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VISCOSITY SOLUTIONS FOR BILEVEL PROBLEMS WITH NASH EQUILIBRIUM CONSTRAINTS

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Abstract

A regularization method for bilevel problems with lower level defined by Nash equilibria is considered and a concept of "viscosity solution" associated to such a regularization is introduced. Sufficient conditions for the existence of viscosity solutions are given and applications to optimistic and pessimistic bilevel optimization problems are presented.

1. Introduction

Let (T, τ) be a Hausdorff topological space, let $V_1, ..., V_k$, $k \geq 1$, be Banach spaces and let $V = V_1 \times \cdots \times V_k$.

If K_i is a nonempty closed subset of V_i, we consider $K = \prod_{j=1,...,k} K_j$

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and, given $\overline{\boldsymbol{x}} \in K$, we denote by $\overline{\boldsymbol{x}}_{-i}$ the point $(\overline{x}_1, ..., \overline{x}_{i-1}, \overline{x}_{i+1}, ..., \overline{x}_k) \in \prod_{j \neq i} K_j = K_{-i}$. If $x_i \in K_i$, we denote by $(x_i, \overline{\boldsymbol{x}}_{-i})$ the point $(\overline{x}_1, ..., \overline{x}_{i-1}, x_i, \overline{x}_{i+1}, ..., \overline{x}_k) \in K$.

Let $f_1, ..., f_k$ be real-valued functions defined on $T \times K$ and bounded from below. Assume that $t \in T$ corresponds to a strategy of a leader playing first and that $\boldsymbol{x}$ corresponds to a strategy profile of k followers playing non-cooperatively (that is, $\boldsymbol{x} = (x_1, ..., x_k)$ means that for all $i = 1, ..., k$, the follower i chooses the strategy x_i). The leader aims to minimize with respect to t the objective function

$$L : (t, \boldsymbol{x}) \in T \times K \to L(t, \boldsymbol{x}) \in \mathbb{R} \cup \{+\infty\},$$

knowing that t is constrained in $T' \subseteq T$ and $\boldsymbol{x}$ belongs to $\mathcal{N}(t)$, the set of solutions to the parametrized Nash equilibrium [28] problem

$$(NE)(t) \text{ find } \overline{\boldsymbol{x}} \in K \text{ such that } f_i(t, \overline{\boldsymbol{x}}) \leq \inf_{x_i \in K_i} f_i(t, x_i, \overline{\boldsymbol{x}}_{-i}), \forall i = 1, ..., k$$

or, equivalently,

$$(NE)(t) \text{ find } \overline{\boldsymbol{x}} \in K \text{ such that } \sum_{i=1}^{k} f_i(t, \overline{\boldsymbol{x}}) \leq \sum_{i=1}^{k} \inf_{x_i \in K_i} f_i(t, x_i, \overline{\boldsymbol{x}}_{-i}).$$

When $(NE)(t)$ does not have a unique solution, which is guaranteed only in restrictive conditions [29], as for bilevel optimization problems two extreme situations can be considered:

- L is minimized with respect to the couple $(t, \boldsymbol{x})$.
- The function $\sup_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x})$ is minimized with respect to t.

This leads to formulate the *optimistic* and the *pessimistic bilevel problem with Nash equilibrium constraints* (see, for example, [22, 24]), also called *strong* and *weak bilevel optimization problem* when $k = 1$ and the lower level corresponds to a minimization problem (see, for example, [21, 10, 8]):

(OBE) find $(\bar{t}, \bar{\boldsymbol{x}}) \in (T' \times K)$ s.t. $\bar{\boldsymbol{x}} \in \mathcal{N}(\bar{t})$ and

$$L(\bar{t}, \bar{\boldsymbol{x}}) = \inf_{t \in T'} \inf_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}),$$

(PBE) find $\bar{t} \in T'$ s.t. $\sup_{\bar{\boldsymbol{x}} \in \mathcal{N}(\bar{t})} L(\bar{t}, \bar{\boldsymbol{x}}) = \inf_{t \in T'} \sup_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}).$

However, following [24, 25, 6], one can have also situations in which the leader minimizes the *intermediate function*

$$C(t) = \eta(t) \inf_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}) + (1 - \eta(t)) \sup_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}), \tag{1}$$

where $\eta(t)$ is a real valued function such that $0 \leq \eta(t) \leq 1$ for any t.

This actually amounts to an intermediate situation since

$$\inf_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}) \leq C(t) \leq \sup_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x})$$

and

$$v_1 = \inf_{t \in T'} \inf_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}) \leq v = \inf_{t \in T'} C(t) \leq v_2 = \inf_{t \in T'} \sup_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}).$$

Then the problem to be solved by the leader is the minimization problem

$$(P) \text{ find } \bar{t} \in T' \text{ s.t. } C(\bar{t}) = \inf_{t \in T'} C(t)$$

that we call *intermediate bilevel problem with Nash equilibrium constraints*.

Observe that when $\eta(t) = 0$ for every t, the problem (P) coincides with the bilevel problem (PBE) and, when $\eta(t) = 1$ for every t, $\bar{t}$ is a solution to (P) if and only if there exists $\bar{\boldsymbol{x}} \in \mathcal{N}(\bar{t})$ such that the couple $(\bar{t}, \bar{\boldsymbol{x}})$ is a solution to (OBE).

Therefore, assuming that the set T' is sequentially compact with respect to τ, it is important to investigate the τ-lower semicontinuity of the function C. However, we show by Example 2.2 that C may fail to be lower semicontinuous due to the possible lack of lower semicontinuity of the function

$$\omega : t \in T \to \sup_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}).$$

Thus, in this paper, we consider, for any $\varepsilon > 0$, the following regularization:

$$C^{\varepsilon}(t) = \eta(t) \inf_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}) + (1 - \eta(t)) \sup_{\boldsymbol{x} \in \widetilde{\mathcal{N}}^{\varepsilon}(t)} L(t, \boldsymbol{x}),$$

where

$$\widetilde{\mathcal{N}}^{\varepsilon}(t) = \left\{ \overline{\boldsymbol{x}} \in K \text{ s.t. } \sum_{i=1}^{k} f_i(t, \overline{\boldsymbol{x}}) < \sum_{i=1}^{k} \inf_{x_i \in K_i} f_i(t, x_i, \overline{\boldsymbol{x}}_{-i}) + \varepsilon \right\}$$

is the set of the *strict* ε*-approximate Nash equilibria* defined in [26].

We emphasize that only the second term of C has to be regularized since the function $\inf_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x})$ is lower semicontinuous (see Proposition 2.1) under very general hypotheses.

We will prove that under sufficiently general assumptions,

- the regularized problem

$$(P^{\varepsilon}) \text{ find } t_{\varepsilon} \in T' \text{ s.t. } C^{\varepsilon}(t_{\varepsilon}) = \inf_{t \in T'} C^{\varepsilon}(t)$$

has at least a solution t_{ε},

- $(t_{\varepsilon})_{\varepsilon}$ has a τ-limit point $\widetilde{t} \in T'$,

- $(C^{\varepsilon})_{\varepsilon}$ epiconverges to $cl(C)$ when ε tends to 0,

- the optimal values $v^{\varepsilon} = \inf_{t \in T'} C^{\varepsilon}(t)$ converge to v when ε tends to zero.

Then, in line with [2], we give the following definition:

Definition 1.1. A point $\widetilde{t} \in T'$ is called a *viscosity solution* for the intermediate bilevel problem with Nash equilibrium constraints whenever it is a τ-limit point of a sequence $(t_{\varepsilon_n})_n$, where $(\varepsilon_n)_n$ decreases to 0, such that:

- t_{ε_n} is a minimum point for C^{ε_n} over T' for any $n \in \mathbb{N}$,
- $\lim_n C^{\varepsilon_n}(t_{\varepsilon_n}) = \inf_{t \in T'} C(t)$.

Note that a similar concept has been considered in [21] for minsup problems with lower level described by optimization problems, i.e. in the case where $\eta(t) = 0$ for every t and $k = 1$.

2. Preliminary Results

First, we recall the definitions that we shall need.

Let (S, σ) be a topological space and $(H_n)_n$ be a sequence of subsets of S. Then:

– $s \in \sigma\text{-}\liminf_n H_n$ if and only if there exists a sequence $(s_n)_n$ σ-converging to s in S such that $s_n \in H_n$ for n sufficiently large,

– $s \in \sigma\text{-}\limsup_n H_n$ if and only if there exists a sequence $(s_n)_n$ σ-converging to s in S such that $s_{n_k} \in H_{n_k}$ for a selection of integers $(n_k)_k$.

Let (U, σ') be a topological space and let Y be a set-valued map from U to S, i.e. a map which associates to any $u \in U$ a subset $Y(u)$ of S.

The map Y is (σ', σ)-*sequentially lower semi-continuous* over $X \subseteq U$ if for every $x \in X$ and every sequence $(x_n)_n$ σ'-converging to x in X, we have $Y(x) \subseteq \sigma\text{-}\liminf_n Y(x_n)$, i.e.,

– for any $x \in X$, any sequence $(x_n)_n$ σ'-converging to x in X and any

$s \in Y(x)$, there exists a sequence $(s_n)_n$ σ-converging to s in S such that $s_n \in Y(x_n)$ for n large.

The map Y is (σ', σ)-*sequentially closed* over X if for every $x \in X$ and every sequence $(x_n)_n$ σ'-converging to x in X we have $\sigma\text{-}\limsup\limits_n Y(x_n) \subseteq Y(x)$, i.e.,

− for any $x \in X$, any sequence $(x_n)_n$ σ'-converging to x in X and any sequence $(s_k)_k$ σ-converging to s in S such that $s_k \in Y(x_{n_k})$ for a selection of integers $(n_k)_k$, we have that $s \in Y(x)$.

The map Y is (σ', σ)-*sequentially subcontinuous* over X if, for every sequence $(x_n)_n$ σ'-converging in X, $\sigma\text{-}\limsup\limits_n Y(x_n) \neq \emptyset$, i.e.,

− for any $x \in X$ and any sequence $(x_n)_n$ σ'-converging in X, every sequence $(s_n)_n$ such that $s_n \in Y(x_n)$ for any $n \in \mathbb{N}$ has a σ-converging subsequence.

For more information about these and related concepts, the reader may refer to [4].

A function $h : U \times S \to \mathbb{R}$ is $(\sigma' \times \sigma)$-*sequentially coercive* over $X \times Y$, where $X \times Y \subseteq U \times S$, if for every $\alpha > \inf\limits_{X \times Y} h$, there exists a sequentially compact set $H_\alpha \subseteq U \times S$ such that

$$\{(x, s) \in X \times Y : f(x, s) \leq \alpha\} \subseteq H_\alpha.$$

Given a function $g : T \to \mathbb{R}$ we denote by $cl_\tau g$ the *lower semicontinuous regularization* of g, that is the greatest τ-lower semicontinuous function which minorizes g. It is known that for every $t \in T$,

$$(cl_\tau g)(t) = \liminf_{t' \to t} g(t) \quad \text{and} \quad \inf_T cl_\tau g = \inf_T g.$$

Finally, a sequence of functions $(g_n)_n$ defined on T τ-*epiconverges* to g in T if

$$(\tau \times \nu) - \limsup_n epi g_n \subseteq epi g \subseteq (\tau \times \nu) - \liminf_n epi g_n,$$

where ν is the usual topology on $\mathbb{R}$. This amounts to say that the following conditions hold together:

- for every $t \in T$ and every sequence $(t_n)_n$ τ-converging to t in T,

$$g(t) \leq \liminf_n g_n(t_n),$$

- for every $t \in T$, there exists a sequence $(t'_n)_n$ τ-converging to t in T such that

$$\limsup_n g_n(t'_n) \leq g(t).$$

Lemma 2.1 [1]**.** *If the sequence $(g_n)_n$ is monotone decreasing and pointwise τ-converges to g on T, then it also epiconverges to $cl_\tau g$.*

For more information and further results, the reader may refer to [1] and [9].

For the sake of shortness, in this paper, we will omit the term sequentially in any of the above conditions as well as in the semicontinuity properties for scalar functions.

If the map Y is nonempty-valued and $g : X \times S \to \mathbb{R}$ is a real-valued function, then the marginal (or value) functions

$$\varphi(x) = \inf_{s \in Y(x)} g(x, s) \quad \text{and} \quad \omega(x) = \sup_{s \in Y(x)} g(x, s)$$

can be defined.

The following results concerning the lower semicontinuity of φ and ω will be used in this paper.

Lemma 2.2 [13]. *If the set-valued map Y is (σ', σ)-closed and (σ', σ)-subcontinuous over X and if g is $(\sigma' \times \sigma)$-lower semicontinuous at (x, s), for every $x \in X$ and every $s \in Y(x)$, then the function φ is σ'-lower semicontinuous over X. When $Y(x) = Y \subseteq S$ for every $x \in X$, then φ is σ'-lower semicontinuous over X whenever g is $(\sigma' \times \sigma)$-lower semicontinuous at (x, s), for every $(x, s) \in X \times Y$, and $(\sigma' \times \sigma)$-coercive over $X \times Y$.*

Lemma 2.3 [13]. *If Y is (σ', σ)-lower semicontinuous over X and g is $(\sigma' \times \sigma)$-lower semicontinuous at (x, s), for every $x \in X$ and every $s \in Y(x)$, then the function ω is σ'-lower semicontinuous over X.*

Therefore, having in mind the expression of the function $C(t)$ in (1), results concerning closedness and lower semicontinuity of the set-valued map $\mathcal{N}$ are crucial. However, while in [7, 26, 18], it has been proven that the map $\mathcal{N}$ may be closed under sufficiently general assumptions, in [27] it has been shown that it may fail to be lower semicontinuous, so that the function C may be not lower semicontinuous.

From now on, given a topology σ_i on the Banach space V_i, we denote by σ the product topology on the Banach space V and by σ_{-i} the product topology on $\prod_{j \neq i} V_j$.

The following proposition gives a sufficient condition for the closedness of the map $\mathcal{N}$.

Proposition 2.1 [18]. *The set-valued map $\mathcal{N}$ is (τ, σ)-closed over a closed set $T' \subseteq T$ if the following assumptions hold*:

(i) $f = \sum_{i=1}^{k} f_i$ *is $(\tau \times \sigma)$-lower semicontinuous over $(T' \times K)$;*

(ii) *for every $(t, \boldsymbol{x}) \in T' \times K$, for every $i \in \{1, ..., k\}$ and every sequence*

$(t_n, \boldsymbol{x}_{-i,n})_n$ $(\tau \times \sigma_{-i})$*-converging to* $(t, \boldsymbol{x}_{-i})$ *in* $T' \times K_{-i}$, *there exists a sequence* $(\tilde{x}_{i,n})_n$ *in* K_i *such that*

$$f_i(t, \boldsymbol{x}) \geq \limsup_n f_i(t_n, \tilde{x}_{i,n}, \boldsymbol{x}_{-i,n}).$$

We point out that assumption (ii) is satisfied if the function f_i is $(\tau \times \sigma)$-upper semicontinuous over $T' \times K$.

Example 2.1 [27]**.** Let $k = 2$, $T = V_1 = V_2 = \mathbb{R}$, $T' = K_1 = K_2 = [0, 1]$ and $f_1(t, x_1, x_2) = f_2(t, x_1, x_2) = tx_1x_2$. Then one easily checks that $(1, 1)$ is a Nash equilibrium for the problem $(NE)(0)$, since $\mathcal{N}(0) = [0, 1]^2$, and that $\liminf_n \mathcal{N}(1/n) = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\})$, so $\mathcal{N}$ is not lower semicontinuous at $t = 0$.

Example 2.2. Consider the data of Example 2.1 and the real-valued function $L(t, x_1, x_2) = x_1 + x_2$ defined on the unitary cube $[0, 1]^3$. One easily checks that

$$\omega(0) = \sup_{\boldsymbol{x} \in \mathcal{N}(0)} L(0, \boldsymbol{x}) = 2 \quad \text{and} \quad \omega(1/n) = \sup_{\boldsymbol{x} \in \mathcal{N}(1/n)} L(1/n, \boldsymbol{x}) = 1$$

so ω is not lower semicontinuous at $t = 0$. Therefore, if $\eta(t) = 0$ for every $t \in [0, 1]$, then we have $C(0) = 2$ and $C(1/n) = 1$, so also C is not lower semicontinuous at $t = 0$.

Nevertheless, it can be proved that the set-valued map $\widetilde{\mathcal{N}}^{\varepsilon}$, considered in the Introduction, is lower semicontinuous under suitable assumptions:

Proposition 2.2 [27]**.** *The set-valued map* $\widetilde{\mathcal{N}}^{\varepsilon}$ *is* (τ, σ)*-lower semicontinuous over* $T' \subseteq T$ *if the following assumption holds*:

(i) $\sum_{i=1}^{k} f_i(\cdot, \boldsymbol{x}) - \sum_{i=1}^{k} \inf_{x_i \in K_i} f_i(\cdot, \boldsymbol{x})$ *is* τ*-upper semicontinuous over* T' *for every* $\boldsymbol{x} \in K$.

Remark 2.1. Using Lemma 2.2, it is easy to see that condition (i) is satisfied whenever the following reasonable conditions hold together:

• the function f_i is $(\tau \times \sigma)$-lower semicontinuous and $(\tau \times \sigma)$-coercive over $T' \times K$, for every $i = 1, ..., k$,

• $\sum_{i=1}^{k} f_i(\cdot, \boldsymbol{x})$ is τ-upper semicontinuous over T' for every $\boldsymbol{x} \in K$.

The following result, that will be also used further on, says that any σ-converging sequence of strict ε_n-Nash equilibria *converges* to a Nash equilibrium when $(\varepsilon_n)_n$ decreases to zero.

Proposition 2.3 [26]. *If, for every* $t \in T$, *the function*

$$\boldsymbol{x} \in K \to \sum_{i=1}^{k} f_i(t, \boldsymbol{x}) - \sum_{i=1}^{k} \inf_{x_i \in K_i} f_i(t, \boldsymbol{x})$$

is σ*-lower semicontinuous over K, then for every sequence* $(\varepsilon_n)_n$ *decreasing to zero, one has*

$$\sigma - \limsup_n \widetilde{\mathcal{N}}^{\varepsilon_n}(t) \subseteq \mathcal{N}(t).$$

We highlight that in this paper, we do not consider neither the "large" version of the approximate ε-Nash equilibrium points

$$\mathcal{N}^{\varepsilon}(t) = \left\{ \overline{\boldsymbol{x}} \in K \text{ s.t. } \sum_{i=1}^{k} f_i(t, \overline{\boldsymbol{x}}) \le \sum_{i=1}^{k} \inf_{x_i \in K_i} f_i(t, x_i, \overline{\boldsymbol{x}}_{-i}) + \varepsilon \right\}$$

neither the classical concept of ε-Nash equilibria:

$$\hat{\mathcal{N}}^{\varepsilon}(t) = \{ \overline{\boldsymbol{x}} \in K \text{ s.t. } f_i(t, \overline{\boldsymbol{x}}) \le \inf_{x_i \in K_i} f_i(t, x_i, \overline{\boldsymbol{x}}_{-i}) + \varepsilon, \forall i = 1, ..., k \}.$$

This is due to the possible lack of lower semicontinuity of these maps under the only assumption (i) and in the absence of further convexity assumptions (see Examples 3.2, 3.3 and 3.6 in [27] and also Example 3.2 in Section 3).

From now on, we assume the following:

(a_1) the function $\eta(t)$ is continuous over T';

(a_2) $(\varepsilon_n)_n$ is a sequence of positive real numbers decreasing to zero;

(a_3) the set of Nash equilibria $\mathcal{N}(t)$ is nonempty for any $t \in T$.

3. Regularization and Viscosity Solutions

In order to prove that the problem (P_ε) admits solutions, we will use the lower semicontinuity of the function C^ε defined by

$$C^\varepsilon(t) = \eta(t) \inf_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}) + (1 - \eta(t)) \sup_{\boldsymbol{x} \in \widetilde{\mathcal{N}}^\varepsilon(t)} L(t, \boldsymbol{x}) \tag{2}$$

which can be derived from the lower semicontinuity of the functions

$$\inf_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}) \quad \text{and} \quad \sup_{\boldsymbol{x} \in \widetilde{\mathcal{N}}^\varepsilon(t)} L(t, \boldsymbol{x})$$

due to assumption (a_1). Then, using the results in Section 2, we have:

Proposition 3.1. *If the following assumptions hold*:

(i) *the set* T' *is* τ*-sequentially compact*;

(ii) *the function* L *is* $(\tau \times \sigma)$*-lower semicontinuous over* $T' \times K$;

(iii) *the function* f_i *is* $(\tau \times \sigma)$*-lower semicontinuous and* $(\tau \times \sigma)$*-coercive over* $T' \times K$ *for every* $i = 1, ..., k$;

(iv) *for every* $(t, \boldsymbol{x}) \in T' \times K$, *for every* $i \in \{1, ..., k\}$ *and every sequence* $(t_n, \boldsymbol{x}_{-i,n})_n$ $(\tau \times \sigma_{-i})$*-converging to* $(t, \boldsymbol{x}_{-i})$ *in* $T' \times K_{-i}$ *there exists a sequence* $(\tilde{x}_{i,n})$ *in* K_i *such that*

$$f_i(t, \boldsymbol{x}) \geq \limsup_n f_i(t_n, \tilde{x}_{i,n}, \boldsymbol{x}_{-i,n});$$

(v) *the function* $\sum_{i=1}^{k} f_i(\cdot, \boldsymbol{x})$ *is* τ*-upper semicontinuous over* T' *for every* $\boldsymbol{x} \in K$;

(vi) *the function*

$$\boldsymbol{x} \in K \to \sum_{i=1}^{k} f_i(t, \boldsymbol{x}) - \sum_{i=1}^{k} \inf_{x_i \in K_i} f_i(t, \boldsymbol{x})$$

is τ*-coercive over* T'; *then*, *the problem* (P_ε) *has at least a solution* t_ε.

Moreover, *for every sequence of positive numbers* $(\varepsilon_n)_n$ *decreasing to zero*, *the sequence* $(t_{\varepsilon_n})_n$ *has a subsequence which* τ*-converges towards* $\tilde{t} \in T'$.

Proof. Assumptions (iii), (iv) and (v) imply that all assumptions of Propositions 2.1 and 2.2 are satisfied, so that the set-valued map $\mathcal{N}$ is (τ, σ)-closed and the set-valued map $\widetilde{\mathcal{N}}^\varepsilon$ is (τ, σ)-lower semicontinuous over $T' \times K$. Assumption (vi) implies that $\mathcal{N}$ is (τ, σ)-subcontinuous over $T' \times K$. Then, using Lemma 2.2 and Lemma 2.3, one concludes that the function C^ε is τ-lower semicontinuous over T', so, from assumption (i), we infer that the problem (P^ε) has a solution t_ε. The last assertion also derives from assumption (i). □

We point out that the problem (P^ε) is Tikhonov well-posed in the generalized sense [11] with respect to τ, which means that it has at least a solution and every minimizing sequence for (P^ε) has a subsequence which τ-converges to a minimum point of C^ε.

Now, we prove that the sequence of functions $(C^{\varepsilon_n})_n$ epiconverges to $cl_\tau C$ and, consequently, the sequence of infima $(v^{\varepsilon_n})_n$ converges to the infimum v.

Proposition 3.2. *If the following assumptions hold*:

(i) *the function* $L(t, \cdot)$ *is* σ-*upper semicontinuous over K*;

(ii) *the function*

$$\boldsymbol{x} \in K \to \sum_{i=1}^{k} f_i(t, \boldsymbol{x}) - \sum_{i=1}^{k} \inf_{x_i \in K_i} f_i(t, \boldsymbol{x})$$

is σ-*lower semicontinuous and* σ-*coercive over K for every* $t \in T'$; *then*, *the sequence* $(C^{\varepsilon_n})_n$ τ-*epiconverges towards* $cl_\tau C$ *and* $\lim_{\varepsilon \to 0} v^\varepsilon = v$.

Proof. We use Lemma 2.1 applied to the functions $g_n = C^{\varepsilon_n}$ and we show that the sequence $(C^{\varepsilon_n})_n$, which is monotonically decreasing with respect to n, τ-pointwise converges to C on T'. Having in mind the expression of C^{ε_n} and C, it is sufficient to prove that

$$\lim_n \sup_{\boldsymbol{x} \in \widetilde{\mathcal{N}}^{\varepsilon_n}(t)} L(t, \boldsymbol{x}) = \sup_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}). \tag{3}$$

The inequality $\sup_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}) \leq \liminf_n \sup_{\boldsymbol{x} \in \widetilde{\mathcal{N}}^{\varepsilon_n}(t)} L(t, \boldsymbol{x})$ follows from the inclusion $\mathcal{N}(t) \subseteq \widetilde{\mathcal{N}}^{\varepsilon_n}(t)$ that is true for every $n \in \mathbb{N}$ and every $t \in T$.

Assume that there exist $t \in T$ and a real number c such that $\sup_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}) < c < \limsup_n \sup_{\boldsymbol{x} \in \widetilde{\mathcal{N}}^{\varepsilon_n}(t)} L(t, \boldsymbol{x})$. We find an increasing sequence $(n_k)_k$ and a sequence $\boldsymbol{x}_k$ such that $\boldsymbol{x}_k \in \widetilde{\mathcal{N}}^{\varepsilon_{n_k}}(t)$ and $\sup_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}) < c < L(t, \boldsymbol{x}_k)$. Assumption (ii) implies that a subsequence $(\boldsymbol{x}_{k'})_{k'}$ of $(\boldsymbol{x}_k)_k$ σ-converges to $\overline{\boldsymbol{x}} \in K$. Then $\overline{\boldsymbol{x}} \in \mathcal{N}(t)$ by Proposition 2.3 and we get the contradiction $\sup_{\boldsymbol{x} \in \mathcal{N}(t)} L(t, \boldsymbol{x}) < L(t, \overline{\boldsymbol{x}})$ since the function $L(t, \cdot)$ is σ-upper semicontinuous at $\overline{\boldsymbol{x}}$. Then (3) is proved.

For the second part, we have

$$v = \inf_{T'} C = \inf_{T'} cl_\tau C = \lim_n \inf_{T'} C^{\varepsilon_n} = \lim_n v^{\varepsilon_n},$$

due to a known property of epiconvergent sequences [1], so, $\lim_{\varepsilon \to 0} v^\varepsilon = v.$ □

Remark 3.1. Using Lemma 2.2, it is easy to see that condition (ii) is satisfied whenever the following condition holds for every $t \in T'$:

• the function $f_i(t, \cdot)$ is σ-continuous and σ-coercive over K for every $i = 1, ..., k$.

Corollary 3.1. *In the same assumptions of Proposition* 3.2, *any sequence* $(t^{\varepsilon_n})_n$ *of solutions to* (P^{ε_n}) *is minimizing for the minimum problem*

$$(\widetilde{P}) \text{ find } \widetilde{t} \in T' \text{ such that } (cl_\tau C)(\widetilde{t}) = \inf_{t \in T'} (cl_\tau C)(t).$$

Proof. We have:

$$\limsup_n (cl_\tau C)(t_{\varepsilon_n}) \le \limsup_n C(t_{\varepsilon_n}) = \lim_n v^{\varepsilon_n} = v = \inf_{t \in T'} (cl_\tau C)(t).$$

Since $\inf_{t \in T'} (cl_\tau C)(t) \le (cl_\tau C)(t_{\varepsilon_n})$ for every $n \in \mathbb{N}$, we get that

$$\lim_n (cl_\tau C)(t_{\varepsilon_n}) = \inf_{T'} cl_\tau C.$$ □

It is worth noting that when the assumptions of Propositions 3.1 and 3.2 hold, then the point $\widetilde{t}$, towards which a subsequence of $(t_{\varepsilon_n})_n$ τ-converges, is a solution to the relaxed minimization problem $(\widetilde{P})$ but it may fail to solve the problem (P).

Example 3.1 [26]**.** Let $k = 2$, $T = V_1 = V_2 = \mathbb{R}$, $T' = K_1 = K_2 = [0, 1]$, $f_1(t, x_1, x_2) = -f_2(t, x_1, x_2) = -x_1(t + x_2)$ and $L(t, \boldsymbol{x}) = t - (x_1 + x_2)$. Then one easily checks that $\mathcal{N}(0) = [0, 1] \times \{0\}$ and $\mathcal{N}(t) = \{(1, 0)\}$ if $0 < t \le 1$, so that for $\eta(t) = 0$, we have $C(0) = 0$, $C(t) = t - 1$ for $t \in]0, 1]$ and the problem (P) does not have solution. However, one has $\widetilde{\mathcal{N}}^{\varepsilon}(t) = [0, 1] \times [0, \varepsilon]$ if $t \in [0, \varepsilon]$ and

$$\widetilde{\mathcal{N}}^{\varepsilon}(t) = \{(x_1, x_2) : 1 - \varepsilon/t < x_1 \leq 1 \text{ and } 0 \leq x_2 < t(x_1 - 1) + \varepsilon\}$$

if $t \in]\varepsilon, 1]$. Therefore, $C^{\varepsilon}(t) = t$ for $t \in [0, \varepsilon]$ and $C^{\varepsilon}(t) = t - 1 + \varepsilon/t$ for $t \in]\varepsilon, 1]$, so, for every $\varepsilon < 1/2$, the minimum point for C^{ε} is $t_{\varepsilon} = \sqrt{\varepsilon}$, the infimum $v^{\varepsilon} = 2\sqrt{\varepsilon} - 1$ and there exists the viscosity solution $\widetilde{t} = 0$ for the intermediate bilevel problem (P).

In conclusion, we get that a viscosity solution for the intermediate bilevel problem with Nash equilibrium constraints (P) will exist whenever all assumptions in Propositions 3.1 and 3.2 hold and, for the sake of completeness, we write the following, more classic, sufficient condition.

Corollary 3.2. *Assume that the following hold*:

(i) *the set* T' *is* τ*-sequentially compact*;

(ii) *the set* K *is* σ*-sequentially compact*;

(iii) *the function* L *is* $(\tau \times \sigma)$*-continuous over* $T' \times K$;

(iv) *the function* f_i *is* $(\tau \times \sigma)$*-continuous over* $T' \times K$ *for every* $i = 1, \ldots, k$.

Then, *there exists a viscosity solution for the intermediate bilevel problem with Nash equilibrium constraints*.

Now, we consider the case where $k = 1$ and there is only one follower at the lower level that solves the minimization problem:

$$\text{find } \bar{x} \in K \text{ such that } f_1(t, \bar{x}) \leq \inf_{x \in K} f_1(t, x),$$

where K is a closed subset of a Banach space V and f_1 is a real valued function bounded from below on $T \times K$. We denote, for any $t \in T$, by

$\mathcal{M}(t)$ the set of minimum points for the function f_1 that is assumed to be nonempty.

As in the multifollower case, there are two extreme situations that are nothing else than the *optimistic* and the *pessimistic bilevel optimization problem*:

$$(OPT) \text{ find } (\bar{t}, \bar{x}) \in (T' \times K) \text{ s.t. } \bar{x} \in \mathcal{M}(\bar{t}) \text{ and}$$

$$L(\bar{t}, \bar{x}) = \inf_{t \in T'} \inf_{x \in \mathcal{M}(t)} L(t, x),$$

$$(PES) \text{ find } \bar{t} \in T' \text{ s.t. } \sup_{\bar{x} \in \mathcal{M}(\bar{t})} L(\bar{t}, \bar{x}) = \inf_{t \in T'} \sup_{x \in \mathcal{M}(t)} L(t, x).$$

The intermediate situation now consists in minimizing the function

$$C(t) = \eta(t) \inf_{x \in \mathcal{M}(t)} L(t, x) + (1 - \eta(t)) \sup_{x \in \mathcal{M}(t)} L(t, x)$$

that may fail to be lower semicontinuous, so the *intermediate bilevel optimization problem*

$$(P) \text{ find } \bar{t} \in T' \text{ s.t. } C(\bar{t}) = \inf_{t \in T'} C(t)$$

may fail to have solutions.

Then we consider the regularized function

$$C^{\varepsilon}(t) = \eta(t) \inf_{x \in \mathcal{M}(t)} L(t, x) + (1 - \eta(t)) \sup_{x \in \widetilde{\mathcal{M}}^{\varepsilon}(t)} L(t, x),$$

where

$$\widetilde{\mathcal{M}}^{\varepsilon}(t) = \{\bar{x} \in K \text{ s.t. } f_1(t, \bar{x}) < \inf_{x \in K} f_1(t, x) + \varepsilon\}$$

is the set of the *strict* ε-*minima* [20], and, applying the previous results, we obtain sufficient conditions for the existence of viscosity solutions to the intermediate bilevel optimization problem. In particular, the existence is guaranteed when the constraints T' and K are compact and the objective functions f_1 and L are continuous.

Remark 3.2. We point out that our regularization approach differs from that suggested in [12] in which the function

$$J^{\varepsilon}(t) = \eta(t) \inf_{x \in \mathcal{M}^{\varepsilon}(t)} L(t, x) + (1 - \eta(t)) \sup_{x \in \mathcal{M}^{\varepsilon}(t)} L(t, x)$$

with

$$\mathcal{M}^{\varepsilon}(t) = \{\bar{x} \in K \text{ s.t. } f_1(t, \bar{x}) \leq \inf_{x \in K} f_1(t, x) + \varepsilon\}$$

is considered. Indeed, as already noted in the Introduction for ε-Nash equilibria, with this type of regularization the function $J^{\varepsilon}(t)$ may fail to be lower semicontinuous and the corresponding minimum problem may fail to have solution even when the objective f_1 and L are continuous.

Example 3.2 [23]. Let $k = 1$, $T = V_1 = \mathbb{R}$, $T' = K_1 = [0, 1]$, $f_1(t, x_1) = -x_1^2 + (1 + t)x_1 - t$ and $L(t, x_1) = t + x_1$. Then one easily checks that $\mathcal{M}(0) = \{0, 1\}$ and $\mathcal{M}(t) = \{0\}$ if $0 < t \leq 1$, so that for $\eta(t) = 0$ for every t, one has $C(0) = 1$, $C(t) = t$ for $0 < t \leq 1$ and the problem (P) does not admit solution. However, since, for $\varepsilon \in]0, 1/4[$,

$$\mathcal{M}^{\varepsilon}(t) = [0, (t + 1)/2 - 1/2\sqrt{(t + 1)^2 - 4\varepsilon}]$$

$$\cup [(t + 1)/2 + 1/2\sqrt{(t + 1)^2 - 4\varepsilon}, 1] \quad \text{if } t \in [0, \varepsilon],$$

$$\mathcal{M}^{\varepsilon}(t) = [0, \varepsilon] \cup \{1\} \quad \text{if } t = \varepsilon,$$

$$\mathcal{M}^{\varepsilon}(t) = [0, (t + 1)/2 - 1/2\sqrt{(t + 1)^2 - 4\varepsilon}] \quad \text{if } t \in]\varepsilon, 1],$$

one has that the function

$$J^{\varepsilon}(t) = t + 1 \quad \text{if } t \in [0, \varepsilon] \text{ and}$$

$$J^{\varepsilon}(t) = t + (t + 1)/2 - 1/2\sqrt{(t + 1)^2 - 4\varepsilon} \quad \text{if } t \in]\varepsilon, 1]$$

is not lower semicontinuous at $t = \varepsilon$ and does not have minima, so the regularization method in [12] cannot be applied.

On the contrary, one can check that the regularization function, introduced in this paper,

$$C^{\varepsilon}(t) = t + 1 \quad \text{if } t \in [0, \varepsilon[\text{ and}$$

$$C^{\varepsilon}(t) = t + (t+1)/2 - 1/2\sqrt{(t+1)^2 - 4\varepsilon} \quad \text{if } t \in [\varepsilon, 1],$$

has minimum for $t_{\varepsilon} = \varepsilon$. Therefore, $\widetilde{t} = 0$ is a viscosity solution for the weak (pessimistic) bilevel optimization problem.

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