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REMARKS ON MONOTONIC OPERATIONS INDUCED BY A SOFT SET

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Abstract

The purpose of this paper is to introduce an operation $\mathcal{F} : P(U) \to P(U)$ induced by $F^{\leftarrow}$ and a given soft set (F, X) over a common universe set U, and to study some basic properties of the operation.

1. Introduction

In 1999, Molodtsov introduced the concept of soft set [8] to solve complicated problems and various types of uncertainties. He introduced that a soft set is an approximate description of an object precisely consisting of two parts, namely predicate and approximate value set. Soft set theory is a mathematical tool for dealing with uncertainties which is free from the

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difficulties of theory of fuzzy sets [11], theory of vague sets [3], and theory of rough sets [9]. Maji et al. [5] introduced several operators for soft set theory: equality of two soft sets, subset and superset of soft set, complement of a soft set, null soft set, and absolute soft set. More, new operations [2] in soft set theory were investigated by using the notions defined in [1]. In [4], we introduced the notions of A_F, A^F and $F^{\leftarrow}$ on a parameter subset *A*, and study some properties of such notions. We also studied various types of subsets on a universe parameter set for a given soft set. In [7], we investigated a monotonic operation $u_F : P(X) \to P(X)$ defined by A_F, $F^{\leftarrow}$ in any soft set (F, X) $(A \subseteq X)$ as the following: $u_F(A) = A_F \cup F^{\leftarrow}(F(A))$ for $A \in P(X)$. Naturally, we have very attentive to research for any operation defined in a given common universe set *U*. For one of the goals of this research, we are going to introduce an operation $\mathcal{F} : P(U) \to P(U)$ induced by $F^{\leftarrow}$ and a given soft set (F, X) over a common universe set *U*, and to study some basic properties of the operation.

2. Preliminaries

Let *U* be an initial universe set and *E* be a collection of all possible parameters with respect to *U*, where parameters are the characteristics or properties of objects in *U*. We will call *E the universe set of parameters* with respect to *U*.

Definition 2.1 [8]**.** A pair (F, A) is called a *soft set* over *U* if $A \subset E$ and $F : A \to P(U)$, where $P(U)$ is the set of all subsets of *U*.

Definition 2.2 [10]**.** Let *U* be an initial universe set and *E* be a universe set of parameters. Let (F, A) and (G, B) be soft sets over a common universe set *U* and $A, B \subseteq E$. Then (F, A) is a *subset* of (G, B), denoted by $(F, A) \tilde{\subseteq} (G, B)$, if

(i) $A \subset B$;

(ii) for all $e \in A$, $F(e) \subseteq G(e)$.

(F, A) equals (G, B), denoted by $(F, A) = (G, B)$, if $(F, A) \tilde{\subseteq} (G, B)$ and $(G, B) \tilde{\subseteq} (F, A)$.

Definition 2.3 [5]**.** A soft set (F, A) over U is said to be a *null soft set* denoted by Φ, if $\forall e \in A, F(e) = \varnothing$.

Definition 2.4 [5]**.** A soft set (F, A) over U is said to be an *absolute soft set* denoted by $\tilde{A}$, if $\forall e \in A, F(e) = U$.

Definition 2.5 [6]**.** Let (F, X) be a soft set over a universe set U. For $A \subseteq X$, we define $F(A) = \bigcup\{F(a) : a \in A\}$.

Definition 2.6 [4]**.** Let (F, X) be a soft set over a universe set U. For $A \subseteq X \subseteq E$ and $S \subseteq U$,

$$A_F = \{a \in A : F(a) = \varnothing\}; \quad A^F = \{a \in A : F(a) \neq \varnothing\};$$

$$F^{\leftarrow}(S) = \{a \in X : F(a) \subseteq S \text{ and } F(a) \neq \varnothing\}.$$

Lemma 2.7 [6]**.** *Let (F, X) be a soft set over a universe set U. Then for $A, B \subseteq X$,*

(i) $A \subseteq B$ *implies* $F(A) \subseteq F(B)$;

(ii) $F(A \bigcup B) = F(A) \bigcup F(B)$;

(iii) $F(A \bigcap B) \subseteq F(A) \bigcap F(B)$.

Theorem 2.8 [4]**.** *Let (F, X) be a soft set over a universe set U. Then for $A, B \subseteq X$,*

(i) $A = A_F \cup A^F$;

(ii) $F(A) = F(A^F)$;

(iii) $A^F \subseteq F^{\leftarrow}(F(A))$;

(iv) *if* $A \subseteq B$, *then* $A_F \subseteq B_F$ *and* $A^F \subseteq B^F$; *moreover*, $F^{\leftarrow}(F(A)) \subseteq F^{\leftarrow}(F(B))$.

Theorem 2.9 [4]**.** *Let* (F, X) *be a soft set over a universe set U. Then*

(i) $F(F^{\leftarrow}(S)) \subseteq S$ *for* $S \subseteq U$;

(ii) $F(F^{\leftarrow}(F(A))) = F(A)$ *for* $A \subseteq X$;

(iii) $F^{\leftarrow}(F(F^{\leftarrow}(F(A)))) = F^{\leftarrow}(F(A))$ *for* $A \subseteq X$.

3. Main Results

In this section, for a fixed parameter subset $X \subseteq E$ and a common universe set *U*, we study an operation $\mathcal{F} : P(U) \to P(U)$ induced by $F^{\leftarrow}$ and a given soft set (F, X) over a common universe set *U*.

Definition 3.1. Let (F, X) be a soft set over a universe set *U*. We define an operation $\mathcal{F} : P(U) \to P(U)$ as follows:

$$\mathcal{F}(S) = F(F^{\leftarrow}(S)) \text{ for } S \in P(U).$$

Theorem 3.2. *Let* (F, X) *be a soft set over a universe set U. Then the operation* $\mathcal{F} : P(U) \to P(U)$ *satisfies the following*:

(i) $\mathcal{F}(\emptyset) = \emptyset$.

(ii) $\mathcal{F}(S) \subseteq S$ *for* $S \in P(U)$.

Proof. Obvious. □

Lemma 3.3 [7]**.** *Let* (F, X) *be a soft set over a common universe U and* $V_1, V_2 \subseteq U$. *Then we have the following*:

(i) $F^{\leftarrow}(V_1) \cap F^{\leftarrow}(V_2) = F^{\leftarrow}(V_1 \cap V_2)$.

(ii) $F^{\leftarrow}(V_1) \cup F^{\leftarrow}(V_2) \subseteq F^{\leftarrow}(V_1 \cup V_2)$.

Proof. (i) For $a \in X$, $a \in F^{\leftarrow}(V_1) \cap F^{\leftarrow}(V_2)$ iff $a \in F^{\leftarrow}(V_1)$ and $a \in F^{\leftarrow}(V_2)$ iff $F(a) \subseteq V_1$ and $F(a) \subseteq V_2$ for $F(a) \neq \varnothing$ iff $F(a) \subseteq V_1 \cap V_2$ for $F(a) \neq \varnothing$ iff $a \in F^{\leftarrow}(V_1 \cap V_2)$.

(ii) For $a \in X$, $a \in F^{\leftarrow}(V_1) \cup F^{\leftarrow}(V_2) \Rightarrow a \in F^{\leftarrow}(V_1)$ or $a \in F^{\leftarrow}(V_2)$ $\Rightarrow F(a) \subseteq V_1$ or $F(a) \subseteq V_2$ for $F(a) \neq \varnothing \Rightarrow F(a) \subseteq V_1 \cup V_2$ for $F(a) \neq \varnothing \Rightarrow a \in F^{\leftarrow}(V_1 \cup V_2)$. □

Example 3.4. Let $U = \{x_1, x_2, x_3, x_4\}$ and a parameter set $E = \{e_1, e_2, e_3, e_4\}$. Consider $X = \{e_1, e_2, e_3\}$ and a soft set (F, X) defined as follows:

$$F(e_1) = \varnothing; \quad F(e_2) = \{x_2\}; \quad F(e_3) = \{x_1, x_3\}.$$

Let $V_1 = \{x_1, x_2\}$ and $V_2 = \{x_2, x_3, x_4\}$. Then $F^{\leftarrow}(V_1 \cup V_2) = \{e_2, e_3\}$. Note that $F(e_3) \nsubseteq V_1, V_2$. So $e_3 \notin F^{\leftarrow}(V_1)$ and $e_3 \notin F^{\leftarrow}(V_2)$. It implies that $F^{\leftarrow}(V_1 \cup V_2) \neq F^{\leftarrow}(V_1) \cup F^{\leftarrow}(V_2)$.

Theorem 3.5. *Let (F, X) be a soft set over a universe set U. Then the operation $\mathcal{F} : P(U) \to P(U)$ satisfies the following*:

$$\mathcal{F}(S_1 \cap S_2) \subseteq \mathcal{F}(S_1) \cap \mathcal{F}(S_2) \textit{ for } S_1, S_2 \in P(U).$$

Proof. Let $S_1, S_2 \in P(U)$. Then by (iii) of Lemma 2.7 and Lemma 3.3, we have

$$\begin{aligned} \mathcal{F}(S_1 \cap S_2) &= F(F^{\leftarrow}(S_1 \cap S_2)) \\ &= F(F^{\leftarrow}(S_1) \cap F^{\leftarrow}(S_2)) \\ &\subseteq F(F^{\leftarrow}(S_1)) \cap F(F^{\leftarrow}(S_2)) \\ &= \mathcal{F}(S_1) \cap \mathcal{F}(S_2). \end{aligned}$$

So $\mathcal{F}(S_1 \cap S_2) \subseteq \mathcal{F}(S_1) \cap \mathcal{F}(S_2)$. □

Example 3.6. Let $U = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8\}$ and a parameter set $E = \{e_1, e_2, e_3, e_4, e_5\}$. Consider $X = E$ and a soft set (F, X) defined as follows:

$$F(e_1) = \{x_1, x_2, x_3\};\ F(e_2) = \{x_2, x_3, x_4\};\ F(e_3) = \{x_3\};$$

$$F(e_4) = \{x_2, x_3, x_5\};\ F(e_5) = \{x_2, x_3, x_6\}.$$

Let $S_1 = \{x_1, x_2, x_3, x_4, x_7\}$ and $S_2 = \{x_2, x_3, x_5, x_6, x_8\}$. Note that:

$$F(F^{\leftarrow}(S_1 \cap S_2)) = F(F^{\leftarrow}(\{x_2, x_3\})) = F(\{e_3\}) = \{x_3\};$$

$$F(F^{\leftarrow}(S_1)) = F(\{e_1, e_2, e_3\}) = \{x_1, x_2, x_3, x_4\} \subseteq S_1;$$

$$F(F^{\leftarrow}(S_2)) = F(\{e_3, e_4, e_5\}) = \{x_2, x_3, x_5, x_6\} \subseteq S_2.$$

So $F(F^{\leftarrow}(S_1 \cap S_2)) \neq F(F^{\leftarrow}(S_1)) \cap F(F^{\leftarrow}(S_2))$.

Theorem 3.7 (Monotonicity)**.** *Let (F, X) be a soft set over a universe set U. Then the operation $\mathcal{F} : P(U) \to P(U)$ satisfies the following*:

$$S_1 \subseteq S_2 \Rightarrow \mathcal{F}(S_1) \subseteq \mathcal{F}(S_2) \text{ for } S_1, S_2 \in P(U).$$

Proof. For $S_1, S_2 \in P(U)$, let $S_1 \subseteq S_2$. Then for each $s \in F^{\leftarrow}(S_1)$, by hypothesis, $F(s) \subseteq S_1 \subseteq S_2$ and so $s \in F^{\leftarrow}(S_2)$. Hence, from Lemma 2.7, we have $\mathcal{F}(S_1) \subseteq \mathcal{F}(S_2)$. □

Theorem 3.8. *Let (F, X) be a soft set over a universe set U. Then the operation $\mathcal{F} : P(U) \to P(U)$ satisfies the following*:

$$\mathcal{F}(S_1) \cup \mathcal{F}(S_2) \subseteq \mathcal{F}(S_1 \cup S_2) \text{ for } S_1, S_2 \in P(U).$$

Proof. From Theorem 3.7, it is obvious. □

Example 3.9. Let $U = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8\}$ and a parameter set $E = \{e_1, e_2, e_3, e_4, e_5\}$. Consider $X = E$ and a soft set (F, X) defined as follows:

$F(e_1) = \{x_1, x_2, x_3\}$; $F(e_2) = \{x_2, x_3, x_4\}$; $F(e_3) = \{x_3\}$;

$F(e_4) = \{x_4, x_5, x_8\}$; $F(e_5) = \{x_6\}$.

Let $S_1 = \{x_1, x_2, x_3, x_4\}$ and $S_2 = \{x_5, x_6, x_7, x_8\}$. Note that:

$F(F^{\leftarrow}(S_1 \cup S_2)) = F(F^{\leftarrow}(U)) = F(E) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_8\}$;

$F(F^{\leftarrow}(S_1)) = F(\{e_1, e_2, e_3\}) = \{x_1, x_2, x_3, x_4\}$;

$F(F^{\leftarrow}(S_2)) = F(\{e_5\}) = \{x_6\}$.

Hence, we know that $F(F^{\leftarrow}(S_1 \cup S_2)) \neq F(F^{\leftarrow}(S_1)) \cup F(F^{\leftarrow}(S_2))$.

Theorem 3.10 (Idempotent)**.** *Let (F, X) be a soft set over a universe set U. Then the operation $\mathcal{F} : P(U) \to P(U)$ satisfies the following*:

$$\mathcal{F}(\mathcal{F}(S)) = \mathcal{F}(S) \text{ for } S \in P(U).$$

Proof. It is obviously obtained $\mathcal{F}(\mathcal{F}(S)) \subseteq \mathcal{F}(S)$ from Theorem 3.2 and the monotonicity of $\mathcal{F}$.

For the other part of proof, first we show that

$$F^{\leftarrow}(S) \subseteq F^{\leftarrow}(F(F^{\leftarrow}(S))).$$

Let $z \in F^{\leftarrow}(S)$. Then $F(z) \neq \varnothing$ and $F(z) \subseteq F(F^{\leftarrow}(S))$. From the definition of $F^{\leftarrow}$, we have $z \in F^{\leftarrow}(F(F^{\leftarrow}(S)))$ and finally $F^{\leftarrow}(S) \subseteq F^{\leftarrow}(F(F^{\leftarrow}(S)))$. From this fact and (i) of Lemma 2.7, it follows $F(F^{\leftarrow}(S)) \subseteq F(F^{\leftarrow}(F(F^{\leftarrow}(S))))$. Hence, we have $\mathcal{F}(S) \subseteq \mathcal{F}(\mathcal{F}(S))$. □

Theorem 3.11. *Let (F, X) be a soft set over a universe set U. Then for $i \in J \neq \varnothing$ and $A_i \subseteq X$, $\bigcup \mathcal{F}(A_i) = \mathcal{F}(\bigcup \mathcal{F}(A_i))$.*

Proof. Let $i \in J \neq \varnothing$ and $A_i \subseteq X$. Then by $\mathcal{F}(A_i) \subseteq \bigcup \mathcal{F}(A_i)$, monotonicity and idempotent of $\mathcal{F}$, $\mathcal{F}(A_i) = \mathcal{F}(\mathcal{F}(A_i)) \subseteq \mathcal{F}(\bigcup \mathcal{F}(A_i)) \subseteq$

$\bigcup \mathcal{F}(A_i)$, and $\bigcup \mathcal{F}(A_i) \subseteq \mathcal{F}(\bigcup \mathcal{F}(A_i)) \subseteq \bigcup \mathcal{F}(A_i)$. Hence $\bigcup \mathcal{F}(A_i) = \mathcal{F}(\bigcup \mathcal{F}(A_i))$. □

Theorem 3.12. *Let (F, X) be a soft set over a universe set U. If $A_i = \mathcal{F}(A_i)$ for $i \in J \neq \emptyset$ and $A_i \subseteq X$, then $\bigcup A_i = \mathcal{F}(\bigcup A_i)$.*

Proof. It is straightforward from Theorem 3.11. □

Definition 3.13. A soft set (F, A) is said to be *distinct* over U if for $a_1, a_2 \in A$, $a_1 \neq a_2 \in A$ implies $F(a_1) \cap F(a_2) = \emptyset$.

Theorem 3.14. *Let (F, X) be a soft set over U. If the soft set (F, X) is distinct over U, then $F(A \cap B) = F(A) \cap F(B)$.*

Proof. From Lemma 2.7, $F(A \cap B) \subseteq F(A) \cap F(B)$. For the proof of converse inclusion, let $z \in F(A) \cap F(B)$. Then for some $a \in A$ and $b \in B$, $z \in F(a)$ and $F(b)$. So $z \in F(a) \cap F(b) \neq \emptyset$ and by the law of contrapositive, $a = b$. This implies $a \in A \cap B$ and $z \in F(A \cap B)$. Hence $F(A) \cap F(B) \subseteq F(A \cap B)$. □

Theorem 3.15. *Let (F, X) be a soft set over a universe set U. If (F, X) is distinct, the operation $\mathcal{F} : P(U) \to P(U)$ satisfies $\mathcal{F}(S_1 \cap S_2) = \mathcal{F}(S_1) \cap \mathcal{F}(S_2)$ for $S_1, S_2 \in P(U)$.*

Proof. For $S_1, S_2 \in P(U)$, by Lemma 3.3, $F^{\leftarrow}(S_1 \cap S_2) = F^{\leftarrow}(S_1) \cap F^{\leftarrow}(S_2)$. From Theorem 3.14, it follows $F(F^{\leftarrow}(S_1 \cap S_2)) = F(F^{\leftarrow}(S_1) \cap F^{\leftarrow}(S_2)) = F(F^{\leftarrow}(S_1)) \cap F(F^{\leftarrow}(S_2))$. Hence, we have $\mathcal{F}(S_1 \cap S_2) = \mathcal{F}(S_1) \cap \mathcal{F}(S_2)$. □

Let *U* be an initial universe set. If a topology τ is given on the universe *U*, we call *U* a *topological universe* [6] with a topology τ (denoted by U_τ). The member of τ is said to be *open* in *U*.

Definition 3.16 [6]**.** Let (F, A) be a soft set over a topological universe set U_τ. We say that (F, A) is a *quasi-open soft set* if $F(A) = \bigcup\{F(a) : a \in A\}$ is open in U_τ.

Theorem 3.17. *If (F, A) is a quasi-open soft set over the topological universe set U_τ, then $(F, F^{\leftarrow}(F(A)))$ is quasi-open such that $F(F^{\leftarrow}(F(A))) = F(A)$.*

Proof. We know that a soft set $(F, F^{\leftarrow}(F(A)))$ is well defined. From (ii) of Theorem 2.9, it follows that $F(F^{\leftarrow}(F(A))) = F(A)$. By hypothesis, $F^{\leftarrow}(F(A))$ is open, and so $(F, F^{\leftarrow}(F(A)))$ is quasi-open. □

In Theorem 3.17, $(F, F^{\leftarrow}(F(A)))$ is not always a soft subset of (F, A) as shown in the next example:

Example 3.18. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and a parameter set $X = \{e_1, e_2, e_3, e_4\}$. Consider a soft set (F, X) defined as follows:

$$F(e_1) = \varnothing;\ \ F(e_2) = \{x_2\};\ \ F(e_3) = \{x_1, x_3\};\ \ F(e_4) = \{x_1, x_2\}.$$

For $A = \{e_1, e_2, e_3\}$ and a soft subset (F, A) of (F, X), $F^{\leftarrow}(F(A)) = \{e_2, e_3, e_4\} \nsubseteq A$ and so $(F, F^{\leftarrow}(F(A)))$ is not a soft subset of (F, A).

Lemma 3.19. *Let (F, X) be a soft set over a universe set U. If (F, X) is distinct, then for $A \subseteq X$ and $x \in X$, $F(x) \subseteq F(A)$ implies $x \in A$.*

Proof. For $A \subseteq X$ and $x \in X$, let $F(x) \subseteq F(A)$. Then since $F(A) = \bigcup\{F(a) : a \in A\}$, there exists an element a in A such that $F(x) \cap F(a) \neq \varnothing$. By hypothesis and the law of contrapositive, $x = a$ and so $x \in A$. □

Theorem 3.20. *Let (F, X) be a soft set over a universe set U. If (F, X) is distinct, then for $A \subseteq X$, $F^{\leftarrow}(F(A)) = A^F$.*

Proof. From (iii) of Theorem 2.8, it is obtained that $A^F \subseteq F^{\leftarrow}(F(A))$. Now we show the other inclusion $F^{\leftarrow}(F(A)) \subseteq A^F$. For the proof, let $x \in F^{\leftarrow}(F(A))$. Then from the definition of $F^{\leftarrow}$, $F(x) \neq \emptyset$ and $F(x) \subseteq F(A)$. From the above lemma, $x \in A$, and since $F(x) \neq \emptyset$, we have $x \in A^F$. □

Theorem 3.21. *If* (F, A) *is a soft subset of* (F, X) *and if* (F, X) *is distinct, then* $(F, F^{\leftarrow}(F(A)))$ *is a soft subset of* (F, A) *such that* $F(F^{\leftarrow}(F(A))) = F(A)$.

Proof. From the above theorem and (ii) of Theorem 2.9, $F^{\leftarrow}(F(A)) = A^F \subseteq A$ and $F(F^{\leftarrow}(F(A))) = F(A)$. Since $(F, F^{\leftarrow}(F(A)))$ is a well defined soft set as $F(x)$ for $x \in F^{\leftarrow}(F(A))$, $(F, F^{\leftarrow}(F(A)))$ is a soft subset of (F, A) satisfying the condition. □

In summary, we have the following theorem from the above lemma and theorems:

Theorem 3.22. *If* (F, A) *is a quasi-open soft set over the topological universe set* U_τ *and if* (F, X) *is distinct, then* $(F, F^{\leftarrow}(F(A)))$ *is a quasi-open soft subset of* (F, A).

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