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MARTINGALE APPROACH FOR CLOSED-FORM EXPRESSION OF AVERAGE RUN LENGTH ON EWMA CHART FOR DETECTION OF A CHANGE IN NUMBER OF DEFECTS

Saowanit Sukparungsee

Department of Applied Statistics
Faculty of Applied Sciences
King Mongkut's University of Technology North Bangkok
Bangkok 10800, Thailand
e-mail: swns@kmutnb.ac.th

Abstract

The closed-form formula of average run length for exponentially weighted moving average (EWMA) control chart is proposed based on the martingale approach. Usually, the number of defective products is defined by binomial distribution. Based on this approach, we prove a martingale identity and apply it for obtaining the closed-form formula for the first passage times. Furthermore, the algorithms for obtaining the optimal parameter of EWMA control chart have been developed in order to design the optimal EWMA control chart for detecting of a change in number of defective products in process. The numerical results are compared with the algorithm which combined the results from the martingale and the Monte Carlo simulations, the proposed algorithm is very time saving.

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1. Introduction

Traditionally, statistical process control (SPC) charts are widely used not only in monitoring and improving quality in manufacturing and industrial statistics but also in finance, medicine, epidemiology, environmental statistics, and in other fields of applications. The SPC is easily used to track the performance of a process in order to bring the process back to a target value as quickly as possible. To detect this change, one needs to apply statistical techniques and constraints. Generally, the most used ones are mean of false alarm time or in-control average run length (ARL_0) - the expectation of the time or observation before the control chart gives a false alarm that an in-control process has gone out-of-control.

A second important characteristic is an out-of-control average run length (ARL_1) which is the expectation of the time or observation between a process going out-of-control and the control chart giving the alarm that the process has gone out-of-control. The ARL_0 of an acceptable chart should be large and the ARL_1 should be small. There must be a trade-off between these two conflicting requirements.

The Shewhart chart is the most commonly used method for detecting a change and it is easily calculated. However, this chart has been proposed to be unsatisfactory in monitoring and detecting small changes. During the past few decades, two extremely effective alternatives to the Shewhart chart have been developed which overcome its shortcomings. These are methods based on the cumulative sum (CUSUM) and exponentially weighted moving average (EWMA). The CUSUM chart was introduced by Page [10] (see Siegmund [15] and Pollak [11]). The EWMA chart was initially presented by Roberts [12] (see more, Hunter [5], Lucas and Saccucci [8] and Srivastava and Wu [16]). Both the CUSUM and EWMA control charts are known to be more sensitive to the detection of small to moderate changes because they pay attention to the historical observations. However, EWMA charts are inherently simpler and are also believed to be more robust with respect to the assumptions than are CUSUM charts (Sukparungsee and Novikov [17]).

In this paper, the closed-form formula and exact lower bound of the ARL for EWMA control chart when the observations are from binomial distribution are presented. A derivation of expression and this bound are based on a martingale approach.

2. The EWMA Control Chart and its Properties

Let $X_1, X_2, ..., X_t, ...$ be observed independent random variables. The martingale approach can be used for different distributions but we restrict our attention in this paper to the most important case of binomial distributed random variables. The model we study is as follows:

$$\begin{cases} X_t \sim bin(n, \alpha_0); & t = 1, 2, ..., \theta - 1, \\ X_t \sim bin(n, \alpha); & t = \theta, \theta + 1, ..., \quad \alpha \neq \alpha_0. \end{cases}$$

We use notation $\theta = \infty$ for the case when there is no change in the distribution of observed data. Note that if $\theta = 1$, then the change occurs at the very beginning.

2.1. The EWMA control chart

The exponentially weighted moving average (EWMA) chart, initiated by Roberts [12], is an effective alternative to the CUSUM chart for detecting small shifts. The EWMA for discrete time case is defined by the following recursion:

$$Z_t = (1 - \lambda) Z_{t-1} + \lambda g(\xi_t), \quad t = 1, 2, ... \tag{1}$$

Typically, $\lambda \in (0, 1)$ is a weighting factor for previous observations and for the function $g(\xi_t) = \xi_t - E_\infty(\xi_t)$. To start the recursion, required for Z_0 is usually assumed that $Z_0 = \alpha_0$. If the anticipated shift in the mean value is positive, then we take the decision that the process is out-of-control when for the first time $Z_t > H$ as follows:

$$\tau = \inf\{t \in N : Z_t > H\}. \tag{2}$$

2.2. The properties of EWMA control chart

There are many other criteria that have been used for optimality of SPC (see, e.g., Shiryaev [13, 14] and Lorden [7]). However, ARL_0 and ARL_1 are still the most popular and commonly used characteristics for evaluating the performance of a control chart. These two criteria will be used as the basis for the comparison of different charts throughout this research. Let $E_\theta(\cdot)$ denote the expectation of τ under the assumption that the change-point occurs at point $\theta \leq \infty$. Then by definition

$$ARL_0 = E_\theta(\tau) = T, \tag{3}$$

where τ is a given number. Another important characteristic of SPC is the quantity $E_1(\tau)$ as

$$E_1(\tau) \leq \sup_\theta E_\theta(\tau - \theta + 1 | \tau \geq \theta). \tag{4}$$

Of course, it would be desirable to minimise on equation (4) but the latter quantity is usually difficult to calculate. One can expect that SPC has a near optimal performance if its ARL_1 is closed to a minimal value. One of the objectives of this paper is to find the optimal combination of parameters (λ^*, H^*) which satisfies to these criterions.

There are several approaches to evaluate the average run length in the literatures, for example, Crowder [3] used a system of Fredholm integral equation for numerical approximation of ARL. Later, Brook and Evans [1] used the Markov Chain approach to evaluate these characteristics. Lucas and Saccucci [8] intensively studied the different pairs (λ, H) to find ARL_1 for different magnitudes of change in mean process.

3. The Closed-form Formula for the ARL by Martingale Approach

In this section, martingale approach is exploited to obtain closed-form relations for the expectations of stopping times of first order autoregressive processes. We start with some general definitions and notation from stochastic analysis.

3.1. Martingales and stopping times

We always assume that given a probability space $(\Omega, \mathbb{F}, P)$ and there is a sequence of σ-algebras $\mathbb{F}_s \in \mathbb{F}_t$ such that $\mathbb{F}_s \in \mathbb{F}_t$ for $t \geq s$. The sequence of σ-algebras $\{\mathbb{F}_t\}$ is called a *filtration* (or *history*, or *information* available to the moment $t = 0, 1, 2, \ldots$). Typically, one may consider $\mathbb{F}_t$ as a collection of all events generated by an underlying observed stochastic process $X_t,\ t = 0, 1, \ldots$. For the latter case, we use notation:

$$\mathbb{F}_t = \sigma\{X_0, \ldots, X_t\}.$$

A random process Y_t is *adapted* to $\mathbb{F}_t$ if Y_t is $\mathbb{F}_t$-measurable for any $t = 0, 1, 2, \ldots$.

Definition 1. A process $M_t,\ 0 \leq t < \infty$ is called a *martingale* with respect to filtration $\mathbb{F}_t$ if

$$E(|M_t|) < \infty \tag{5}$$

and

$$E(M_t \mid \mathbb{F}_{t-1}) = M_{t-1}. \tag{6}$$

Definition 2 (Stopping Time)**.** A stopping time τ is a nonnegative-integer-valued random variable such that the event $\{\tau \leq t\} \in \mathbb{F}_t$ for any $t \geq 0$.

Further, it will be convenient to use notation $M(\mathbb{F}_t, P)$ for the class of all martingales on the filtered probability space $(\Omega, \mathbb{F}_t, \mathbb{F}, P)$.

Proposition 1. *If* $M_t \in M(\mathbb{F}_t, P)$, *then*

$$E(M_t) = E(M_0),$$

$$E(M_s) = E(E(M_t \mid \mathbb{F}_s)) = E(M_t) = \cdots = E(M_0). \tag{7}$$

Theorem 1 (Martingale Stopping Theorem)**.** *If* $M_t \in M(\mathbb{F}_t, P)$, $t \geq 0$, *then for any stopping time* τ,

$$X_t := M_{\min(\tau, t)} \in M(\mathbb{F}_t, P)$$

and hence for any fixed $t \geq 0$,

$$E(M_{\min(\tau, t)}) = E(M_0). \tag{8}$$

We define an AR(1) process as a recursive equation

$$X_t = \phi X_{t-1} + \xi_t, \quad t \in 0, 1, 2, ..., \tag{9}$$

where ξ_t is a sequence of independent identically distributed random variables (innovation), $X_0 = x$ and ϕ are nonrandom constants, $0 < \phi < 1$. Note that EWMA statistic Z_t coincides with X_t for a particular case of $\phi = 1 - \lambda$ and independent random variables $\xi_t = \lambda g(\xi_t)$. Set

$$E_\theta e^{u\lambda\xi_t} := e^{u\psi_\theta(u)} < \infty \text{ for any } u \in (0, \infty),$$

where $\theta = \infty$ (in-control) or $\theta = 1$ (out-of-control) and let

$$\varphi_\theta(u) = \sum_{k=0}^{\infty} \psi_\theta((1-\lambda)^k u). \tag{10}$$

Now make further assumptions that ξ_t has the normal distribution, $\xi_t \sim N(\alpha, \sigma^2)$. Then we obtain

$$\varphi_\theta(u) = u\alpha + \frac{\lambda u^2 \sigma^2}{(4 - 2\lambda)}.$$

Indeed, since for the case under consideration

$$\varphi_\theta(u) = \alpha\lambda u + \frac{\lambda^2}{2}\sigma^2 u^2,$$

$$\varphi_\theta(u) = \sum_{k=0}^{\infty} \psi_\theta((1-\lambda)^k u)$$

$$= \sum_{k=0}^{\infty}\left[\alpha((1-\lambda)^k \lambda u) - \frac{1}{2}\sigma^2((1-\lambda)^{2k}\lambda^2 u^2)\right]$$

$$= \frac{\alpha\lambda u}{1-(1-\lambda)} + \frac{1}{2}\frac{\lambda^2\sigma^2 u^2}{1-(1-\lambda)}$$

$$= u\alpha + \frac{\lambda\sigma^2 u^2}{(4-2\lambda)}.$$

Without losing generality, it is usually supposed that $E_\infty(\xi_t) = \alpha_0 = 0.$ Here we also use this assumption and therefore we have

$$\varphi_\infty(u) = \frac{\lambda u^2\sigma^2}{(4-2\lambda)}, \quad \varphi_1(u) = u\alpha + \frac{\lambda u^2\sigma^2}{(4-2\lambda)}.$$

Consider now the case of binomial EWMA which is based on the first passage time

$$\tau_H = \inf\{t \geq 0 : Z_t > H\},$$

assuming that $H > z = 0.$ Then

$$E_{\theta(\tau_H)} = \frac{1}{|\ln(1-\lambda)|}\int_0^\infty E_\theta(e^{uZ_{\tau_H}} - 1)e^{-\varphi_\theta(u)}du. \tag{11}$$

3.2. Expectation of first passage times for binomial EWMA

If ξ_t are independent random variables with outcomes 0 and 1 such that the probability of "success" $P(\xi_t = 1) = \alpha,$ then

$$\psi_\theta(u) = \log((\alpha e^{u\lambda} + (1-\alpha)))^n$$

and

$$\varphi_\theta(u) = -u\alpha_0 + \log\prod_{k=0}^{\infty}((\alpha e^{(u(1-\lambda)^k\lambda)} + (1-\alpha)))^n, \tag{12}$$

where

$$\varphi_\theta(u) = -u\alpha_0 + \log \prod_{k=0}^{\infty} ((\alpha_0 e^{(u(1-\lambda)^k \lambda)} + (1-\alpha_0)))^n,$$

$$\varphi_1(u) = -u\alpha_0 + \log \prod_{k=0}^{\infty} ((\alpha e^{(u(1-\lambda)^k \lambda)} + (1-\alpha)))^n.$$

Therefore, the closed-form expressions of ARL for binomial EWMA control chart are

$$ARL_0 = E_\infty(\tau_H) = \frac{1}{|\ln(1-\lambda)|} \int_0^\infty \frac{E_\theta(e^{uZ_{\tau_H}} - 1)e^{u\alpha_0}}{u \prod_{k=0}^{\infty} ((\alpha_0 e^{(u(1-\lambda)^k \lambda)} + (1-\alpha_0)))^n} du, \quad (13)$$

$$ARL_1 = E_1(\tau_H) = \frac{1}{|\ln(1-\lambda)|} \int_0^\infty \frac{E_1(e^{uZ_{\tau_H}} - 1)e^{u\alpha_0}}{u \prod_{k=0}^{\infty} ((\alpha e^{(u(1-\lambda)^k \lambda)} + (1-\alpha)))^n} du. \quad (14)$$

3.3. Corrected approximation

The closed-form formulas from equations (14) and (15) include the overshoot $\chi_H = Z_{\tau_H} - H$ whose distribution is, generally speaking, unknown. Neglecting by overshoot, we obtained the explicit lower-bounds for ARL_0 and ARL_1 which are easy to calculate but it may be not an accurate approximation. Here we provide some considerations about how to find a correction term to improve the accuracy.

Recall that the process EWMA is governed by the following equation:

$$Z_t = (1-\lambda)Z_{t-1} + \lambda\xi_t, \quad Z_0 = z.$$

Setting $\widetilde{Z}_t = \frac{Z_t}{\lambda}$, we obtain

$$\widetilde{Z}_t = (1-\lambda)\widetilde{Z}_{t-1} + \xi_t, \quad \widetilde{Z}_0 = z/\lambda \quad (15)$$

and consequently, the alarm time is

$$\tau_H = \inf\left\{t > 0 : \widetilde{Z}_t > \frac{H}{\lambda}\right\}.$$

If λ is close to zero, then $\widetilde{Z}_t$ is close to a random walk $S_t = \sum_{k=1}^{t} \xi_k$ with i.i.d. increments ξ_t. Based on these considerations, we suggest to use the following approximation:

$$\widetilde{Z}_{\tau_H} \simeq \frac{H}{\lambda} + \widetilde{\chi}_H,$$

where $\widetilde{\chi}_H$ is the overshoot of the random walk $S_t = \sum_{k=1}^{t} \xi_k$ over level $\frac{H}{\lambda}$. This leads to the following approximation:

$$Z_{\tau_H} = \lambda \widetilde{Z}_{\tau_H} \simeq \lambda\left(\frac{H}{\lambda} + \widetilde{\chi}_H\right) = H + \lambda \widetilde{\chi}_H$$

and correspondingly, for $u \geq 0$,

$$E_\theta e^{uZ_{\tau_H}} \simeq E_\theta e^{u(H + \lambda \widetilde{\chi}_H)} = e^{uH} E_\theta e^{\lambda u \widetilde{\chi}_H}.$$

Note that due to Jensen inequality $E_\theta e^{\lambda u \widetilde{\chi}_H} \geq e^{\lambda u E_\theta \widetilde{\chi}_H}$ and also due to Taylor expansion $e^x = 1 + x + o(x)$, $x \to 0$, we have for $\lambda \to 0$,

$$E_\theta e^{\lambda u \widetilde{\chi}_H} = 1 + \lambda u E_\theta \widetilde{\chi}_H + o(\lambda) = e^{\lambda u E_\theta \widetilde{\chi}_H} + o(\lambda).$$

For the cases when the level $\frac{H}{\lambda}$ is high, we can use the following well-known result from the theory of random walks (see, e.g., Siegmund [15]).

Theorem 2. *Let* $S_t = \sum_{k=1}^{t} \xi_k$, $\tau_b = \inf\{t > 0 : S_t > b\}$ *the distribution of* ξ_t *be non-lattice*, $E\xi_k = a \geq 0$, $0 < E|\xi_k|^2 < \infty$. *Then there exists limit*

$$\lim_{b \to \infty} E(S_{\tau_b} - b) = C.$$

The analytical calculation of the constant C could be a hard problem. Note that the constant C depends on $\alpha = E|\xi_1|$. Under the assumption $E|\xi_k|^3 < \infty$, the constant C can be represented in the following form:

$$C = \frac{E(S_{\tau_+}^2)}{2E(S_{\tau_+})}, \tag{16}$$

where $\tau_+ = \inf\{t > 0 : S_t > 0\}$, S_{τ_+} is the so-called positive ladder variable of a random walk S_t. If ξ_t is nonnegative, then $S_{\tau_+} = \xi_1$.

Remark 1. If the distribution of ξ_t is lattice (e.g., Poisson and Bernoulli distribution), then the statement of Theorem 2 still holds.

The monograph of Siegmund [15] and the papers of Chang and Peres [2] contain a lot of results concerning properties of limiting distribution of $P(S_{\tau_b} - b > x)$ as $b \to \infty$ for the case of Gaussian random walk. In particular, it is known that if $\xi_t \sim N(0, 1)$, then

$$C = -\frac{\varsigma(1/2)}{\sqrt{2\pi}} = 0.5826, \tag{17}$$

where $\varsigma(x)$ is the Riemann zeta function.

Summarizing the above considerations, we suggest for the case $\xi_t \sim N(0, 1)$ to use as the formula

$$E_\theta e^{uZ_{\tau_H}} \simeq e^{u(H+C\lambda)}, \tag{18}$$

where $C = 0.5826$ for $\lambda = 0$ or when λ is close to zero.

In this paper, we also use considerations to obtain approximations for an overshoot. Note that if $\xi_t \sim \mathit{binomial}(n, \alpha_0)$, then

$$C = \frac{E(S_{\tau_+}^2)}{2E(S_{\tau_+})} = \frac{E(\xi_t^2)}{2E(\xi_t)} = \frac{1 - \alpha_0 + n\alpha_0}{2}. \tag{19}$$

Proposition 2. *The closed-form formulas for binomial EWMA*:

$$ARL_0 = E_{\infty}(\tau_H) \simeq \frac{1}{|\ln(1-\lambda)|}\int_0^{\infty} \frac{E_{\theta}(e^{uZ_{\tau_H}} - 1)e^{u\alpha_0}}{u\prod_{k=0}^{\infty}((\alpha_0 e^{(u(1-\lambda)^k\lambda)} + (1-\alpha_0)))^n}du, \quad (20)$$

$$ARL_1 = E_{1}(\tau_H) \simeq \frac{1}{|\ln(1-\lambda)|}\int_0^{\infty} \frac{E_{1}(e^{uZ_{\tau_H}} - 1)e^{u\alpha_0}}{u\prod_{k=0}^{\infty}((\alpha e^{(u(1-\lambda)^k\lambda)} + (1-\alpha)))^n}du. \quad (21)$$

If the overshoot is neglected, then we obtain exact lower-bounds:

$$ARL_0 \geq \frac{1}{|\ln(1-\lambda)|}\int_0^{\infty} \frac{(e^{uZ_{\tau_H}} - 1)e^{u\alpha_0}}{u\prod_{k=0}^{\infty}((\alpha e^{(u(1-\lambda)^k\lambda)} + (1-\alpha)))^n}du, \quad (22)$$

$$ARL_1 \geq \frac{1}{|\ln(1-\lambda)|}\int_0^{\infty} \frac{(e^{uH} - 1)e^{u\alpha_0}}{u\prod_{k=0}^{\infty}((\alpha e^{(u(1-\lambda)^k\lambda)} + (1-\alpha)))^n}du. \quad (23)$$

4. Numerical Results

The accuracy of the lower-bounds is reasonable for small λ. The numerical results of approximations ARL_1, lower-bounds are presented in Table 1 with in-control parameter $\alpha_0 = 0.01$ and including the "first approximation" with constant $C = 0.745$ and MC. However, the accuracy of the first approximation differs significantly from MC. The closed-form formulas for ARL_0 and ARL_1 presented above contain an overshoot. To get more accurate numerical approximations, we suggest using a combination of MC and a martingale closed-from formula applying nonlinear. We then obtain the constant $\widetilde{C} = 0.596487$ for evaluating a "second approximation" which is more certainly accurate approximation.

Note that similar approximations are often used in many other problems of sequential analysis. The theoretical justification of such an approximation is a very hard problem and the value $C = 0.745$ should be used only as a "first approximation" for binomial distribution in case $\lambda = 0.01$ is presented in Table 1 when fixed $n = 50$, $\alpha_0 = 0.01$. The first approximation of the overshoot produces usually a good approximation for small λ. A more accurate approximation can be obtained with Monte Carlo simulations and fitting with the non-linear least-square methods as a "second approximation".

Table 1. Comparison of numerical results of ARL_1 between lower-bounds, first approximation and MC for binomial EWMA when fixed $n = 50$, $\alpha_0 = 0.01$ and $\lambda = 0.01$

H	Lower-bounds equation (23)	First approximation ($C = 0.745$)	Second approximation ($\widetilde{C} = 0.596487$)	MC
0.10	21.929	23.771	23.392	23.492 ± 0.024
0.15	34.987	37.085	36.649	36.806 ± 0.032
0.20	49.977	52.412	51.903	51.983 ± 0.042
0.25	67.553	70.450	69.90	69.843 ± 0.053
0.30	88.757	92.322	91.559	91.559 ± 0.069
0.35	115.371	119.974	119.036	119.04 ± 0.093
0.40	150.789	157.185	155.844	155.91 ± 0.132
0.45	202.455	212.450	210.353	210.13 ± 0.207
0.50	290.341	309.426	305.394	305.44 ± 0.379

4.1. Comparison of analytical approximation with Monte Carlo simulations

We have numerically calculated based on the closed-form formula suggested above and compared these values with the results obtained from MC. The accuracy of the approach is confirmed by the simulations as shown in Table 2.

Table 2. Comparison of the approximations with Monte Carlo simulations for the Bernoulli EWMA when fixed $n = 50$ and $\alpha_0 = 0.01$

N	α	$T = 100$				
		$\tilde{\lambda}$	$\tilde{H}$	$\tilde{C}$	ARL_1^*	MC
50	0.01010	0.109773	0.814459	0.727083	95.6534	95.6576 ± 0.271
	0.01050	0.117894	0.836782	0.73718	80.8942	80.8942 ± 0.274
	0.01100	0.128177	0.863633	0.754808	66.9327	66.9327 ± 0.279
	0.01250	0.159631	0.941007	0.80750	41.9089	41.9089 ± 0.287
	0.01500	0.213393	1.06635	0.870672	23.7304	23.7304 ± 0.295
n	α	$T = 370$				
		$\tilde{\lambda}$	$\tilde{H}$	$\tilde{C}$	ARL_1^*	MC
50	0.01010	0.038408	0.703383	0.652939	342.53	340.871 ± 0.970
	0.01050	0.043579	0.727204	0.657833	258.61	258.021 ± 0.659
	0.01100	0.050316	0.756717	0.663189	191.438	192.531 ± 0.484
	0.01250	0.071701	0.840745	0.703237	96.7764	98.069 ± 0.234
	0.01500	0.109204	0.968894	0.771411	45.8983	47.151 ± 0.109

4.2. Choices of optimal parameter of binomial EWMA designs

Tables 3 and 4 contain the approximations for optimal value parameters $(\tilde{\lambda}, \tilde{H})$ when observations are from binomial distribution. The values were calculated numerically for the one-side EWMA case. These optimal values were obtained by minimising ARL_1 values when fixed ARL_0 values of 100 and 370 in-control parameter $n = 50$ and 100, $\alpha_0 = 0.01$ and the sizes of parameter changes, $\alpha = 0.0101,\ 0.01025,\ 0.0105,\ 0.01075,\ 0.011,\ 0.015,$ 0.0175. The numerical results from the martingale technique approximations are as good as the results from the Monte Carlo simulations. The suggested algorithms can be easily used to create curves of ARL_1 for a range of magnitudes of change.

Table 3. Optimal parameter values and ARL_1 of binomial EWMA when fixed $n = 50$, $\alpha_0 = 0.01$ and $ARL_0 = 100$ and 370

ARL_0	α	$\tilde{\lambda}$	$\tilde{H}$	$\tilde{C}$	ARL_1^*	MC
100	0.01010	0.109773	0.814459	0.727083	95.6534	95.6576 ± 0.271
	0.01050	0.117894	0.836782	0.73718	80.8942	80.8942 ± 0.274
	0.01075	0.123018	0.850425	0.744797	73.3956	73.3956 ± 0.276
	0.0110	0.128177	0.863633	0.754808	66.9327	66.9327 ± 0.279
	0.0150	0.213393	1.06635	0.870672	23.7304	23.7304 ± 0.295
	0.0175	0.269317	1.19612	0.905754	15.6254	15.6254 ± 0.301
	0.020	0.330223	1.32976	0.959683	11.2413	11.2413 ± 0.301
370	0.01010	0.038408	0.703383	0.652939	342.53	340.871 ± 0.970
	0.01050	0.043579	0.727204	0.657833	258.61	258.027 ± 0.659
	0.01075	0.046914	0.742542	0.649166	221.144	223.336 ± 0.569
	0.0110	0.050316	0.756717	0.663189	191.438	192.531 ± 0.484
	0.0150	0.109204	0.968894	0.771411	45.8983	47.151 ± 0.109
	0.0175	0.147519	1.08860	0.812893	27.5776	28.726 ± 0.064
	0.020	0.186182	1.20620	0.831984	18.7764	19.738 ± 0.044

Table 4. Optimal parameter values and ARL_1 of binomial EWMA when fixed $n = 100$, $\alpha_0 = 0.01$ and $ARL_0 = 100$ and 370

ARL_0	p	$\tilde{\lambda}$	$\tilde{H}$	$\tilde{C}$	ARL_1^*	MC
100	0.01010	0.123236	1.47935	0.915069	97.2307	94.4671 ± 0.251
	0.01050	0.135439	1.52263	0.933812	75.5766	76.1648 ± 0.199
	0.01075	0.143203	1.54873	0.950169	66.6535	66.9576 ± 0.173
	0.0110	0.151057	1.57508	0.962824	59.2623	59.6703 ± 0.152
	0.0150	0.284887	1.97611	1.14816	17.0882	17.8888 ± 0.043
	0.0175	0.380823	2.24857	1.12379	10.6272	11.4632 ± 0.027
370	0.01010	0.042864	1.30863	0.813591	333.446	334.955 ± 0.872
	0.01050	0.050780	1.34719	0.866811	230.388	231.388 ± 0.578
	0.01075	0.059595	1.38450	0.842689	189.115	190.023 ± 0.469
	0.0110	0.061284	1.41359	0.847562	158.045	160.523 ± 0.392
	0.0150	0.155607	1.55607	0.991610	30.452	31.4703 ± 0.068
	0.0175	0.217927	2.06065	1.083740	17.5074	18.4426 ± 0.040

5. Conclusions

In this research, a martingale technique has been used to obtain analytical proofs of approximations for ARL_0 and ARL_1 for EWMA charts. The approximations are presented as closed-form formulas. We have shown that our martingale approximations are easy to program and calculate. The approximations also produce accurate results and reduce the computational time when compared with other standard methods such as Monte Carlo simulation methods. In addition, we have shown that the martingale approach can be adopted and expanded to non-Gaussian distributions such as the binomial distribution. We have also developed algorithms and Mathematica® programs that we have used to obtain optimal parameter for optimal EWMA designs for binomial distribution.

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References

[1] D. Brook and D. A. Evans, An approach to the probability distribution of CUSUM run length, Biometrika 59 (1972), 539-548.

[2] J. Chang and Y. Peres, Ladder heights, Gaussian random walks, and the Riemann zeta function, Ann. Probab. 25 (1997), 787-802.

[3] S. V. Crowder, A simple method for studying run-length distributions of exponentially weighted moving average charts, Technometrics 29 (1987), 401-407.

[4] M. Frisen and C. Sonesson, Optimal surveillance based on exponentially weighted moving averages, Sequential Anal. 25 (2006), 379-403.

[5] J. S. Hunter, The exponentially weighted moving average, Journal of Quality Technology 18 (1986), 203-210.

[6] S. Knoth, Accurate ARL calculation for EWMA control charts monitoring normal mean and variance simultaneously, Sequential Anal. 26 (2007), 251-263.

[7] G. Lorden, Procedures for reacting to a change in distribution, Annals of Mathematical Statistics 42 (1971), 1897-1908.

[8] J. M. Lucas and M. S. Saccucci, Exponentially weighted moving average control schemes: properties and enhancements (with discussion), Technometrics 32 (1990), 1-29.

[9] G. V. Moustakides, Optimal stopping times for detecting changes in distributions, Ann. Statist. 14 (1986), 1379-1387.

[10] E. S. Page, Continuous inspection schemes, Biometrika 41 (1954), 100-114.

[11] M. Pollak, Optimal stopping times for detecting changes in distributions, Ann. Statist. 13 (1985), 206-227.

[12] S. W. Roberts, Control chart tests based on geometric moving average, Technometrics 1 (1959), 239-250.

[13] A. N. Shiryaev, The problem of the most rapid detection of a distribution in a stationary process, Soviet Mathematics 2 (1961), 795-799.

[14] A. N. Shiryaev, Optimal Stopping Rules, 2nd ed., Springer-Verlag, New York, 1978.

[15] D. Siegmund, Sequential Analysis: Tests and Confidence Intervals, 1st ed., Springer-Verlag, New York, 1985.

[16] M. S. Srivastava and Y. Wu, Comparison of EWMA, CUSUM and Shiryayev-Roberts procedures for detecting a shift in the mean, Ann. Statist. 21 (1993), 645-670.

[17] S. Sukparungsee and A. A. Novikov, On EWMA procedure for detection of a change in observations via martingale approach, KMITL Science Journal 6 (2006), 373-380.