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ON A DISCRETE BALEEN WHALE MODEL

Chunxue Wu*, Liying Wang, Decun Zhang and Xiaobao Li

*School of Mathematics and Information Sciences
Yantai University
P. R. China
e-mail: wuchunxue1020@163.com

Institute of Applied Mathematics
Naval Aeronautical and Astronautical University
Yantai, Shandong 264001
P. R. China
e-mail: ytliyingwang@163.com

Abstract

In this paper, we investigate the oscillation behavior and the global stability character of all positive solutions of the discrete baleen whale model

$$N_{n+1} = (1-\mu)N_n + \mu N_{n-k}\left[1 + q\left(1 - \left(\frac{N_{n-k}}{\bar{k}}\right)^z\right)\right]_+ \quad \text{for } n = 0, 1, \ldots$$

and answer the Research Projects which were proposed by V. L. Kocić and G. Ladas.

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1. Introduction

Our aim in this paper is to study the oscillation and the global asymptotic stability of a discrete baleen whale model, which has been proposed by Kocić and Ladas in [1] as the open Research Projects.

Research Project 4.7.1. Study the oscillatory behavior of the following equation:

$$N_{n+1} = (1-\mu)N_n + \mu N_{n-k}\left[1 + q\left(1 - \left(\frac{N_{n-k}}{\bar{k}}\right)^z\right)\right]_+ \quad \text{for} \quad n = 0, 1, \ldots, \tag{1}$$

where k is a nonnegative integer, $[\cdot]_+ = \max\{\cdot, 0\}$ and

$$\bar{k},\ q,\ z \in (0, \infty), \quad \mu \in (0, 1). \tag{2}$$

Research Project 4.7.2. Obtain conditions for the global asymptotic stability of the equilibrium $\bar{k}$ of equation (1) relative to the interval $(0, N^*)$, where $N^* = \bar{k}\left(\frac{1+q}{q}\right)^{\frac{1}{z}}$.

It is easy to see that if (2) holds and the initial conditions satisfy (3),

$$N_{-k}, \ldots, N_{-1} \in [0, \infty) \quad \text{and} \quad N_0 \in (0, \infty), \tag{3}$$

then the solutions $\{N_n\}$ of equation (1) are positive.

Besides their theoretical interests, difference equations play an important role in economic sphere, mathematical biology. Many scholars have studied the difference equation with biological background, for examples [2-5].

In this work, we obtain a sufficient condition for every positive solution of equation (1) is global asymptotic stability under condition (2) and (3) by using the Lyapunov function method, and answer the above open problems.

2. Some Lemmas

Definition 2.1. Lyapunov functions.

Consider the equation

$$x_{n+1} = G(x_n). \tag{4}$$

Let $S \subset \mathbb{R}^{k+1}$ be contained in the $\operatorname{dom} G$. A function $V : \mathbb{R} \mapsto [0, \infty)$ is said to be a *Lyapunov function* of equation (1) on *S* if *V* is continuous and $V(G(x)) \leq V(x)$ for all $x \in S$.

Lemma 2.1. *Let $S \subset \mathbb{R}^{k+1}$ be a bounded open set such that $G(S) \subset S$. Assume that $\bar{x}$ is the only equilibrium point of equation* (4) *in S. Suppose that V is a Lyapunov function of equation* (4) *on S such that*

$$V(G(x)) < V(x) \ \textit{for all} \ x \in S \ \textit{with} \ x \neq \bar{x}.$$

Let E_0 be the set of all points on the boundary of S such that $V(G(x)) = G(x)$ and M_0 be the set of all points such that $V(x) = x$ for $x \in M_0$. Denote $M = M_0 \bigcup \{\bar{x}\}$. Then the following statements are true:

(a) *if $M = \{\bar{x}\}$, then $\bar{x}$ is globally asymptotically stable relative to S*;

(b) *if there is no solution $\{x_n\}$ of equation* (4) *such that $\inf_{z \in M_0} \{\| x_n - z \|\} \to 0$ when $n \to \infty$, then $\bar{x}$ is globally asymptotically stable relative to S.*

One can refer to Kocić and Ladas [1, P_{10}, Theorem 1.3.1].

Next, we consider the delay difference equations

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + (1 - A) F\left(\sum_{i=0}^{m} b_i x_{n-i}\right) \ \text{ for } \ n = 0, 1, \ldots, \tag{5}$$

where

$$a_0, \ldots, a_k, b_0, \ldots, b_m \in [0, \infty), \ r = \max\{k, m\}, \ A = \sum_{i=0}^{k} a_i < 1, \ \sum_{i=0}^{m} b_i = 1 \tag{6}$$

and

$$F \in C[[0, \infty), (0, \infty)], \ F \text{ has a unique positive fixed point } \bar{x}$$

and

$$(x - \bar{x})(F(x) - x) < 0 \quad \text{for} \quad 0 < x \neq \bar{x} \tag{7}$$

and

$$z_{n+1} = F(z_n) \quad \text{for} \quad n = 0, 1, \dots . \tag{8}$$

Now, we have the following lemma:

Lemma 2.2. *Assume that* (6) *and* (7) *hold and suppose that there exists a convex function v which is a Lyapunov function of the first-order difference equation* (8) *on* $(0, \infty)$ *such that* $V(F(x)) < V(x)$ *for* $x \neq \bar{x}$. *Then the positive equilibrium* $\bar{x}$ *of equation* (5) *is globally asymptotically stable*.

One can refer to Kocić and Ladas [1, P_{51}, Theorem 2.5.2].

Lemma 2.3. *Assume that* $p_i \in (0, \infty)$ *and* $k_i \in \{0, 1, \dots\}$ *with* $\sum_{i=1}^{m}(p_i + k_i) \neq 1$. *Let* $\{p_i(n)\}$ *be a sequence of positive numbers such that*

$$\liminf_{n \to \infty} p_i(n) \geq p_i \quad \text{for} \quad i = 1, \dots, m.$$

Suppose that the linear difference inequality

$$z_{n+1} - z_n + \sum_{i=1}^{m} p_i(n) z_{n-k_i} \leq 0 \quad \text{for} \quad n = 0, 1, \dots$$

has an eventually positive solution. *Then the difference equation*

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0 \quad \text{for} \quad n = 0, 1, \dots$$

has a positive solution.

One can refer to Kocić and Ladas [1, P_6, Theorem 1.2.2].

Lemma 2.4. *Assume that* $p \in \mathbb{R}$ *and k is a nonnegative integer*. *Then every solution of the difference equation*

$$y_{n+1} - y_n + py_{n-k} = 0 \quad \textit{for} \quad n = 0, 1, \ldots$$

oscillates if and only if

$$p \geq 1 \quad \textit{if} \quad k = 0,$$

$$p > \frac{k^k}{(k+1)^{k+1}} \quad \textit{if} \quad k \geq 1.$$

One can refer to Kocić and Ladas [1, P_6, Corollary 1.2.1].

Lemma 2.5. *Consider the difference equation*

$$x_{n+1} = x_n\left[1 + q\left(1 - \left(\frac{x_n}{\bar{k}}\right)^z\right)\right]_+ \quad \textit{for} \quad n = 0, 1, \ldots, \tag{9}$$

where $\bar{k}, q, z \in (0, \infty)$ *and*

$$q < \frac{(1+z)^{1+\frac{1}{z}}}{z} - 1.$$

Set $N^* = \bar{k}\left(\frac{1+q}{q}\right)^{\frac{1}{z}}$. *If* $x_0 \in (0, N^*)$, *then* $x_n \in (0, N^*)$ *for all n.*

One can refer to Kocić and Ladas [1, P_{120}, Corollary 4.7.1].

3. Main Results

Theorem 3.1. *Assume that* (2) *holds and*

$$\frac{(qz-1)\mu}{(1-\mu)^{k+1}} > \frac{k^k}{(k+1)^{k+1}}.$$

Then every positive solution of equation (1) *oscillates about the positive equilibrium* $\bar{k}$.

Proof. Assume for the sake of contradiction that equation (1) has a positive solution which does not oscillate about the positive equilibrium $\bar{k}$.

Now, we set

$$N_n = \bar{k} + x_n \quad \text{for} \quad n = -k, -k+1, \ldots,$$

then $\{x_n\}$ is a nonoscillatory solution of the difference equation

$$x_{n+1} - x_n + \mu x_n - \mu\left\{(\bar{k} + x_{n-k})\left[1 + q\left(1 - \left(\frac{\bar{k} + x_{n-k}}{\bar{k}}\right)^z\right)\right]_+ - \bar{k}\right\} = 0$$

$$\text{for} \quad n = 0, 1, \ldots. \quad (10)$$

Without loss of generality, we assume that $\{x_n\}$ is eventually positive. Let n_0 be an integer such that $x_n > 0$ for $n \geq n_0$.

First, we will claim that $\{x_n\}$ is a bounded sequence. Otherwise, there exists a subsequence $\{x_{n_i}\}$ such that for $n_i \geq n_0$ and $i = 1, 2, \ldots,$

$$\lim_{i\to\infty} x_{n_i} = +\infty \quad \text{and} \quad x_{n_i+1} - x_{n_i} \geq 0.$$

It follows from (10) that for i sufficiently large

$$x_{n_i+1} \leq (\bar{k} + x_{n_i-k})\left[1 + q\left(1 - \left(\frac{\bar{k} + x_{n-k}}{\bar{k}}\right)^z\right)\right]_+. \quad (11)$$

Furthermore, we have the following inequality holds:

$$\left[1 + q\left(1 - \left(\frac{\bar{k} + x_{n-k}}{\bar{k}}\right)^z\right)\right]_+ \leq 1,$$

so

$$x_{n_i+1} \leq \bar{k} + x_{n_i-k}. \quad (12)$$

By (12), we can see that $\lim_{i\to\infty} x_{n_i-k} = +\infty$. But, equation (11) leads to a contradiction as $i \to \infty$.

Second, we claim that

$$\lim_{n\to\infty} x_n = 0. \tag{13}$$

Otherwise, let

$$\lambda = \limsup_{n\to\infty} x_n. \tag{14}$$

Then $\lambda > 0$ and there exists a subsequence $\{x_{n_i}\}$ such that for $n_i \geq n_0$ and $i = 1, 2, \ldots,$

$$\limsup_{i\to\infty} x_{n_i} = \lambda \quad \text{and} \quad x_{n_i+1} - x_{n_i} \geq 0,$$

and (11)-(12) also hold.

From (12), we see that $\lambda \leq \limsup_{i\to\infty} x_{n_i}$. Owing to (14), we have $\lim_{i\to\infty} x_{n_i} = \lambda$. But from (11), we can see that

$$\lambda + \bar{k} \leq (\lambda + \bar{k})\left[1 + q\left(1 - \left(\frac{\lambda + \bar{k}}{\bar{k}}\right)^z\right)\right]_+ < \lambda + \bar{k},$$

which is impossible. Hence, (13) holds.

Now, we rewrite equation (10) as follows:

$$x_{n+1} - x_n + \mu x_n + p(n)x_{n-k} = 0 \quad \text{for} \quad n = 0, 1, \ldots, \tag{15}$$

where

$$p(n) = \frac{-\mu\left\{(\bar{k} + x_{n-k})\left[1 + q\left(1 - \left(\frac{\bar{k} + x_{n-k}}{\bar{k}}\right)^z\right)\right]_+ - \bar{k}\right\}}{x_{n-k}}$$

and $(qz - 1)\mu > 0$.

One can easily see that the hypotheses of Lemma 2.3 are satisfied. So the linear equation

$$y_{n+1} - y_n + \mu y_n + (qz - 1)\mu y_{n-k} = 0 \quad \text{for} \quad n = 0, 1, \ldots \tag{16}$$

has an eventually positive solution.

Let $\{y_n\}$ be an eventually positive solution of equation (16). Then $z_n = (1 - \mu)^n y_n$ is an eventually positive solution of the following difference equation:

$$z_{n+1} - z_n + (1 - \mu)^{-k-1}(qz - 1)\mu z_{n-k} = 0 \quad \text{for} \quad n = 0, 1, \ldots.$$

According to Lemma 2.4, we know that equation (16) has no nonoscillatory solutions. This is a contradiction and we complete the proof.

Theorem 3.2. *Assume that*

$$q < \frac{2}{z} < \frac{(1 + z)^{\frac{1}{z}}}{z} - 1 \quad \textit{and} \quad \left(\frac{1 + q}{q}\right)^{\frac{1}{z}} \geq 2.$$

Then the positive equilibrium $\bar{k}$ *of equation* (1) *is globally asymptotically stable relative to the interval* $(0, N^*)$.

Proof. First, we show that the function defined by

$$V(x) = | x - \bar{k} | \quad \text{for} \quad x \in (0, N^*) \tag{17}$$

is a Lyapunov function for equation (9).

Clearly, V is a nonnegative and continuous function, and then the remaining is to show that

$$V(f(x)) < V(x) \quad \text{for} \quad x \in (0, N^*) \quad \text{and} \quad x \neq \bar{x}, \tag{18}$$

where

$$f(x) = x\left[1 + q\left(1 - \left(\frac{x}{\bar{k}}\right)^z\right)\right]_+ \quad \text{for} \quad x \in [0, \infty).$$

Let $x \in (0, \bar{k})$. Then (18) is equivalent to

$$\left\{x\left[1 + q\left(1 - \left(\frac{x}{\bar{k}}\right)^z\right)\right] - \bar{k}\right\}^2 < (x - \bar{k})^2,$$

i.e.,

$$qx\left(1 - \left(\frac{x}{\bar{k}}\right)^z\right) + 2x - 2\bar{k} < 0.$$

Set

$$g(x) = 2x + qx\left(1 - \left(\frac{x}{\bar{k}}\right)^z\right) - 2\bar{k} \quad \text{for} \quad x \in (0, \bar{k}),$$

then

$$g'(x) = 2 + q - q(1 + z)\left(\frac{x}{\bar{k}}\right)^z > 2 + q - q(1 + z) \geq 0.$$

We can easily know that $g(0) = -2\bar{k} < 0$ and $g(\bar{k}) = 0$. Thus, $g(x) < 0$ for $x \in (0, \bar{k})$, and this shows that (18) holds.

Furthermore, let $x \in (\bar{k}, N^*)$. Then (18) is equivalent to

$$\left\{x\left[1 + q\left(1 - \left(\frac{x}{\bar{k}}\right)^z\right)\right] - \bar{k}\right\}^2 < (x - \bar{k})^2$$

and this is true provided that $g(x) > 0$ for $x \in (\bar{k}, N^*)$.

Indeed, we can get

$$g(\bar{k}) = 0, \quad g(N^*) = N^* - 2k \geq 0, \quad g''(x) < 0.$$

Therefore, $g(x)$ is a convex function in $x \in (\bar{k}, N^*)$. By the properties of convex functions, we know that $f((0, N^*)) \subset (0, N^*)$.

Thus, by Lemma 2.1, we know that $\bar{k}$ is globally asymptotically stable relative to the interval $(0, N^*)$, and by Lemma 2.2, we can complete the proof.

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