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ON EXTENDED BIRECURRENT MANIFOLDS

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Abstract

In this paper, we introduce a type of Riemannian manifolds (namely, extended birecurrent manifold) and study its various geometric properties. Also the existence of such a manifold is ensured by a proper example.

1. Introduction

During the last six decades, the notion of locally symmetric manifolds which are a natural generalization of manifolds of constant curvature has been weakened in various ways such as recurrent manifold [13], 2-recurrent manifold [5], generalized recurrent manifold [1, 3, 4, 7, 12], generalized 2-recurrent manifold [8], quasi generalized recurrent manifold [10] and weakly generalized recurrent manifold [11].

The present paper introduces a type of Riemannian manifolds called extended birecurrent manifold realizing the following relation:

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$$(\nabla_U \nabla_V R)(X, Y, Z, W)$$
$$= A(U, V)R(X, Y, Z, W) + B(U, V)(g \bullet g)(X, Y, Z, W), \qquad (1.1)$$

where ∇ and R denote the Levi-Civita connection and the curvature tensor respectively, A and B are $(0, 2)$-tensors of which B is nonzero, the symbol $\bullet$ is the Nomizu-Kulkarni product of symmetric $(0, 2)$-tensors generating a curvature type tensor:

$$(h \bullet k)(X, Y, Z, W) = h(X, Z)k(Y, W) + h(Y, W)k(X, Z)$$
$$- h(X, W)k(Y, Z) - h(Y, Z)k(X, W).$$

In particular, if $B = 0$, then the manifold reduces to a birecurrent manifold. This justifies the name extended birecurrent manifold. In this paper, we study some properties of extended birecurrent manifold and give a proper example.

2. Some Properties of Extended Birecurrent Manifold

First, we have

Theorem 2.1. *Let* (M^n, g) *be an extended birecurrent manifold of dimension* n. *Then its scalar curvature* s *does not vanish, and the associated* $(0, 2)$-*tensor* A *in* (1.1) *is symmetric if and only if the associated* $(0, 2)$-*tensor* B *in* (1.1) *is symmetric*.

Proof. In local coordinates, (1.1) can be expressed as

$$R_{ijkl;\,qp} = A_{pq}R_{ijkl} + B_{pq}(g \bullet g)_{ijkl}, \qquad (2.2)$$

where the "semicolon" denotes covariant derivative with respect to the metric.

Contracting (2.2) over i and l, and then contracting the relation obtained thus over j and k, we obtain

$$s_{;\,qp} = A_{pq}s + B_{pq}2(1 - n)n. \qquad (2.3)$$

Suppose $s = 0$, then we have from (2.3) $B = 0$, which is inadmissible by the defining condition of extended birecurrent manifold. Therefore we conclude $s \neq 0$. On the other hand, it follows from (2.3) that

$$0 = (A_{pq} - A_{qp})s + (B_{pq} - B_{qp})2(1-n)n, \tag{2.4}$$

which yields

$$A_{pq} - A_{qp} = \frac{2(n-1)n}{s}(B_{pq} - B_{qp}).$$

And so from the last relation we conclude that

$$A_{pq} = A_{qp}$$

is equivalent to

$$B_{pq} = B_{qp}.$$

This completes the proof of Theorem 2.1.

Theorem 2.2. *Let* (M^n, g) *be an extended birecurrent manifold. If its scalar curvature s is constant, then either the associated* $(0, 2)$*-tensors A, B in* (1.1) *are symmetric or the manifold is a space of constant curvature. In addition, if A is skew-symmetric, then B is skew-symmetric, and vice versa.*

Proof. Taking account of (2.3), we have

$$s_{;pq} + s_{;qp} = (A_{pq} + A_{qp})s + (B_{pq} + B_{qp})2(1-n)n. \tag{2.5}$$

From $s =$ constant and (2.5), it follows that

$$A_{pq} + A_{qp} = \frac{2(n-1)n}{s}(B_{pq} + B_{qp}), \tag{2.6}$$

which implies that

$$A_{pq} = -A_{qp}$$

is equivalent to

$$B_{pq} = -B_{qp}.$$

Furthermore, by virtue of (2.3) and $s =$ constant, we have

$$B_{pq} = \frac{s}{2(n-1)n} A_{pq}. \tag{2.7}$$

Considering both (2.2) and (2.7), we get

$$R_{ijkl;\,qp} = A_{pq}\left[R_{ijkl} + \frac{s}{2(n-1)n}(g \bullet g)_{ijkl}\right]. \tag{2.8}$$

By virtue of (2.8), we have

$$R_{ijkl;\,qp} - R_{ijkl;\,pq} = (A_{pq} - A_{qp})\left[R_{ijkl} + \frac{s}{2(n-1)n}(g \bullet g)_{ijkl}\right]. \tag{2.9}$$

From Walker's lemma [13] and (2.9), it follows

$$\begin{aligned} 0 &= R_{ijkl;\,qp} - R_{ijkl;\,pq} + R_{klpq;\,ji} - R_{klpq;\,ij} + R_{pqij;\,lk} - R_{pqij;\,kl} \\ &= (A_{pq} - A_{qp})\left[R_{ijkl} + \frac{s}{2(n-1)n}(g \bullet g)_{ijkl}\right] \\ &\quad + (A_{ij} - A_{ji})\left[R_{klpq} + \frac{s}{2(n-1)n}(g \bullet g)_{klpq}\right] \\ &\quad + (A_{kl} - A_{lk})\left[R_{pqij} + \frac{s}{2(n-1)n}(g \bullet g)_{pqij}\right]. \end{aligned} \tag{2.10}$$

Note that $L_{ijkl} = R_{ijkl} + \frac{s}{2(n-1)n}(g \bullet g)_{ijkl}$ is a symmetric $(0, 4)$-tensor with respect to the first pair of two indices and the last pair of two indices. Using Walker's lemma [13] we immediately deduce that either $A_{pq} - A_{qp} = 0$ (and so $B_{pq} - B_{qp} = 0$ by (2.7)) or $L_{ijkl} = 0$, showing that either the associated $(0, 2)$-tensors A, B in (1.1) are symmetric or the manifold is a space of constant curvature. This completes the proof of Theorem 2.2.

A Riemannian manifold (M^n, g) is said to be *Einstein* [2] if its Ricci tensor r is proportional to the metric g, that is, $r = \frac{s}{n} g$. Note that if $n \geq 3$, then its scalar curvature s is constant [2]. We call a Riemannian manifold *Einstein and extended birecurrent* if the manifold is simultaneously an Einstein manifold and an extended birecurrent manifold.

Theorem 2.3. *Let* (M^n, g) $(n > 3)$ *be an Einstein and extended birecurrent manifold. Then the associated* $(0, 2)$*-tensors A, B in* (1.1) *are collinear, and the associated* $(0, 2)$*-tensor A does not vanish.*

Proof. By virtue of the second Bianchi identity, we have

$$(\nabla_V R)(X, Y, Z, W) + (\nabla_W R)(X, Y, V, Z) + (\nabla_Z R)(X, Y, W, V) = 0, \tag{2.11}$$

which implies

$$(\nabla_U \nabla_V R)(X, Y, Z, W) + (\nabla_U \nabla_W R)(X, Y, V, Z) + (\nabla_U \nabla_Z R)(X, Y, W, V) = 0. \tag{2.12}$$

Taking account of (1.1) and (2.12), we have

$$\begin{aligned} &A(U, V)R(X, Y, Z, W) + B(U, V)(g \bullet g)(X, Y, Z, W) \\ &+ A(U, W)R(X, Y, V, Z) + B(U, W)(g \bullet g)(X, Y, V, Z) \\ &+ A(U, Z)R(X, Y, W, V) + B(U, Z)(g \bullet g)(X, Y, W, V) = 0. \end{aligned} \tag{2.13}$$

Contracting (2.13) on X, W and then contracting the relation obtained thus on Y, Z, we obtain from the Einstein condition

$$\frac{s}{n}(n-2)A(U, V) + (-2)(n-1)(n-3)B(U, V) = 0. \tag{2.14}$$

Taking account of $B \neq 0$, $n > 3$ and (2.14), we have

$$A \neq 0$$

and

$$B = \frac{(n-2)s}{2n(n-1)(n-3)} A.$$

This completes the proof of Theorem 2.3.

A Riemannian manifold (M^n, g) is said to be *extended Ricci birecurrent* if its Ricci tensor r satisfies the relation

$$(\nabla_U \nabla_V r)(Y, Z) = D(U, V) r(Y, Z) + E(U, V) g(Y, Z), \tag{2.15}$$

where D and E are the associated $(0, 2)$-tensors of which E is nonzero. Now we have

Theorem 2.4. *Every extended birecurrent manifold is extended Ricci birecurrent*.

Proof. By contracting (1.1) on X and W, we have

$$(\nabla_U \nabla_V r)(Y, Z) = A(U, V) r(Y, Z) + B(U, V) 2(1-n) g(Y, Z),$$

showing that the manifold is extended Ricci birecurrent. This completes the proof of Theorem 2.4.

The Weyl curvature tensor C of type $(0, 4)$ is defined as follows:

$$\begin{aligned} C(X, Y, Z, W) = R(X, Y, Z, W) - \frac{1}{(n-2)} [&g(Y, Z) r(X, W) \\ &- g(X, Z) r(Y, W) + r(Y, Z) g(X, W) - r(X, Z) g(Y, W)] \\ &+ \frac{s}{(n-1)(n-2)} [g(Y, Z) g(X, W) - g(X, Z) g(Y, W)]. \end{aligned} \tag{2.16}$$

Notice that $C = 0$ if and only if (M^n, g) is conformally flat [2].

A Riemannian manifold is called *Weyl birecurrent* if its Weyl curvature tensor C satisfies the relation

$$(\nabla_U \nabla_V C)(X, Y, Z, W) = A(U, V) C(X, Y, Z, W).$$

Then we have

Theorem 2.5. *Every extended birecurrent manifold is Weyl birecurrent.*

Proof. Contracting (1.1) on X and W, we get

$$(\nabla_U \nabla_V r)(Y, Z) = A(U, V) r(Y, Z) + B(U, V) 2(1-n) g(Y, Z). \quad (2.17)$$

Again contracting (2.17) on Y and Z, we obtain

$$\nabla_U \nabla_V s = A(U, V) s + B(U, V) 2(1-n) n. \quad (2.18)$$

From (2.16), it follows that

$$\begin{aligned}
&(\nabla_U \nabla_V C)(X, Y, Z, W) \\
&= (\nabla_U \nabla_V R)(X, Y, Z, W) - \frac{1}{(n-2)}[g(Y, Z)(\nabla_U \nabla_V r)(X, W) \\
&\quad - g(X, Z)(\nabla_U \nabla_V r)(Y, W) + (\nabla_U \nabla_V r)(Y, Z) g(X, W) \\
&\quad - (\nabla_U \nabla_V r)(X, Z) g(Y, W)] \\
&\quad + \frac{\nabla_U \nabla_V s}{(n-1)(n-2)}[g(Y, Z) g(X, W) - g(X, Z) g(Y, W)]. \quad (2.19)
\end{aligned}$$

Taking account of (1.1), (2.16), (2.17), (2.18) and (2.19), we have

$$(\nabla_U \nabla_V C)(X, Y, Z, W) = A(U, V) C(X, Y, Z, W).$$

This completes the proof of Theorem 2.5.

A Riemannian manifold (M^n, g) is called *generalized recurrent* if its curvature tensor R satisfies the relation

$$\begin{aligned}
&(\nabla_V R)(X, Y, Z, W) \\
&= \alpha(V) R(X, Y, Z, W) + \beta(V)(g \bullet g)(X, Y, Z, W), \quad (2.20)
\end{aligned}$$

where α and β are the associated 1-forms of which β is nonzero.

Then we have

Theorem 2.6. *Every generalized recurrent manifold is extended birecurrent*.

Proof. From (2.20), it follows that

$$\begin{aligned}
&(\nabla_U \nabla_V R)(X, Y, Z, W) \\
&= ((\nabla_U \alpha)(V) + \alpha(V)\alpha(U))R(X, Y, Z, W) \\
&\quad + ((\nabla_U \beta)(V) + \alpha(V)\beta(U))(g \bullet g)(X, Y, Z, W) \\
&= A(U, V)R(X, Y, Z, W) + B(U, V)(g \bullet g)(X, Y, Z, W),
\end{aligned}$$

which implies that the manifold is extended birecurrent.

Theorem 2.7. *Let* (M^n, g) *be an extended Ricci birecurrent manifold. If* (M^n, g) *is conformally flat, then the manifold is extended birecurrent*.

Proof. From (2.16) and conformal flatness, it follows that

$$\begin{aligned}
&(\nabla_U \nabla_V R)(X, Y, Z, W) \\
&= \frac{1}{(n-2)}[g(Y, Z)(\nabla_U \nabla_V r)(X, W) - g(X, Z)(\nabla_U \nabla_V r)(Y, W) \\
&\quad + (\nabla_U \nabla_V r)(Y, Z)g(X, W) - (\nabla_U \nabla_V r)(X, Z)g(Y, W)] \\
&\quad - \frac{\nabla_U \nabla_V s}{(n-1)(n-2)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \qquad (2.21)
\end{aligned}$$

Contracting (2.15) on Y and Z, we get

$$\nabla_U \nabla_V s = D(U, V)s + E(U, V)n. \qquad (2.22)$$

Taking account of (2.15), (2.21) and (2.22), we have

$$\begin{aligned}
&(\nabla_U \nabla_V R)(X, Y, Z, W) \\
&= \frac{D(U, V)}{(n-2)}[g(Y, Z)r(X, W) - g(X, Z)r(Y, W) + r(Y, Z)g(X, W)
\end{aligned}$$

$$- r(X, Z)g(Y, W)] - \frac{D(U, V)s}{(n-1)(n-2)}[g(Y, Z)g(X, W)$$

$$- g(X, Z)g(Y, W)] + \frac{E(U, V)}{(n-2)}[g(Y, Z)g(X, W)$$

$$- g(X, Z)g(Y, W) + g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$$

$$- \frac{E(U, V)n}{(n-1)(n-2)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)],$$

which reduces to

$$(\nabla_U \nabla_V R)(X, Y, Z, W)$$

$$= A(U, V)R(X, Y, Z, W) + B(U, V)(g \bullet g)(X, Y, Z, W),$$

where $A = D$ and $B = \frac{1}{2(1-n)}E.$

Therefore the manifold is extended birecurrent and this completes the proof of Theorem 2.7.

A Riemannian manifold is said to be an *extended Weyl birecurrent manifold* if its Weyl curvature tensor C satisfies the relation

$$(\nabla_U \nabla_V C)(X, Y, Z, W)$$

$$= F(U, V)C(X, Y, Z, W) + G(U, V)(g \bullet g)(X, Y, Z, W), \qquad (2.23)$$

where F and G are the associated $(0, 2)$-tensors.

Now we have

Theorem 2.8. *Let* (M^n, g) *be an extended Weyl birecurrent manifold. If* (M^n, g) *is Einstein, then the manifold is extended birecurrent.*

Proof. Taking account of (2.16) and the Einstein condition $r = \frac{s}{n}g,$ we get

$$C(X, Y, Z, W)$$

$$= R(X, Y, Z, W) - \frac{s}{n(n-1)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \quad (2.24)$$

From (2.23) and (2.24), it follows that

$$(\nabla_U \nabla_V R)(X, Y, Z, W)$$

$$= F(U, V)R(X, Y, Z, W) - \frac{F(U, V)s}{n(n-1)}[g(Y, Z)g(X, W)$$

$$- g(X, Z)g(Y, W)] + G(U, V)(g \bullet g)(X, Y, Z, W)$$

$$= A(U, V)R(X, Y, Z, W) + B(U, V)(g \bullet g)(X, Y, Z, W),$$

where $A = F$ and $B = \frac{s}{2n(n-1)}F + G.$

This shows that the manifold is extended birecurrent and so completes the proof of Theorem 2.8.

Theorem 2.9. *On an extended birecurrent manifold, there does not exist a nontrivial parallel vector field.*

Proof. Suppose that a vector field P is parallel, that is, $\nabla P = 0$ [9, 14], then from the definition of curvature tensor R we have

$$R(X, Y)P = \nabla_X \nabla_Y P - \nabla_Y \nabla_X P - \nabla_{[X, Y]} P = 0$$

and so

$$g(R(X, Y)P, W) = R(X, Y, P, W) = 0. \quad (2.25)$$

By virtue of (1.1) and (2.25), we have

$$0 = (\nabla_U \nabla_V R)(X, Y, P, W)$$

$$= A(U, V)R(X, Y, P, W) + B(U, V)(g \bullet g)(X, Y, P, W)$$

$$= B(U, V)(g \bullet g)(X, Y, P, W). \quad (2.26)$$

Contracting (2.26) on X, W and then putting $Y = P$ in the relation obtained thus, we get

$$2(1 - n)B(U, V)\| P \|^2 = 0,$$

which yields $P = 0$ because $B = 0$ is inadmissible by the defining condition of extended birecurrent manifold. Therefore the manifold does not allow a nontrivial parallel vector field. This completes the proof of Theorem 2.9.

3. An Example of Extended Birecurrent Manifold

In this section, the existence of an extended birecurrent manifold is ensured by the following nontrivial example.

Example. We define a Riemannian manifold (R^4_+, g) as follows:

$$R^4_+ = \{(x_1, x_2, x_3, x_4) | x^1 > 1\},$$

$$g = (g_{ij}) = (1 + e^{x^1})\delta_{ij},$$

where $i, j = 1, ..., 4$ and δ_{ij} is the Kronecker delta $(0$ if $i \neq j$, 1 if $i = j)$.

The only nonvanishing components of the Christoffel symbols, the curvature tensor and their covariant derivatives (of second order) are

$$\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{1}{2}\left(\frac{e^{x^1}}{1 + e^{x^1}}\right),$$

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{1}{2}\left(\frac{e^{x^1}}{1 + e^{x^1}}\right),$$

$$R_{1221} = R_{1331} = R_{1441} = \frac{1}{2}\left(\frac{e^{x^1}}{1 + e^{x^1}}\right),$$

$$R_{2332} = R_{2442} = R_{3443} = \frac{1}{4}\left(\frac{e^{2x^1}}{1 + e^{x^1}}\right),$$

$$R_{1221;1} = R_{1331;1} = R_{1441;1} = \frac{e^{x^1}(1-2e^{x^1})}{2(1+e^{x^1})^2},$$

$$R_{2332;1} = R_{2442;1} = R_{3443;1} = \frac{e^{2x^1}(2-e^{x^1})}{4(1+e^{x^1})^2},$$

$$R_{1221;11} = R_{1331;11} = R_{1441;11} = \frac{e^{x^1}(2-15e^{x^1}+10e^{2x^1})}{4(1+e^{x^1})^3},$$

$$R_{2332;11} = R_{2442;11} = R_{3443;11} = \frac{e^{2x^1}(8-16e^{x^1}+3e^{2x^1})}{8(1+e^{x^1})^3},$$

and the components obtained by the symmetry properties.

We consider the $(0,\ 2)$-tensors A and B as follows:

If $i = j = 1$,

$$A_{ij} = \frac{b}{a}, \quad B_{ij} = \frac{d}{c}$$

and otherwise

$$A_{ij} = 0, \quad B_{ij} = 0,$$

where

$$a = 8(1+e^{x^1})\left(-e^{x^1} - \frac{1}{2}e^{2x^1} + \frac{1}{2}e^{3x^1}\right),$$

$$b = -8e^{x^1} + 76e^{2x^1} - 72e^{3x^1} + 6e^{4x^1},$$

$$c = 16(1+e^{x^1})^4\left(-e^{x^1} - \frac{1}{2}e^{2x^1} + \frac{1}{2}e^{3x^1}\right)$$

and

$$d = 6e^{3x^1} - e^{4x^1} - 7e^{5x^1}.$$

Notice that a and c are nonzero on R_{+}^{4}. It follows from straightforward computations that the above tensors satisfy the relation

$$R_{ijkl;\,pq} = A_{pq} R_{ijkl} + B_{pq} (g \bullet g)_{ijkl},$$

showing that the manifold is extended birecurrent.

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