x\in[0,\infty)$. Consider the sequences (x_{mn}) and (y_{mn}) are defined by

$$x_{mn}=\frac{1}{(m+n)!\,(mn)^{m+n}}\quad\text{and}\quad y_{mn}=(m+n)!\,(mn)^{m+n}\quad\text{for all}\quad m,n\in\mathbb{N}.$$

Then $(x_{mn})\in[\chi_M^{2F},\ \|\,(d(x_1),\ d(x_2),\ \ldots,\ d(x_{n-1}))\,\|_p]$, but

$$(y_{mn})\notin[\chi_M^{2F},\ \|\,(d(x_1),\ d(x_2),\ \ldots,\ d(x_{n-1}))\,\|_p].$$

Hence, the space $[\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$ is not convergence free.

Theorem 3.5. *The space* $[\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$ *is not symmetric*.

Proof. Let $M(x) = x$ for all $x \in [0, \infty)$. If

$$(x_{mn}) = \begin{cases} \frac{1^{m+n}}{(m+n)!}, & \text{if } m, n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $(x_{mn}) \in [\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$. Let $K \subset \mathbb{N} \times \mathbb{N}$; $\phi : K \to A$ and $\psi : \mathbb{N} \times \mathbb{N} - K \to \mathbb{N} \times \mathbb{N} - A$ be bijections. Then the map $\pi : \mathbb{N} \times \mathbb{N}$ defined by

$$(\pi_{mn}) = \begin{cases} \phi(mn), & \text{for } m, n \in K, \\ \psi(mn), & \text{otherwise,} \end{cases}$$

is a permutation on $\mathbb{N} \times \mathbb{N}$, but

$$x_{\pi(mn)} \notin [\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p].$$

Hence, $[\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$ is not symmetric.

Theorem 3.6. *The spaces* $[\chi_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p] \subset [\Lambda_M^{2F}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p]$ *for moduli Fibonacci* $F = (f_{k\ell}^{mn})$ *and the inclusion are proper*.

Proof. It is easy to prove. Therefore, omit the proof.

Theorem 3.7. *The function* $h : [\Lambda^{2F}_{\chi_M^{2F}}, \| (d(x_1), d(x_2), ..., d(x_{n-1})) \|_p] \to \mathbb{R}$ *is the Lipschitz function*, *where*

$$[\Lambda^2_M, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p]$$

$$= [\chi^{2F}_M, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p]$$

$$\cap [\Lambda^2_M, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p],$$

and hence they are uniformly continuous.

Proof. Let $x, y \in [\Lambda^{2F}_{\chi^{2F}_M}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p]$, $x \neq y$. Then the sets

$$A_x = \{[M(f^{mn}_{k\ell}((m+n)!|x_{mn} - h(x)|)^{1/m+n},$$
$$\|(d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p))] \geq d(x, y)\},$$

$$A_y = \{[M(f^{mn}_{k\ell}((m+n)!|y_{mn} - h(x)|)^{1/m+n},$$
$$\|(d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p))] \geq d(x, y)\}.$$

Thus, the sets

$$B_x = \{[M(f^{mn}_{k\ell}((m+n)!|x_{mn} - h(x)|)^{1/m+n},$$
$$\|(d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p))] < d(x, y)\}$$
$$\in [\Lambda^{2F}_{\chi^{2F}_M}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p],$$

$$B_y = \{[M(f^{mn}_{k\ell}((m+n)!|y_{mn} - h(x)|)^{1/m+n},$$
$$\|(d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p))] < d(x, y)\}$$
$$\in [\Lambda^{2F}_{\chi^{2F}_M}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p].$$

Here also $B = B_x \cap B_y \in [\Lambda^{2F}_{\chi^{2F}_M}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p]$, so that $B \neq \phi$. Now taking $m, n \in B$,

$$\{[M(f_{k\ell}^{mn}((m+n)!|h(x)-h(y)|)^{1/m+n}, \|(d(x_1), d(x_2), ..., d(x_{n-1})\|_p))]\}$$

$$\leq \{[M(f_{k\ell}^{mn}((m+n)!|h(x)-x_{mn}|)^{1/m+n}, \|(d(x_1), d(x_2), ..., d(x_{n-1})\|_p))]\}$$

$$+ \{[M(f_{k\ell}^{mn}((m+n)!|x_{mn}-y_{mn}|)^{1/m+n}, \|(d(x_1), d(x_2), ..., d(x_{n-1})\|_p))]\}$$

$$+ \{[M(f_{k\ell}^{mn}((m+n)!|y_{mn}-h(y)|)^{1/m+n}, \|(d(x_1), d(x_2), ..., d(x_{n-1})\|_p))]\}$$

$$\leq 3d(x, y).$$

Thus, h is a Lipschitz function.

Proposition 3.8. *If* $x, y \in [\Lambda^{2F}_{\chi^{2F}_M}, \|(d(x_1), d(x_2), ..., d(x_{n-1}))\|_p]$, *then* $(x \cdot y) \in [\Lambda^{2F}_{\chi^{2F}_M}, \|(d(x_1), d(x_2), ..., d(x_{n-1}))\|_p]$ *and* $h(xy) = h(x)h(y)$.

Proof. Let $\varepsilon > 0$. Then

$$B_x = \{[M(f_{k\ell}^{mn}((m+n)!|x_{mn}-h(x)|)^{1/m+n},$$

$$\|(d(x_1), d(x_2), ..., d(x_{n-1})\|_p))] < \varepsilon\}$$

$$\in [\Lambda^{2F}_{\chi^{2F}_M}, \|(d(x_1), d(x_2), ..., d(x_{n-1}))\|_p],$$

$$B_y = \{[M(f_{k\ell}^{mn}((m+n)!|y_{mn}-h(x)|)^{1/m+n},$$

$$\|(d(x_1), d(x_2), ..., d(x_{n-1})\|_p))] < \varepsilon\}$$

$$\in [\Lambda^{2F}_{\chi^{2F}_M}, \|(d(x_1), d(x_2), ..., d(x_{n-1}))\|_p].$$

Now,

$$\{[M(f_{k\ell}^{mn}((m+n)!|x_{mn}y_{mn}-h(x)h(y)|)^{1/m+n},$$

$$\|(d(x_1), d(x_2), ..., d(x_{n-1})\|_p))]\}$$

$$= \{[M(f_{k\ell}^{mn}((m+n)!\,|\,x_{mn}y_{mn} - x_{mn}h(y) + x_{mn}h(y) - h(x)h(y)\,|)^{1/m+n},$$

$$\|\,(d(x_1), d(x_2), ..., d(x_{n-1})\,\|_p))]\}$$

$$\{[M(f_{k\ell}^{mn}((m+n)!\,|\,x_{mn}\,||\,y_{mn} - h(y)\,|)^{1/m+n},$$

$$\|\,(d(x_1), d(x_2), ..., d(x_{n-1})\,\|_p))]\}$$

$$+ \{[M(f_{k\ell}^{mn}((m+n)!\,|\,h(y)\,||\,x_{mn} - h(x)\,|)^{1/m+n},$$

$$\|\,(d(x_1), d(x_2), ..., d(x_{n-1})\,\|_p))]\}.$$

As

$$[\Lambda^{2F}_{\chi^{2F}_M}, \|\,(d(x_1), d(x_2), ..., d(x_{n-1}))\,\|_p]$$

$$\subseteq [\Lambda^2_M, \|\,(d(x_1), d(x_2), ..., d(x_{n-1}))\,\|_p],$$

there exists an $L \in \mathbb{R}$ such that $|\,x_{mn}\,|^{1/m+n} < L$ and $|\,h(y)\,|^{1/m+n} < L.$ Therefore, using the above equation, we get

$$\{[M(f_{k\ell}^{mn}((m+n)!\,|\,x_{mn}y_{mn} - h(x)h(y)\,|)^{1/m+n},$$

$$\|\,(d(x_1), d(x_2), ..., d(x_{n-1})\,\|_p))]\}$$

$$\leq L\varepsilon + L\varepsilon = 2L\varepsilon$$

for all $m, n \in B_x \cap B_y \in [\Lambda^{2F}_{\chi^{2F}_M}, \|\,(d(x_1), d(x_2), ..., d(x_{n-1}))\,\|_p].$ Hence, $(x \cdot y) \in [\Lambda^{2F}_{\chi^{2F}_M}, \|\,(d(x_1), d(x_2), ..., d(x_{n-1}))\,\|_p]$ and $h(xy) = h(x)h(y).$

Acknowledgement

I wish to thank the referees for their several remarks and valuable suggestions that improved the presentation of the paper.

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