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RELATION BETWEEN SUM OF 2*m*th POWERS AND POLYNOMIALS OF TRIANGULAR NUMBERS

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Abstract

Let $\Phi_{(m,k)}(n)$ denote the number of representations of an integer n as a sum of k $2m$th powers and $\Psi_{(m,k)}(n)$ denote the number of representations of an integer n as a sum of k polynomial $P_m(\gamma)$, where γ is a triangular number. We show that $\Phi_{(2,k)}(8n+k) = 2^k \Psi_{(2,k)}(n)$ for $1 \leq k \leq 7$. A general relation between the number of representations $\sum_{i=1}^{k} x^{2m}$ and the sum of its associated polynomial of triangular numbers for any degree $m \geq 2$ is given as $\Phi_{(m,k)}(8n+k) = 2^k \Psi_{(m,k)}(n)$.

Introduction

Let m be a positive integer, x_i be an integer and γ_i denote the triangular

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numbers $\gamma_i = \frac{(x_i)(x_i+1)}{2}$, where $i = 1, 2, \ldots, k$. Let $\Phi_{(m,k)}(n)$ and $\Psi_{(m,k)}(n)$ denote the number of representations of a non-negative integer n as a sum of k $2m$th powers and as a sum of k associated polynomials of triangular numbers denoted by $P_m(\gamma)$ of degree m, respectively. In [1], Barrucand et al. gave a relation between $\Phi_{(m,k)}(n)$ and $\Psi_{(m,k)}(n)$ when $m = 1$ and $P_1(\gamma) = \gamma$ as

$$\Phi_{(1,k)}(8n+k) = a_k \Psi_{(1,k)}(n), \text{ where } a_k = 2^{k-1}\left\{1 + \binom{k}{4}\right\}$$

for $1 \leq k \leq 7$. They proved this result by applying generating functions in [1]. Later, a combinatorial proof was given in [3]. Bateman et al. proved in [2] that this result does not hold for any value $k \geq 8$. Here, we give a relation between $\Phi_{(m,k)}(n)$ and $\Psi_{(m,k)}(n)$ when $m \geq 2$.

A Relation between Sum of *k* Fourth Powers and its Associated Polynomial of Triangular Numbers of Degree 2

Let $\Phi_{(2,k)}(n)$ and $\Psi_{(2,k)}(n)$ denote the number of representations of an integer n as $\sum_{i=1}^{k} x_i^4$ and as a sum of k polynomials of the form $8\gamma^2 + 2\gamma$, where γ is a triangular number, respectively. In other words, $\Phi_{(2,k)}(n)$ is the number of solutions in integers of the equation

$$x_1^4 + x_2^4 + \cdots + x_k^4 = n$$

and $\Psi_{(2,k)}(n)$ is the number of solutions in non-negative integers of the equation

$$\sum_{i=1}^{k}\left[8\left(\frac{x_i(x_i+1)}{2}\right)^2 + 2\left(\frac{x_i(x_i+1)}{2}\right)\right] = n.$$

Theorem 1 gives a relation between $\Phi_{(2,k)}(n)$ and $\Psi_{(2,k)}(n)$ for any non-negative integer n and $1 \leq k \leq 7$. The following lemma is needed for the proof of this theorem.

Lemma 1. *Let*

$$\alpha(q) = 1 + 2\sum_{i=1}^{\infty} q^{i^4}$$

and

$$\beta(q) = \sum_{n=0}^{\infty} q^{8\left(\frac{i(i+1)}{2}\right)^2 + 2\left(\frac{i(i+1)}{2}\right)}.$$

Then, we have

$$\alpha(q) + \alpha(-q) = 2\alpha(q^{16})$$

and

$$\alpha(q) - \alpha(-q) = 4q\beta(q^8).$$

Proof.

$$\alpha(q) + \alpha(-q) = \left(1 + 2\sum_{i=1}^{\infty} q^{i^4}\right) + \left(1 + 2\sum_{i=1}^{\infty} (-q)^{i^4}\right)$$

$$= 2 + 2\sum_{i=1}^{\infty} (q^{i^4} + (-q)^{i^4})$$

$$= 2 + 4\sum_{i=1}^{\infty} q^{(2i)^4}$$

$$= 2 + 4\sum_{i=1}^{\infty} q^{16i^4}$$

$$= 2\left[1 + 2\sum_{i=1}^{\infty} (q^{16})^{i^4}\right]$$

$$= 2\alpha(q^{16}),$$

$$\alpha(q) - \alpha(-q) = \left(1 + 2\sum_{i=1}^{\infty} q^{i^4}\right) - \left(1 + 2 = \sum_{i=1}^{\infty} (-q)^{i^4}\right)$$

$$= 2\sum_{i=1}^{\infty} (q^{i^4} - (-q)^{i^4})$$

$$= 4\sum_{i=1}^{\infty} q^{(2i-1)^4}$$

$$= 4\sum_{i=0}^{\infty} q^{(2i+1)^4}$$

$$= 4\sum_{i=0}^{\infty} q^{16i^4 + 32i^3 + 24i^2 + 8i + 1}$$

$$= 4q\sum_{i=0}^{\infty} q^{16i^4 + 32i^3 + 24i^2 + 8i}$$

$$= 4q\sum_{i=0}^{\infty} q^{64\left(\frac{i(i+1)}{2}\right)^2 + 16\left(\frac{i(i+1)}{2}\right)}$$

$$= 4q\beta(q^8). \qquad \square$$

From Lemma 1, we provide a relation between $\Phi_{(2,k)}(n)$ and $\Psi_{(2,k)}(n)$ in the following theorem.

Theorem 1. *For any non-negative integer n,*

$$\Phi_{(2,k)}(8n + k) = 2^k \Psi_{(2,k)}(n), \qquad 1 \le k \le 7.$$

Proof. From Lemma 1, we have

$$\alpha(q) + \alpha(-q) = 2\alpha(q^{16}) \tag{0.1}$$

and

$$\alpha(q) - \alpha(-q) = 4q\beta(q^8). \tag{0.2}$$

Using Equation (0.1) and Equation (0.2), we obtain

$$\alpha(q) = 2q\beta(q^8) + \alpha(q^{16}).$$

Then, we have

$$\begin{aligned}\alpha(q)^k &= (2q\beta(q^8) + \alpha(q^{16}))^k \\ &= \sum_{r=0}^{k} \binom{k}{r} 2^{k-r} q^{k-r} \beta(q^8)^{k-r} \alpha(q^{16})^r \\ &= 2^k q^k \beta(q^8)^k + \sum_{r=1}^{k} \binom{k}{r} 2^{k-r} q^{k-r} \beta(q^8)^{k-r} \alpha(q^{16})^r.\end{aligned}$$

Extracting all terms in which the exponents are congruent to $k(\bmod 8)$, we have

$$\sum_{n=0}^{\infty} \Phi_{(2,\, k)}(8n + k) q^{8n+k} = 2^k q^k \beta(q^8)^k.$$

Followed by dividing by q^k and replacing q^8 by q, we obtain

$$\sum_{n=0}^{\infty} \Phi_{(2,\, k)}(8n + k) q^n = 2^k \beta(q)^k,$$

$$\Phi_{(2,\, k)}(8n + k) = 2^k \Psi_{(2,\, k)}(n). \qquad \square$$

A Relation between $\sum_{i=1}^{k} x_i^{2m}$ and its Associated Polynomial of Triangular Numbers of Degree $m \geq 2$

In this section, we extend our discussion to cases in which $m \geq 2$ and give a general relation between number of representations of a non-negative integer n as $\sum_{i=1}^{k} x_i^{2m}$ and the sum of its associated polynomial of triangular numbers $P_m(\gamma)$ of degree m, where

$$P_m(\gamma) = a_{m,m}\gamma^m + a_{m,m-1}\gamma^{m-1} + \cdots + a_{m,1}\gamma$$

with γ is a triangular number and

$$a_{m,\theta} = 2^{3(\theta-1)}(m-\theta+1) + \sum_{i=m-\theta+1}^{m-1} 2^{3(\theta-i-1)} a_{i,i-(m-\theta-1)} \text{ for } m \geq \theta.$$

In order to show a relation between $\Phi_{(m,k)}(n)$ and $\Psi_{(m,k)}(n)$, the following lemmas are needed.

Lemma 2. *For any positive integer n, we have*

$$(2x+1)^{2n} = b_{n,n}\gamma^n + b_{n,n-1}\gamma^{n-1} + \cdots + b_{n,1}\gamma + 1,$$

where

$$\gamma = \frac{x(x+1)}{2}$$

and

$$b_{n,\theta} = 2^{3\theta}(n-\theta+1) + \sum_{i=n-\theta+1}^{n-1} 2^{3(\theta-i)} b_{i,i-(n-\theta-1)} \textit{ for } 1 \leq \theta \leq n.$$

Proof. We prove the following identity by induction on n:

$$(2x+1)^{2n} = 2^{3n}\gamma^n + \left(2^{3(n-1)}(2) + \sum_{i=2}^{n-1} 2^{3(n-i-1)} b_{i,i}\right)\gamma^{n-1}$$

$$+ \left(2^{3(n-2)}(3) + \sum_{i=3}^{n-1} 2^{3(n-i-2)} b_{i,i-1}\right)\gamma^{n-2}$$

$$+ \cdots + b_{n,1}\gamma + 1,$$

where

$$\gamma = \frac{x(x+1)}{2}.$$

When $n = 1$

$$(2x+1)^2 = b_{1,1}\gamma + 1$$

$$= 2^3\gamma + 1.$$

Assume that assertion is true for $n = k.$ That is

$$(2x+1)^{2k} = 2^{3k}\gamma^k + \left(2^{3(k-1)}(2) + \sum_{i=2}^{k-1} 2^{3(k-i-1)} b_{i,i}\right)\gamma^{k-1}$$

$$+ \left(2^{3(k-2)}(3) + \sum_{i=3}^{k-1} 2^{3(k-i-2)} b_{i,i-1}\right)\gamma^{k-2} + \cdots + b_{k,1}\gamma + 1.$$

When $n = k+1,$

$$(2x+1)^{2(k+1)} = (2x+1)^{2k}(2x+1)^2$$

$$= \left[2^{3k}\gamma^k + \left(2^{3(k-1)}(2) + \sum_{i=2}^{k-1} 2^{3(k-i-1)} b_{i,i}\right)\gamma^{k-1}\right.$$

$$\left. + \left(2^{3(k-2)}(3) + \sum_{i=3}^{k-1} 2^{3(k-i-2)} b_{i,i-1}\right)\gamma^{k-2} + \cdots + b_{k,1}\gamma + 1\right][2^3\gamma + 1]$$

$$= 2^{3(k+1)}\gamma^{k+1} + \left(2^{3(k)}(2) + \sum_{i=2}^{k-1} 2^{3(k-i)} b_{i,i} + 2^{3(k)}\right)\gamma^k$$

$$+ \left(2^{3(k-1)}(3) + \sum_{i=3}^{k-1} 2^{3(k-i-1)} b_{i,i-1} + 2^{3(k-1)}(2) + \sum_{i=2}^{k-1} 2^{3(k-i-1)} b_{i,i}\right)\gamma^{k-1}$$

$$+ \cdots + 2^3(k+1)\gamma + 1$$

$$= 2^{3(k+1)}\gamma^{k+1} + \left(2^{3(k)}(2) + \sum_{i=2}^{k-1} 2^{3(k-i)} b_{i,i} + b_{k,k}\right)\gamma^k$$

$$+ \left(2^{3(k-1)}(3) + \sum_{i=3}^{k-1} 2^{3(k-i-1)} b_{i,i-1} + b_{k,k-1}\right)\gamma^{k-1}$$

$$+ \cdots + 2^3(k+1)\gamma + 1$$

$$= 2^{3(k+1)}\gamma^{k+1} + \left(2^{3(k)}(2) + \sum_{i=2}^{k} 2^{3(k-i)} b_{i,i}\right)\gamma^k$$

$$+ \left(2^{3(k-1)}(3) + \sum_{i=3}^{k-1} 2^{3(k-i-1)} b_{i,i-1}\right)\gamma^{k-1} + \cdots + b_{k+1,1}\gamma + 1.$$

By the above induction, it is clear that the assertion is true for all $n \geq 1$. $\square$

Lemma 3. *Let* $\alpha(q) = 1 + 2\sum_{i=1}^{\infty} q^{i^{2m}}$ *and* $\beta(q) = \sum_{i=0}^{\infty} q^{P_m(\gamma_i)}$. *Then we have*

$$\alpha(q) + \alpha(-q) = 2\alpha(q^{2^{2m}})$$

and

$$\alpha(q) - \alpha(-q) = 4q\beta(q^8).$$

Proof.

$$\alpha(q) + \alpha(-q) = \left(1 + 2\sum_{i=1}^{\infty} q^{i^{2m}}\right) + \left(1 + 2\sum_{i=1}^{\infty} (-q)^{i^{2m}}\right).$$

In the summations $\sum_{i=1}^{\infty} q^{i^{2m}}$ and $\sum_{i=1}^{\infty} (-q)^{i^{2m}}$, clearly $q^{i^{2m}} = (-q)^{i^{2m}}$ for even values of *i* and will cancel out for odd values of *i*. Hence

$$\alpha(q) + \alpha(-q) = 2 + 2\sum_{i=1}^{\infty} (q^{i^{2m}} + (-q)^{i^{2m}})$$

$$= 2 + 4\sum_{i=1}^{\infty} q^{(2i)^{2m}}$$

$$= 2 + 4\sum_{i=1}^{\infty} q^{2^{2m}(i^{2m})}$$

$$= 2\left[1 + 2\sum_{i=1}^{\infty} (q^{2^{2m}})^{(i^{2m})}\right]$$

$$= 2\alpha(q^{2^{2m}}),$$

$$\alpha(q) - \alpha(-q) = \left(1 + 2\sum_{i=1}^{\infty} q^{i^{2m}}\right) - \left(1 + 2\sum_{i=1}^{\infty} (-q)^{i^{2m}}\right).$$

In the subtraction $\sum_{i=1}^{\infty} q^{i^{2m}}$ and $\sum_{i=1}^{\infty} (-q)^{i^{2m}}$, clearly $q^{i^{2m}} = (-q)^{i^{2m}}$ for even values of *i* and will cancel out when *i* is even. Hence

$$\alpha(q) - \alpha(-q) = 2\sum_{i=1}^{\infty} (q^{i^{2m}} - (-q)^{i^{2m}})$$

$$= 4\sum_{i=1}^{\infty} q^{(2i-1)^{2m}}$$

$$= 4\sum_{i=0}^{\infty} q^{(2i+1)^{2m}}.$$

From Lemma 2, we have

$$\alpha(q) - \alpha(-q) = 4\sum_{i=0}^{\infty} q^{b_{m,m}\gamma_i^m + b_{m,m-1}\gamma_i^{m-1} + \cdots + b_{m,1}\gamma_i + 1}$$

$$= 4q\sum_{i=0}^{\infty} q^{b_{m,m}\gamma_i^m + b_{m,m-1}\gamma_i^{m-1} + \cdots + b_{m,1}\gamma_i}$$

$$= 4q\sum_{i=0}^{\infty} q^{2^3(a_{m,m}\gamma_i^m + a_{m,m-1}\gamma_i^{m-1} + \cdots + a_{m,1}\gamma_i)}$$

$$= 4q\beta(q^8). \qquad \square$$

By applying Lemma 3, the relation between $\Phi_{(m,k)}(n)$ and $\Psi_{(m,k)}(n)$ for $m \geq 2$ is obtained as in the following theorem.

Theorem 2. *For any non-negative integer n*

$$\Phi_{(m,k)}(8n+k) = 2^k \Psi_{(m,k)}(n), \quad 1 \leq k \leq 7.$$

Proof. From Lemma 3, we have

$$\alpha(q) + \alpha(-q) = 2\alpha(q^{2^{2m}}) \tag{0.3}$$

and

$$\alpha(q) - \alpha(-q) = 4q\beta(q^8). \tag{0.4}$$

By applying Equation (0.3) and Equation (0.4), we obtain

$$\alpha(q) = 2q\beta(q^8) + \alpha(q^{2^{2m}}).$$

It follows that

$$\begin{aligned}
\alpha(q)^k &= (2q\beta(q^8) + \alpha(q^{2^{2m}}))^k \\
&= \sum_{r=0}^{k} \binom{k}{r} 2^{k-r} q^{k-r} \beta(q^8)^{k-r} \alpha(q^{2^{2m}})^r \\
&= 2^k q^k \beta(q^8)^k + \sum_{r=1}^{k} \binom{k}{r} 2^{k-r} q^{k-r} \beta(q^8)^{k-r} \alpha(q^{2^{2m}})^r.
\end{aligned}$$

For $m \geq 2$, extracting all terms in which the exponents are congruent to $k(\mathrm{mod}\, 8)$, we obtain

$$\sum_{n=0}^{\infty} \Phi_{(m,k)}(8n+k) q 8n + k = 2^k q^k \beta(q^8)^k.$$

Dividing by q^k and replacing q^8 by q, we have

$$\sum_{n=0}^{\infty} \Phi_{(m,k)}(8n+k)q^n = 2^k \beta(q)^k,$$

$$\Phi_{(m,k)}(8n+k) = 2^k \Psi_{(m,k)}(n). \qquad \square$$

Conclusion

In this paper, a general relation between the number of representations of non-negative integer n as a $\sum_{i=1}^{k} x_i^{2m}$ and as a sum of its associated polynomial of triangular numbers $P_m(\gamma)$ is given by $\Phi_{(m,k)}(8n+k) = 2^k \Psi_{(m,k)}(n)$ when $m \geq 2$ and $1 \leq k \leq 7$.

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