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EXCITING FORCES AND HYDRODYNAMIC COEFFICIENTS DUE TO RADIATION FOR A SPHERE IN WATER OF FINITE DEPTH

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Abstract

The evaluation of hydrodynamic coefficients and loads on submerged bodies has a lot of significance in designing these structures. Analytical expressions for the exciting forces, added-mass and damping coefficients due to the effects of diffraction and radiation arising out of interaction of water waves with a submerged sphere are derived. Theory of multi-pole expansions is used in obtaining the velocity potential in terms of an infinite series of associated Legendre functions with unknown coefficients. Two motions, namely surge and heave motions, are considered. Numerical results for the various expressions are presented for various depth to radius ratios.

1. Introduction

The forces exerted by the surface waves on a structure in water are very important for designing these structures. Accurate prediction of wave loads becomes indispensable in order to design safe structures. The researchers have been trying to evaluate the various loads and coefficients associated with the interaction of water waves with a submerged sphere. The motions of a floating or submerged body are

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influenced by the added-mass effect in the water and the damping introduced by the motion of the structure.

A number of notable works have been done over the last few decades on analytical solutions for the linear or first order forces acting on a floating or submerged body of spherical, hemispherical or spheroidal shape in water. Havelock [3] can be considered the pioneer in the area of hydrodynamic loading on spherical structures. He started with calculating the wave resistance of a submerged spheroid by replacing it with a distribution of sources and sinks, or of doublets. Gray [2] considered a fully submerged, rigid, stationary sphere reducing the problem to the solution of an infinite set of linear algebraic equations for the expansion coefficients in spherical harmonics of the velocity potential.

Hulme [4] considered the heave and surge motions of a floating hemisphere to derive added-mass and damping coefficients associated with the periodic motions. Wang [9] discussed the free motions of submerged vehicle with spherical hull form but with different metacentric heights. The works of Hulme and Wang were based on the multi-pole expansions of Thorne [8] which proves to be very successful for periodic motions without forward speed but this method did not seem to be applicable to the problem of a body with forward speed. As forward speed significantly affects the body-surface, free-surface and radiation boundary conditions imposed on the velocity potential corresponding to the oscillations of the body, Wu and Eatock Taylor [10] have discussed the hydrodynamic problems of a submerged spheroid in waves based on linearized potential theory and in spheroidal coordinates. But these solutions could not be extended for the cases with forward speed.

Recent mathematical developments in potential theories applied to diffraction and radiation of water waves by the floating or submerged spherical bodies can be found in the work of Bora et al. [1]. Based on the linear diffraction theory, Rahman [6] has presented a simulation of diffraction of ocean waves by a submerged sphere in finite depth. The method of multi-pole expansions is used to obtain the fluid velocity potential in the form of double series of the associated Legendre functions with the unknown coefficients of the infinite set of infinite matrix equations.

In this paper we have presented an analytical procedure for the boundary value problem to evaluate the hydrodynamic coefficients and motions for a submerged sphere in finite depth due to surge and heave motions. We have considered the boundary value problem to consist of two distinct problems: the diffraction problem and the radiation problem. The diffraction velocity potential is expressed in terms of an infinite series of associated Legendre functions with unknown coefficients. Using the body boundary condition, we set up a linear system of equations. By solving the linear system, we find the velocity potential and hence the exciting forces along horizontal and vertical directions are evaluated. Similarly the radiation potentials due to the surge and heave motions are found using the same technique and that helps us to evaluate the coefficients related with the radiation problem. The analytical expressions for the added-mass and damping coefficients due to surge and heave motions are computed for different depth to radius ratios. These results are displayed in tabular and graphical forms.

2. Mathematical Formulation

We assume that the fluid is homogeneous, inviscid and incompressible and the fluid motion is irrotational. The waves are also assumed to be of small amplitude. Here we consider the coefficients related to the motion with two degrees of freedom, namely, the two translational motions in the x and z directions, i.e., surge and heave motions, respectively. We consider a surface wave of amplitude A incident on a sphere of radius a submerged in water of finite depth d. The body is assumed to have motions with three degrees of freedom in the presence of incident wave with angular frequency σ. The wave is parallel to x-axis at the time of incidence on the sphere and is propagating along the positive direction.

We consider two sets of coordinate systems. One is a right-handed Cartesian coordinate system (x, y, z), in which the x-y plane coincides with the undisturbed free surface and the z-axis is taken vertically downwards from the Still Water Level (SWL). The other coordinate system is the spherical coordinate system (r, θ, ψ) with the origin at the geometric centre of the sphere. The depiction in Figure 1 shows the axes systems along a sphere of radius a in water of depth d with its geometric centre located at $(0, 0, h)$ with respect to the Cartesian coordinate system, and $H = d - h$ is the depth below the centreline of the sphere.

The relationship between the coordinate systems is

$$R = \sqrt{x^2 + y^2}$$

$$r = \sqrt{R^2 + (z - h)^2}$$

$$\tan\theta = \frac{R}{z - h} \text{ for } 0 \le \theta \le \pi$$

$$\tan\psi = \frac{y}{x} \text{ for } -\pi \le \psi \le \pi.$$

For an incompressible and inviscid fluid, and for small amplitude wave theory with irrotational motion, we can express the fluid motion by introducing a velocity potential $\Phi(r, \theta, \psi, t)$. This Φ can be written as

$$\Phi(r, \theta, \psi, t) = \text{Re}[\phi(r, \theta, \psi)e^{-i\sigma t}], \tag{1}$$

where Re stands for the real part.

The motion is assumed harmonic. Also, from Bernoulli's equation, we get pressure, $P(r, \theta, \psi, t)$ as

$$P = -\rho \frac{\partial \Phi}{\partial t}. \tag{2}$$

Now, the problem can be considered as a combination of two fundamental problems: the diffraction problem of an incident wave interacting with a fixed body; and the radiation problem of a body forced to oscillate in otherwise still water. Because of the linearity of the situation, the time-independent velocity potential $\phi(r, \theta, \psi)$ can be decomposed into four velocity potentials ϕ_I, ϕ_D, ϕ_1 and ϕ_3, where ϕ_I is the incident potential, ϕ_D is the velocity potential due to the diffraction of an incident wave acting on the sphere; and ϕ_1 and ϕ_3 are the velocity potentials due to the radiation of surge and heave, respectively.

Thus ϕ can be written as $\phi = \phi_I + \phi_D + X_1\phi_1 + X_3\phi_3$, where X_1 and X_2 are the displacements for surge and heave motions, respectively. Here

$\phi_I, \phi_D, \phi_j, j = 1, 3$ are all functions of r, θ and ψ and $X_j, j = 1, 3$ is the independent parameter.

To obtain the velocity potential ϕ, the following boundary problem is to be solved:

(I) Laplace's equation in spherical coordinates:

$$\nabla^2 \phi = 0. \tag{3}$$

(II) Free surface condition:

$$\frac{\partial \phi}{\partial z} + K\phi = 0 \text{ on } z = 0. \tag{4}$$

(III) Bottom boundary condition:

$$\frac{\partial \phi}{\partial z} = 0, \quad z = d. \tag{5}$$

(IV) Radiation condition:

$$\lim_{R \to \infty} \sqrt{R}\left(\frac{\partial}{\partial R} - ik_0\right)\phi = 0, \tag{6}$$

where $K = \frac{\sigma^2}{g}$ and k_0 is the finite depth wave number defined by

$$k_0 \sinh k_0 d - K \cosh k_0 d = 0 \tag{7}$$

and the incident and diffraction potentials satisfy the body surface condition

$$\frac{\partial \phi_I}{\partial \mathbf{n}} = -\frac{\partial \phi_D}{\partial \mathbf{n}} \text{ on } r = a, \tag{8}$$

where **n** denotes the normal vector from body surface to fluid.

The radiation potentials satisfy the body surface condition

(a) for surge motion:

$$\frac{\partial \phi}{\partial r} = i\sigma \sin\theta \cos\psi \text{ on } r = a \tag{9}$$

(b) for heave motion:

$$\frac{\partial \phi}{\partial r} = i\sigma \cos\theta \text{ on } r = a. \tag{10}$$

The boundary conditions (9) and (10) have arisen from the equation

$$\frac{\partial \phi_j}{\partial \mathbf{n}} = (-i\sigma) n_j, \quad j = 1, 3. \tag{11}$$

2.1. Incident potential

The incoming waves of amplitude A and frequency σ propagating in the positive x-direction can be described by the following incident velocity potential:

$$\phi_I = \frac{Ag}{\sigma} \frac{\cosh k_0 (z-d)}{\cosh k_0 d} e^{ik_0 R \cos\psi}. \tag{12}$$

Using McLachlan [5] and Thorne [8], the incident potential can be expressed in terms of associated Legendre function as

$$\phi_I = \frac{Ag}{2\sigma \cosh k_0 d} \sum_{m=0}^{\infty} \varepsilon_m i^m \cos m\psi$$

$$\times \sum_{s=m}^{\infty} \{(-1)^{s+m} e^{k_0(d-h)} + e^{k_0(h-d)}\} \frac{(k_0 r)^s}{(s+m)!} P_s^m(\cos\theta), \tag{13}$$

where $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ for $m \geq 1$ or we can write for our convenience,

$$\phi_I(r, \theta, \psi) = \sum_{m=0}^{\infty} \hat{\phi}_I(r, \theta) \cos m\psi, \tag{14}$$

where

$$\hat{\phi}_I(r, \theta) = \frac{Ag}{\sigma} \varepsilon_m i^m \sum_{s=0}^{\infty} \chi_s \frac{(k_0 r)^{s+m}}{(s+2m)!} P_{s+m}^m(\cos\theta) \tag{15}$$

with

$$\chi_s = \frac{(-1)^s e^{k_0(d-h)} + e^{-k_0(d-h)}}{2\cosh k_0 d}$$

$$= \begin{cases} \dfrac{\cosh k_0(d-h)}{\cosh k_0 d}, & s = 0, 2, 4, 6, \ldots, \\ -\dfrac{\sinh k_0(d-h)}{\cosh k_0 d}, & s = 1, 3, 5, \ldots. \end{cases} \tag{16}$$

2.2. Diffraction potential

The diffraction velocity potential ϕ_D satisfies equations (3)-(6) and (8). We can express this potential by making it ψ-independent as

$$\phi_D(r, \theta, \psi) = \sum_{m=0}^{\infty} \hat{\phi}_D(r, \theta) \cos m\psi, \tag{17}$$

where the ψ-independent potential is

$$\hat{\phi}_D(r, \theta) = \sum_{n=m}^{\infty} a^{n+2} A_{mn} G_n^m. \tag{18}$$

Here A_{mn} are the unknown complex coefficients and G_n^m are the multi-pole potentials. Multi-pole potentials are solutions of Laplace's equation which satisfy the free surface and bottom boundary conditions and behave like outgoing waves from the singular point which in this case is the centre of the sphere.

G_n^m can be expressed as

$$G_n^m = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{P_n^m(\cos\alpha)}{r_1^{n+1}} + \frac{1}{(n-m)!}$$
$$\times \int_0^{\infty} \frac{(K+k)\left[e^{-k(d+H)} + (-1)^{n+m} e^{-kh}\right]}{k \sinh kd - K \cosh kd} k^n \cosh k(z-d) J_m(kR) dk. \tag{19}$$

The quantities α and r_1 are defined as

$$r_1 = \sqrt{R^2 + (d+H-z)^2}$$
$$\tan\alpha = \frac{R}{d+H-z},$$

where R, d and H have already been defined.

The line integration in the expression for G_n^m passes under the singular point of the integrand at $k = k_0$. The potentials G_n^m and ϕ_D

satisfy Laplace's equation, free surface condition, bottom surface condition and the radiation condition.

The second and third terms in equation (19) can be expanded in the region near the body surface into a series of associated Legendre functions by

$$\frac{P_n^m(\cos\alpha)}{r_1^{n+1}} = \sum_{s=0}^{\infty} B_{ns}^m \left(\frac{r}{2H}\right)^{s+m} P_{s+m}^m(\cos\theta) \tag{20}$$

and

$$\frac{1}{(n-m)!}\int_0^\infty \frac{(K+k)[e^{-k(d+H)} + (-1)^{n+m}e^{-kh}]}{k\sinh kd - K\cosh kd} k^n \cosh k(z-d) J_m(kR)dk$$

$$= \sum_{s=0}^{\infty} C_s(n,\ m)\left(\frac{r}{2H}\right)^{s+m} P_{s+m}^m(\cos\theta), \tag{21}$$

where B_{ns}^m and $C_s(n,\ m)$ are given by

$$B_{ns}^m = \frac{1}{(2H)^{n+1}} \frac{(s+n+m)!}{(s+2m)!(n-m)!} \tag{22}$$

$C_s(n,\ m)$

$$= \frac{(2H)^{s+m}}{(n-m)!(s+2m)!}\int_0^\infty \frac{(K+k)[e^{-k(d+H)} + (-1)^{n+m}e^{-kh}]}{k\sinh kd - K\cosh kd} u_s(kH)dk \tag{23}$$

with $u_s(kH)$ as

$$u_s(kH) = \begin{cases} \cosh kH, & s = 0,\ 2,\ 4,\ \ldots, \\ -\sinh kH, & s = 1,\ 3,\ 5,\ \ldots. \end{cases} \tag{24}$$

Hence the multi-pole potentials G_n^m can finally be written as

$$G_n^m = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{s=0}^{\infty}[B_{ns}^m + C_s(n,\ m)]\left(\frac{r}{2H}\right)^{s+m} P_{s+m}^m(\cos\theta). \tag{25}$$

Now the body boundary condition (8) becomes equivalent to

$$\sum_{n=m}^{\infty} a^{n+2} A_{mn} \frac{\partial G_n^m}{\partial r}\bigg|_{r=a} = -\frac{\partial \hat{\phi}_I}{\partial r}\bigg|_{r=a}. \tag{26}$$

From the expressions for G_n^m and $\hat{\phi}_I$ from equations (25) and (15) respectively and using the orthogonality property of associated Legendre functions, we arrive at

$$\sum_{n=m}^{\infty} A_{mn} E_{ns}^m = T_s^m \text{ for } s = m, m+1, m+2, \ldots, \tag{27}$$

where

$$T_s^m = -\frac{Agk_0}{\sigma} \varepsilon_m i^m (k_0 a)^{s-1} \frac{s}{(s+m)!} \chi_{s-m} \tag{28}$$

$$E_{ns}^m = -(n+1)\delta_{ns} + D_n^m(s-m) \tag{29}$$

$$D_n^m(s) = a^{n+1}(s+m)\left(\frac{a}{2H}\right)^{s+m} [C_s(n, m) + B_{ns}^m]. \tag{30}$$

Equation (27) is a complex matrix equation in the unknowns A_{mn}. Since the infinite series appearing in equations (28) and (30) have excellent truncation property, the infinite matrices can be truncated at a certain term to solve equation (27) numerically. Commercially available complex matrix inversion routines are used to obtain the solution of the modified equation. Once these coefficients are known, the diffraction problem is completely known.

3. Exciting Forces

The forces associated with the incident and diffraction potentials are the exciting forces which play a very important role in the wave field for a structure in water. The exciting forces $F_j^{(e)}$ can be obtained from

$$F_j^{(e)} = 2i\rho a^2 \sigma A \int_0^{\pi} \int_0^{\pi} \phi_{ID}\,|_{r=a}\; n_j \sin\theta d\theta d\psi, \tag{31}$$

where $j = 0$ corresponds to heave motion and $j = 1$ corresponds to surge motion and we have written $\phi_{ID} = \phi_I + \phi_D$,

$$n_j = -P_1^j(\cos\theta)\cos j\psi, \quad j = 0, 1. \tag{32}$$

Applying the body surface condition $\frac{\partial\phi_D}{\partial r} = -\frac{\partial\phi_I}{\partial r}$ at $r = a$ and after some simplifications, we have

$$\phi_{ID}|_{r=a} = a\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{2n+1}{n}A_{mn}P_n^m(\cos\theta)\cos m\psi. \tag{33}$$

Now, the exciting forces are given by

$$F_j^{(e)} = -\frac{2i\rho\sigma a^2 A\pi}{\varepsilon_j}\int_0^{\pi}\sum_{n=j}^{\infty}a\frac{2n+1}{n}A_{jn}P_n^j(\cos\theta)\sin\theta d\theta, \tag{34}$$

where $\varepsilon_j = 1$ for $j = 0$, $\varepsilon_j = 2$ for $j \geq 1$.

Using the orthogonality property of associated Legendre functions,

$$F_j^{(e)} = -4i\rho\sigma\pi a^3 AA_{j1} \tag{35}$$

since the terms ε_j and $\frac{(1+j)!}{(1-j)!}$ cancel out for the respective values of j.

Hence the surge exciting force $F_x^{(e)} = f_{xd}$ and the heave exciting force $F_z^{(e)} = f_{zd}$ are given by

$$\frac{f_{xd}}{4i\rho\sigma A\pi a^3} = -A_{11} \tag{36}$$

and

$$\frac{f_{zd}}{4i\rho\sigma A\pi a^3} = -A_{01}. \tag{37}$$

4. Radiation Potentials

Having solved the diffraction problem for the submerged sphere, now we turn our attention to the radiation problem. As mentioned earlier we will consider surge and heave potentials only. Both these potentials mainly satisfy the same set of equations except for the body boundary condition which is different for each motion. Both being related with translational motions, surge and heave potentials have resemblance in their expressions.

The radiation velocity potential ϕ_m must satisfy

$$\nabla^2 \phi_m = 0 \text{ in the fluid} \tag{38}$$

$$\frac{\partial \phi_m}{\partial z} + k\phi_m = 0 \text{ on } z = 0 \tag{39}$$

$$\frac{\partial \phi_m}{\partial z} = 0 \text{ on } z = d \tag{40}$$

$$\frac{\partial \phi_m}{\partial r} = (-i\sigma)n_j, \quad j = 1, 3 \text{ on } r = a \tag{41}$$

$$\lim_{R \to \infty} R^{\frac{1}{2}} \left\{ \frac{\partial}{\partial R} - ik \right\} \phi_m = 0. \tag{42}$$

The kinematic boundary condition on the body surface for the radiation problem in case of surge and heave motions can be written as

$$\frac{\partial \phi_m}{\partial r} = i\sigma P_1^m (\cos \theta) \cos m\psi, \tag{43}$$

$m = 0$ and $m = 1$ correspond to heave and surge motion, respectively.

The ψ-dependence of ϕ_m can be removed by assuming

$$\phi_m(r, \theta, \psi) = \hat{\phi}_m(r, \theta) \cos m\psi. \tag{44}$$

The velocity potential $\hat{\phi}_m(r, \theta)$ will be expanded in multi-pole potentials which have already been discussed while dealing with the diffraction potential. Now, removing the time dependence term,

$$\hat{\phi}_m(r, \theta) = \frac{P_n^m(\cos \theta)}{r^{n+1}} + \frac{(-1)^{n+m-1}}{(n-m)!} \int_0^\infty \frac{K + k}{K - k} k^n e^{-k(z+d)} J_m(kR) dk$$

$$+ i \frac{(-1)^{m+n}}{(n-m)!} 2\pi K^{n+1} e^{-k(z+d)} J_m(KR), \tag{45}$$

$\hat{\phi}_m$ can be finally expressed as

$$\hat{\phi}_m(r, \theta) = \frac{P_n^m(\cos\theta)}{r^{n+1}}$$

$$+ \sum_{s=m}^{\infty} \frac{(-1)^{m+s-1}}{(n-m)!(s+m)!} r^s P_s^m(\cos\theta) PV \int_0^{\infty} \frac{K+k}{K-k} k^{n+s} e^{-2kd} dk$$

$$+ i \sum_{s=m}^{\infty} \frac{(-1)^{n+s}}{(n-m)!(s+m)!} 2\pi K^{n+s+1} e^{-2Kd} r^s P_s^m(\cos\theta), \tag{46}$$

where PV means the principal value of the integral is to be considered. Or we can write $\hat{\phi}_m$ as

$$\hat{\phi}_m(r, \theta) = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{s=m}^{\infty} [A_s^m + iB_s^m] r^s P_s^m(\cos\theta), \tag{47}$$

where

$$A_s^m = \frac{(-1)^{m+s-1}}{(n-m)!(s+m)!} \int_0^{\infty} \frac{K+k}{K-k} k^{n+s} e^{-2kd} dk, \tag{48}$$

$$B_s^m = \frac{(-1)^{n+s}}{(n-m)!(s+m)!} 2\pi K^{n+s+1} e^{-2Kd}. \tag{49}$$

Hence the radiation potential ϕ_m can be written as

$$\phi_m(r, \theta, \psi) = \left[\frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{s=m}^{\infty} (A_s^m + iB_s^m) r^s P_s^m(\cos\theta) \right] \cos m\psi. \tag{50}$$

Applying the body boundary condition, simplifying and using the orthogonality of associated Legendre functions, we obtain

$$\sum_{n=m}^{\infty} \frac{2na^{n-1}}{2n+1} \frac{(n+m)!}{(n-m)!} (A_n^m + iB_n^m)$$

$$= \frac{2(n+1)}{(2n+1)a^{n+2}} \frac{(n+m)!}{(n-m)!} + \frac{2i\sigma}{3} \frac{(1+m)!}{(1-m)!}, \quad m = 0, 1 \tag{51}$$

which is an infinite system of linear algebraic equations in an infinite number of unknowns. Solution of these will enable us to find the radiation potentials and subsequently the surge and heave hydrodynamic coefficients.

We can also write $A_n^m + iB_n^m = D_n^m$ to have complex coefficients D_n^m. Then, we can rewrite equation (51) as

$$\sum_{n=m}^{\infty} \frac{2na^{n-1}}{2n+1} \frac{(n+m)!}{(n-m)!} D_n^m$$

$$= \frac{2(n+1)}{(2n+1)a^{n+2}} \frac{(n+m)!}{(n-m)!} + \frac{2i\sigma}{3} \frac{(1+m)!}{(1-m)!}, \quad m = 0, 1 \tag{52}$$

or we can equate real and imaginary parts from equation (51),

$$\sum_{n=m}^{\infty} \frac{2na^{n-1}}{2n+1} \frac{(n+m)!}{(n-m)!} A_n^m = \frac{2(n+1)}{(2n+1)a^{n+2}} \frac{(n+m)!}{(n-m)!} \tag{53}$$

$$\sum_{n=m}^{\infty} \frac{2na^{n-1}}{2n+1} \frac{(n+m)!}{(n-m)!} B_n^m = \frac{2}{3} \sigma \frac{(1+m)!}{(1-m)!}. \tag{54}$$

5. Determination of Hydrodynamic Coefficients and Motion

The coefficients related with the radiation play an important role in allowing us to know the impact of motions due to radiation. The evaluation of added-mass and damping coefficients is of utmost importance in analyzing the contribution of radiation to the total boundary value problem.

5.1. Surge hydrodynamic coefficients

From Sarpkaya and Isaacson [7], the components of the radiated force can be written as

$$F_i^{(R)} = -\sum_j \left(\mu_{ij} \frac{\partial^2 X_j}{\partial t^2} + \lambda_{ij} \frac{\partial X_j}{\partial t} \right), \tag{55}$$

where μ_{ij} and λ_{ij} are respectively called the *added-mass* and *damping coefficients*. Those coefficients are taken to be real.

The equation of motion can be written as

$$(M_{ij} + \mu_{ij})\frac{\partial^2 X_j}{\partial t^2} + \lambda_{ij}\frac{\partial X_j}{\partial t} + C_{ij}X_j = F_i^{(e)}, \tag{56}$$

where M_{ij} is the mass matrix, C_{ij} is the hydrodynamic stiffness matrix and $F_i^{(e)}$ are the exciting forces associated with the diffraction potential.

The radiated force F_{r1} due to the surge motion can be written as the real part of $f_{r1}e^{-i\sigma t}$, where f_{r1} is given by

$$f_{r1} = -2i\rho a^2 \sigma A \int_0^{\pi}\int_0^{\pi} \hat{X}_1 \phi_1(a, \theta, \psi)\sin^2\theta\cos\psi d\theta d\psi. \tag{57}$$

This radiated force can be conveniently decomposed into components in phase with the velocity and the acceleration,

$$F_{r1} = -\left(\mu_{11}\frac{\partial^2 X_1}{\partial t^2} + \lambda_{11}\frac{\partial X_1}{\partial t}\right). \tag{58}$$

Also, $X_1 = \mathrm{Re}\{\hat{X}_1 e^{-i\sigma t}\}$. Hence, we can write

$$F_{r1} = \mathrm{Re}\{\sigma^2\mu_{11}\hat{X}_1 + i\sigma\lambda\hat{X}_1\}. \tag{59}$$

From equations (58) and (59) we can write

$$\mu_{11} + i\frac{\lambda_{11}}{\sigma} = -\frac{2i\rho a^2 A}{\sigma}\int_0^{\pi}\int_0^{\pi}\phi_1(a, \theta, \psi)\sin^2\theta\cos\psi d\theta d\psi. \tag{60}$$

Hence, the added-mass and damping coefficients are respectively given by

$$\mu_{11} = -\frac{2\rho A a^2}{\sigma}\int_0^{\pi}\int_0^{\pi}\mathrm{Re}[i\phi_1(a, \theta, \psi)]\sin^2\theta\cos\psi d\theta d\psi \tag{61}$$

$$\lambda_{11} = -2\rho A a^2\int_0^{\pi}\int_0^{\pi}\mathrm{Im}[i\phi_1(a, \theta, \psi)]\sin^2\theta\cos\psi d\theta d\psi. \tag{62}$$

The surge potential $\phi_1(r, \theta, \psi)$ can be written from equation (50) and at $r = a$, $(m = 1)$,

$$\phi_1(a, \theta, \psi) = \left[\frac{P_n^1(\cos\theta)}{a^{n+1}} + \sum_{n=1}^{\infty} D_n^1 a^n P_n^1(\cos\theta)\right]\cos\psi. \tag{63}$$

Hence using equation (63) in equations (61) and (62) and simplifying by the use of associated Legendre functions, we obtain the added-mass and damping coefficients as

$$\mu_{11} = \frac{4}{3}\frac{\rho a^3 \pi A}{\sigma} B_1^1 \tag{64}$$

$$\lambda_{11} = -\frac{4}{3}\rho\pi A[1 + A_1^1 a^3]. \tag{65}$$

Or else we can represent μ_{11} and λ_{11} as

$$\frac{\mu_{11}}{\frac{4}{3}\frac{\rho a^3 \pi A}{\sigma}} = B_1^1$$

and

$$\frac{\lambda_{11}}{\frac{4}{3}\rho\pi A} = -[1 + A_1^1 a^3].$$

5.2. Heave hydrodynamic coefficients

The radiated force F_{r3} due to the heave motion can be written as the real part of $f_{r3}e^{-i\sigma t}$, where f_{r3} is given by

$$f_{r3} = -2i\rho a^2 A\sigma \int_0^{\pi}\int_0^{\pi} \hat{X}_3 \phi_3(a, \theta, \psi)\sin\theta\cos\theta d\theta d\psi. \tag{66}$$

Considering $X_3 = \mathrm{Re}\{\hat{X}_3 e^{-i\sigma t}\}$, we have, proceeding as in the previous subsection,

$$\mu_{33} + i\frac{\lambda_{33}}{\sigma} = -\frac{2i\rho A a^2}{\sigma}\int_0^{\pi}\int_0^{\pi} \phi_3(a, \theta, \psi)\sin\theta\cos\theta d\theta d\psi, \tag{67}$$

where μ_{33} and λ_{33} are the heave added-mass and damping coefficient due to heave motion, respectively. Hence,

$$\mu_{33} = -\frac{2\rho A a^2}{\sigma}\int_0^{\pi}\int_0^{\pi} \mathrm{Re}[i\phi_3(a, \theta, \psi)]\sin\theta\cos\theta d\theta d\psi \tag{68}$$

$$\lambda_{33} = -2\rho A a^2 \int_0^{\pi}\int_0^{\pi} \mathrm{Im}[i\phi_3(a, \theta, \psi)]\sin\theta\cos\theta d\theta d\psi. \tag{69}$$

The heave potential $\phi_3(r, \theta, \psi)$ at $r = a$ is $(m = 0)$,

$$\phi_3(a, \theta, \psi) = \frac{P_n^0(\cos\theta)}{a^{n+1}} + \sum_{n=0}^{\infty} D_n^0 a^n P_n^0(\cos\theta). \tag{70}$$

Therefore, proceeding similarly as in the previous subsection, we obtain the coefficients as

$$\mu_{33} = \frac{4}{3}\frac{\rho a^3 \pi A}{\sigma} B_1^0 \tag{71}$$

and

$$\lambda_{33} = -\frac{4}{3}\rho\pi A(1 + A_1^0 a^3). \tag{72}$$

Or else we can represent μ_{33} and λ_{33} as

$$\frac{\mu_{33}}{\frac{4}{3}\frac{\rho a^3 A\pi}{\sigma}} = B_1^0$$

and

$$\frac{\lambda_{33}}{\frac{4}{3}\rho A\pi} = -[1 + A_1^0 a^3].$$

6. Results and Conclusions

In this section we present the results of the analytical expressions for the exciting forces, added-mass and damping coefficients due to surge and heave motions. The complex matrix equation (27) is to be solved in order to determine the unknown coefficients A_{mn} for $m = 0$ and $m = 1$. To compute the horizontal exciting force, f_{xd}, we need to solve the equation (36) and the vertical exciting force, f_{zd}, is evaluated by solving equation (37). This infinite system of equations represented by equation (27) is made finite by truncating as

$$\sum_{n=0}^{N_p} A_{mn} E_{ns}^m = T_s^m, \tag{73}$$

where E_{ns}^m and T_s^m are respectively given by equations (28) and (29).

To compute the hydrodynamic coefficients due to the surge and heave motions, we need to find the coefficients $D_n^m = A_n^m + iB_n^m$ from equation (52). Once the coefficients A_n^m and B_n^m are found, then the surge and heave added-mass and the damping coefficients can be computed.

Tables 1-4 give us the exciting force coefficients for both fixed submergence and fixed depth. The results are compared with the results of Wang [9] and they agree with those sets of results. The first two tables present the surge and heave exciting forces for a fixed submergence $h/a = 1.25$ for various depths, e.g., $d/a = 2.5$, 3.0, 5.0, 11.0 and $d/a = 20.0$. Tables 3 and 4 present us the surge and heave exciting forces for a fixed depth $d/a = 6.0$ but for a set of different submergence value. Figures 2-5 represent those results in graphical forms.

Tables 5-8 give us the results for the added-mass and damping coefficients for both surge and heave motions for different submergence values. These results show good agreement with those obtained by Wang [9]. The added-mass μ_{11} and μ_{33} steadily decrease after reaching the maximum values in the range $0.4 \leq Ka \leq 0.5$. The damping coefficients start from zero and then decrease uniformly to zero again. Also, the damping coefficients are smaller compared to the added-mass for all the submergence values. These numerical behaviours are confirmed by the graphical plots in Figures 6-9.

The work is motivated by the need for analytical solutions for the exciting forces, the added-mass and the damping coefficients. It has been shown that the body submergence and depth have influence on the forces and the coefficients. The use of associated Legendre functions reduces the solutions to simpler forms. The pitch motion due to the radiation was not considered as for a spherical body the moment acting on it becomes automatically zero. It will be interesting to extend the investigation further to consider two or more submerged spheres. The analysis of interaction among several structures nearer to each other would help in many practical cases.

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Table 1. Surge exciting force $(h/a = 1.25)$

	d/a				
Ka	2.5	3.0	5.0	11.0	20.0
0.10	3.1539	2.7864	2.1872	1.5893	1.4897
0.20	2.1152	2.1152	1.5902	1.3151	1.2621
0.30	1.6347	1.3976	1.1861	1.1361	1.1102
0.40	1.2862	1.1471	0.9861	0.9858	0.9826
0.50	1.1134	0.9876	0.8862	0.8852	0.8834
0.60	0.9217	0.8692	0.8682	0.7809	0.8124
0.70	0.7692	0.7418	0.7398	0.6947	0.7395
0.80	0.6824	0.6675	0.6482	0.6345	0.6315
0.90	0.5824	0.5791	0.5789	0.5786	0.5785
1.00	0.5037	0.4981	0.4925	0.4911	0.4901
1.20	0.3476	0.3403	0.3391	0.3379	0.3377

Table 2. Heave exciting force $(h/a = 1.25)$

	d/a				
Ka	2.50	3.00	5.00	11.00	20.00
0.10	0.8241	0.9582	1.2041	1.3979	1.4671
0.20	0.7965	0.9297	1.1505	1.3192	1.3294
0.30	0.7752	0.9042	1.1421	1.2547	1.2609
0.40	0.7598	0.8847	1.1167	1.1147	1.1162
0.50	0.7421	0.8624	0.9917	0.9867	0.9872
0.60	0.7134	0.8261	0.9256	0.9269	0.9283
0.70	0.6790	0.7931	0.8291	0.8304	0.8317
0.80	0.6224	0.7391	0.7398	0.7404	0.7409
0.90	0.5631	0.6112	0.6123	0.6136	0.6149
1.00	0.4832	0.4841	0.4846	0.4850	0.4852
1.10	0.4162	0.4221	0.4247	0.4261	0.4275
1.20	0.3281	0.3289	0.3286	0.3284	0.3283

Table 3. Surge exciting force $(d/a = 6)$

	h/a		
Ka	1.25	1.75	3.00
0.10	2.0117	1.8694	1.7021
0.20	1.5106	1.2864	0.9462
0.30	1.2461	0.9862	0.6741
0.40	1.0967	0.7421	0.3909
0.50	0.8984	0.6842	0.3646
0.60	0.7791	0.5098	0.2517
0.70	0.7364	0.4726	0.2021
0.80	0.6274	0.3622	0.1271
0.90	0.5097	0.2671	0.0983
1.00	0.4892	0.2491	0.0608
1.20	0.3972	0.1977	0.0323
1.40	0.2947	0.1389	0.0086
1.60	0.2566	0.1082	0.0016
1.80	0.2314	0.0627	0.0009

Table 4. Heave exciting force $(d/a = 6)$

	h/a		
Ka	1.25	1.75	3.00
0.10	1.2561	1.0692	0.6841
0.20	1.2293	0.9542	0.6194
0.30	1.2007	0.7781	0.4382
0.40	1.1467	0.7392	0.3922
0.50	0.9724	0.6107	0.2965
0.60	0.8862	0.5566	0.2264
0.70	0.6833	0.4192	0.1791
0.80	0.6374	0.3643	0.1267
0.90	0.5277	0.2818	0.1082

1.00	0.4721	0.2364	0.0927
1.40	0.2021	0.1028	0.0237
1.80	0.1161	0.0711	0.0081
2.00	0.0986	0.0529	0.0072
2.40	0.0583	0.0294	0.0039
2.80	0.0185	0.0129	0.0014
3.00	0.0011	0.0081	0.0005

Table 5. Surge added-mass μ_{11} for different submergence values

	h/a			
Ka	1.50	1.75	2.00	3.00
0.00	0.5287	0.5179	0.5118	0.5034
0.10	0.5403	0.5266	0.5187	0.5066
0.20	0.5545	0.5363	0.5255	0.5082
0.30	0.5656	0.5422	0.5283	0.5069
0.40	0.5693	0.5416	0.5255	0.5030
0.50	0.5646	0.5347	0.5187	0.4986
0.60	0.5527	0.5234	0.5092	0.4949
0.70	0.5359	0.5107	0.4989	0.4920
0.80	0.5160	0.4966	0.4895	0.4905
0.90	0.4962	0.4841	0.4815	0.4893
1.00	0.4776	0.4732	0.4752	0.4896
1.20	0.4475	0.4578	0.4675	0.4903
1.40	0.4286	0.4497	0.4648	0.4915
1.60	0.4189	0.4475	0.4652	0.4925
1.80	0.4158	0.4481	0.4676	0.4930
2.00	0.4171	0.4505	0.4698	0.4938
3.00	0.4381	0.4653	0.4787	0.4950
4.00	0.4523	0.4721	0.4825	0.4955
5.00	0.4582	0.4750	0.4839	0.4966

Table 6. Surge damping coefficients λ_{11} for different submergence values

	h/a			
Ka	1.50	1.75	2.00	3.00
0.00	0.0000	0.0000	0.0000	0.0000
0.10	0.0018	0.0017	0.0016	0.0013
0.20	0.0113	0.0098	0.0088	0.0057
0.30	0.0285	0.0237	0.0200	0.0106
0.40	0.0506	0.0398	0.0317	0.0138
0.50	0.0734	0.0544	0.0412	0.0147
0.60	0.0934	0.0655	0.0472	0.0138
0.70	0.1082	0.0722	0.0496	0.0120
0.80	0.1172	0.0745	0.0489	0.0099
0.90	0.1205	0.0733	0.0460	0.0076
1.00	0.1190	0.0695	0.0418	0.0057
1.20	0.1063	0.0574	0.0317	0.0030
1.40	0.0873	0.0438	0.0223	0.0014
1.60	0.0678	0.0318	0.0148	0.0006
1.80	0.0504	0.0220	0.0094	0.0003
2.00	0.0363	0.0148	0.0058	0.0001
3.00	0.0053	0.0015	0.0004	0.0000
4.00	0.0005	0.0001	0.0000	0.0000
5.00	0.0000	0.0000	0.0000	0.0000

Table 7. Heave added-mass μ_{33} for different submergence values

	h/a			
Ka	1.50	1.75	2.00	3.00
0.00	0.5586	0.5362	0.5239	0.5070
0.10	0.5834	0.5539	0.5375	0.5131
0.20	0.6139	0.5742	0.5518	0.5166
0.30	0.6365	0.5859	0.5570	0.5133

0.40	0.6421	0.5831	0.5506	0.5055
0.50	0.6272	0.5667	0.5350	0.4969
0.60	0.5955	0.5414	0.5147	0.4895
0.70	0.5541	0.5127	0.4939	0.4845
0.80	0.5095	0.4846	0.4752	0.4890
0.90	0.4680	0.4598	0.4598	0.4794
1.00	0.4316	0.4394	0.4481	0.4793
1.20	0.3788	0.4123	0.4346	0.4805
1.40	0.3497	0.3998	0.4306	0.4827
1.60	0.3381	0.3971	0.4321	0.4847
1.80	0.3374	0.4000	0.4362	0.4863
2.00	0.3428	0.4055	0.4412	0.4874
3.00	0.3852	0.4331	0.4587	0.4901
4.00	0.4091	0.4457	0.4654	0.4910
5.00	0.4203	0.4513	0.4686	0.4918

Table 8. Heave damping coefficients λ_{33} for different submergence values

	h/a			
Ka	1.50	1.75	2.00	3.00
0.00	0.0000	0.0000	0.0000	0.0000
0.10	0.0040	0.0036	0.0033	0.0026
0.20	0.0245	0.0208	0.0182	0.0116
0.30	0.0631	0.0505	0.0416	0.0215
0.40	0.1129	0.0847	0.0658	0.0276
0.50	0.1627	0.1149	0.0848	0.0293
0.60	0.2037	0.1361	0.0958	0.0275
0.70	0.2304	0.1473	0.0991	0.0237
0.80	0.2423	0.1490	0.0964	0.0193
0.90	0.2414	0.1439	0.0896	0.0150
1.00	0.2318	0.1340	0.0805	0.0115
1.20	0.1966	0.1078	0.0604	0.0059

1.40	0.1554	0.0809	0.0421	0.0028
1.60	0.1172	0.0579	0.0279	0.0013
1.80	0.0856	0.0399	0.0177	0.0005
2.00	0.0609	0.0267	0.0109	0.0002
3.00	0.0085	0.0026	0.0007	0.0000
4.00	0.0009	0.0002	0.0003	0.0000
5.00	0.0001	0.0000	0.0000	0.0000

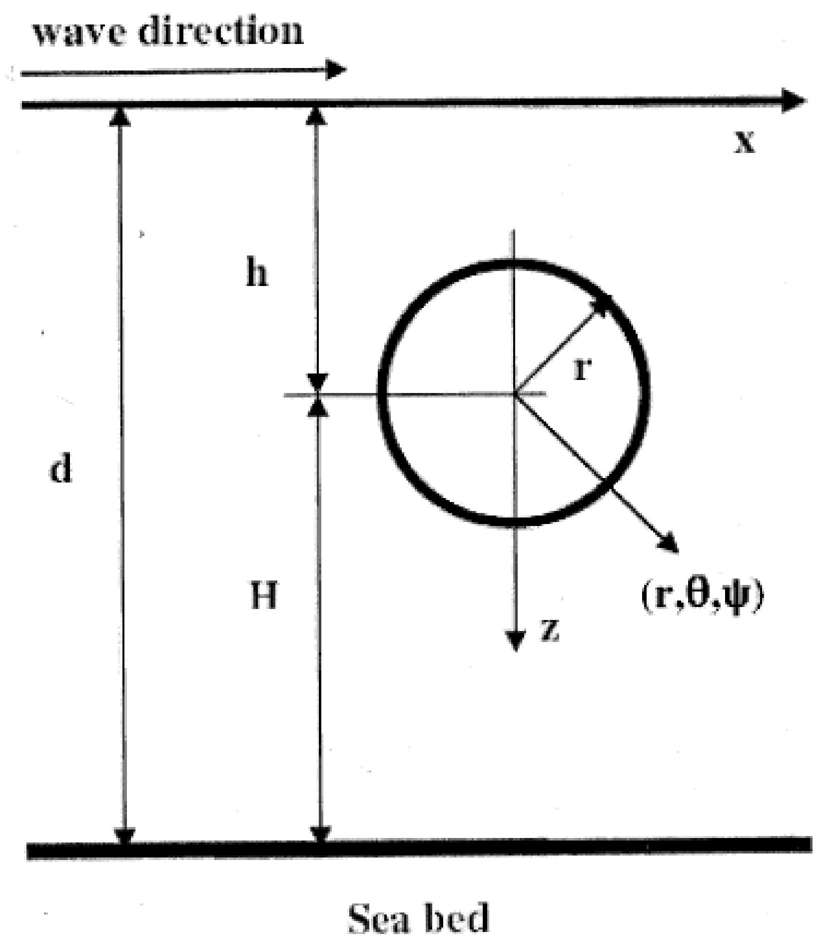

Figure 1. Reference coordinate system

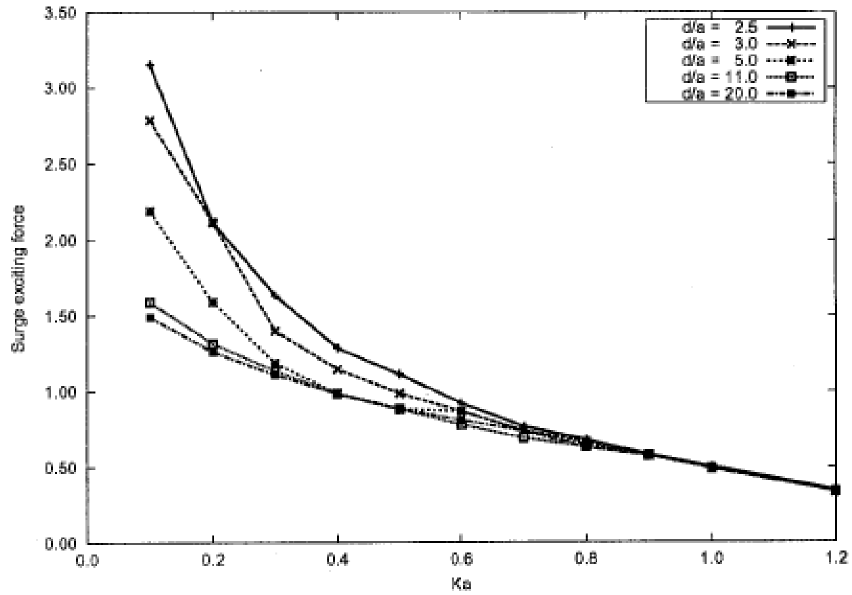

Figure 2. Surge exciting force for a fixed submergence $h/a = 1.25$

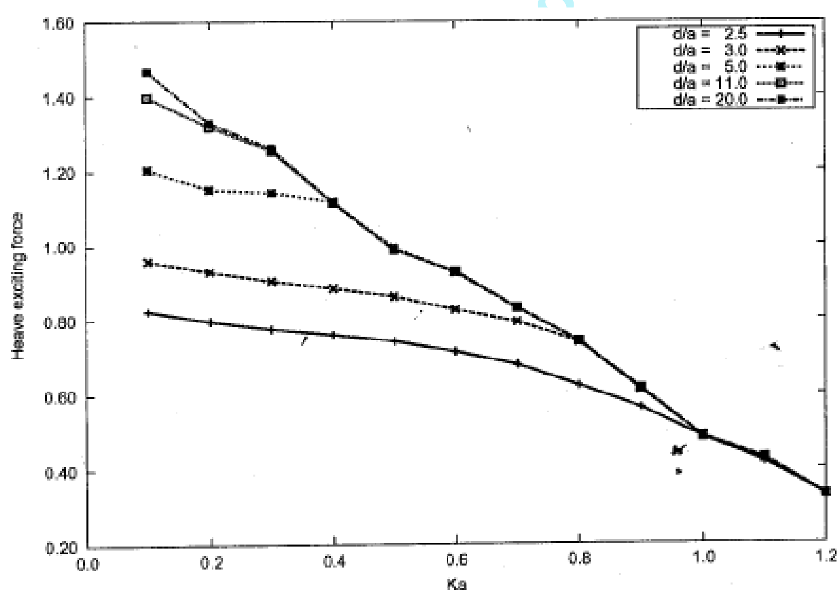

Figure 3. Heave exciting force for a fixed submergence $h/a = 1.25$

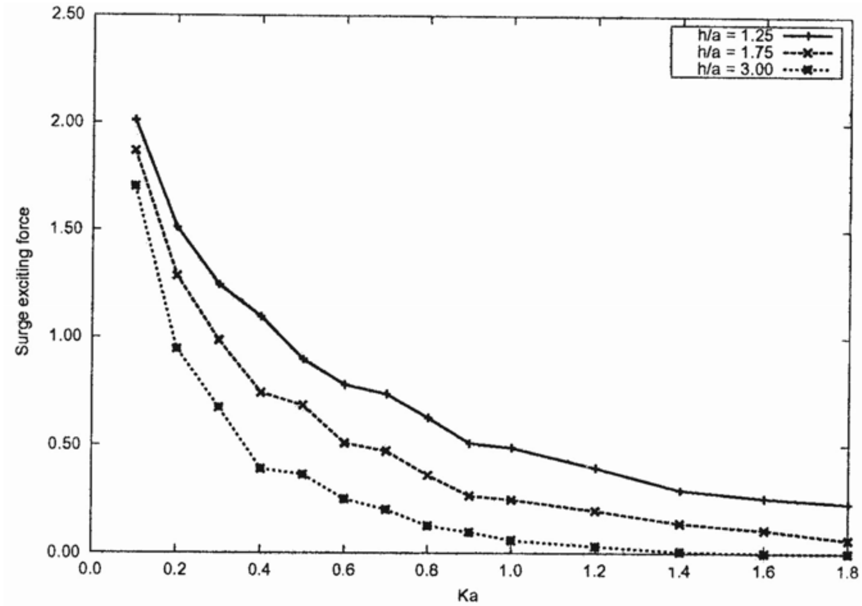

Figure 4. Surge exciting force for a fixed depth $d/a = 6.0$

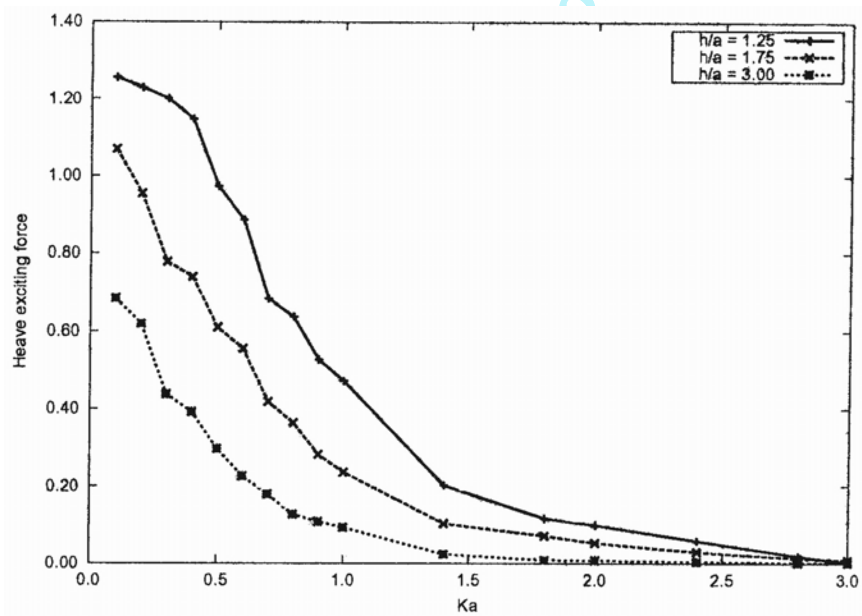

Figure 5. Heave exciting force for a fixed depth $d/a = 6.0$

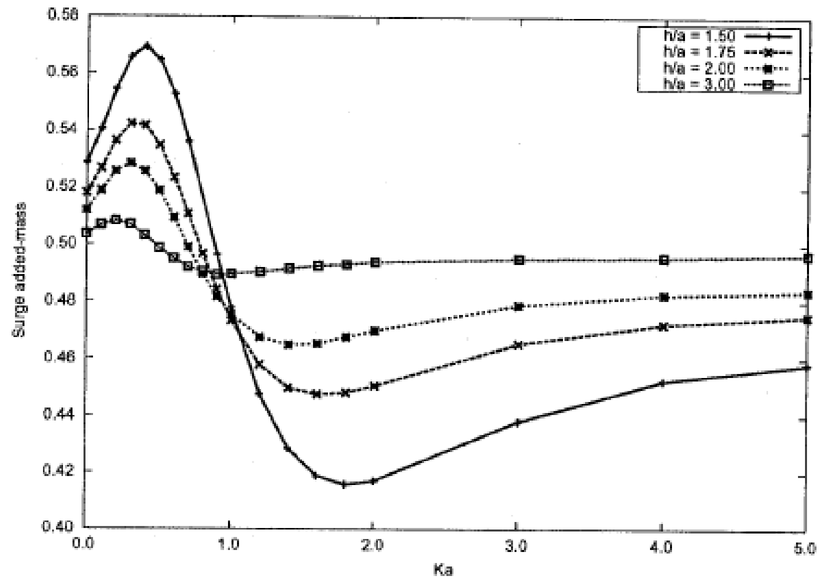

Figure 6. Surge added-mass μ_{11} for different submergence values

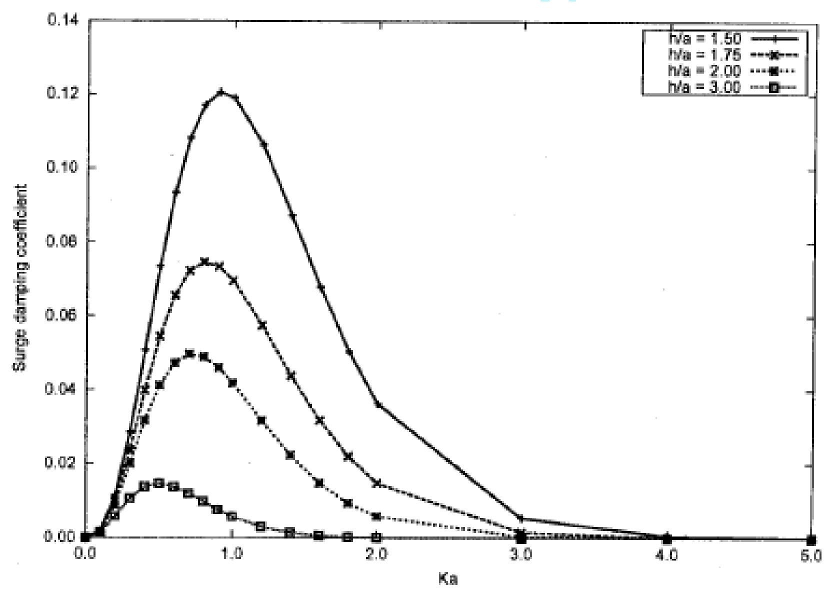

Figure 7. Surge damping coefficient λ_{11} for different submergence values

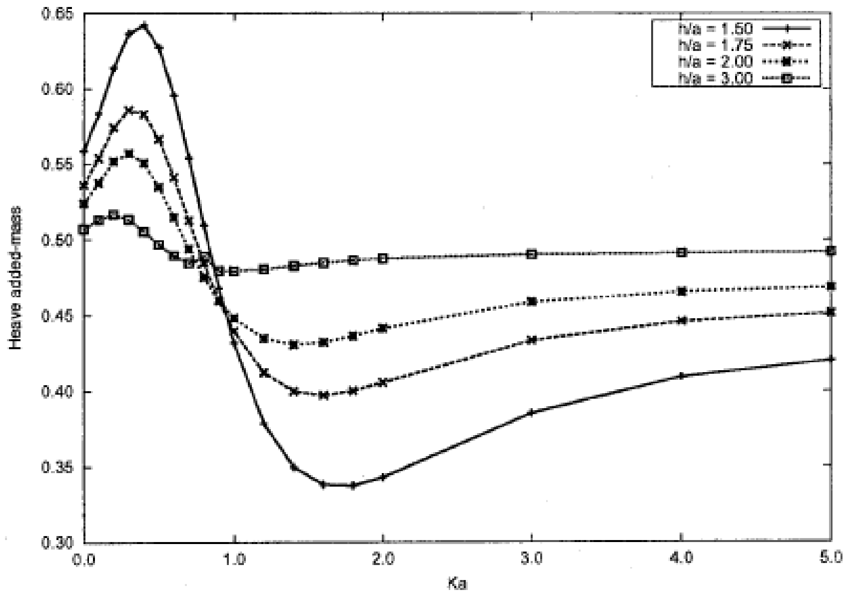

Figure 8. Heave added-mass μ_{33} for different submergence values

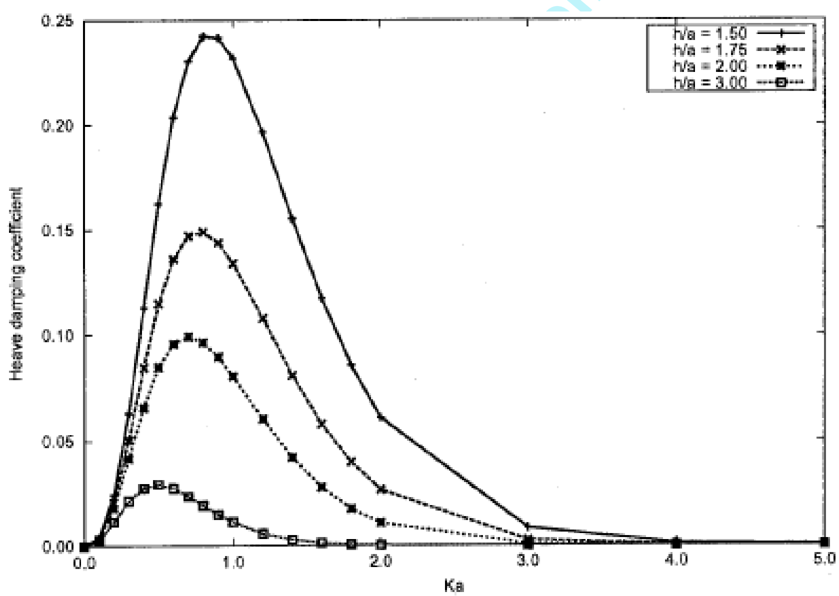

Figure 9. Heave damping coefficient λ_{33} for different submergence values

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