Internat. J. Functional Analysis, Operator Theory and Applications
© 2014 Pushpa Publishing House, Allahabad, India
Published Online: September 2014
Available online at http://pphmj.com/journals/ijaota.htm
Volume 6, Number 2, 2014, Pages 97-118

ON DUALS OF COUNTABLY SEMINORMED SPACES

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Abstract

It is well known that a normed space E is a uniformly convex (smooth) normed space if and only if its dual E^* is uniformly smooth (convex). We extend these geometric properties to seminormed spaces and then introduce definitions of uniformly convex (smooth) countably seminormed spaces. A new vision of the completion of countably seminormed space was helpful in our task. We get some fundamental links between Lindenstrauss duality formulas. A duality property between uniform convexity and uniform smoothness of countably seminormed space is also given.

1. Definitions and Examples

Definition 1.1 (Countably seminormed space) [5, 3]**.** A linear space E

Received: April 10, 2014; Revised: April 29, 2014; Accepted: June 19, 2014
2010 Mathematics Subject Classification: 46A04.
Keywords and phrases: countably seminormed space, Fréchet spaces, uniformly convex (smooth) normed space, completion of countably seminormed space, uniformly convex (smooth) countably seminormed space, the dual of a countably seminormed space.
Communicated by Nasir Nabievich Ganikhodjaev

equipped with a total countable family of seminorms $\{p_n,\ n \in \mathcal{N}\}$ is said to be a *countably seminormed space*. Totality $(p_n(x) = 0,\ \forall n \Rightarrow x = \theta)$ guarantees that E is Hausdorff. A complete countably seminormed space is called a *Fréchet space*. If E is equipped with one seminorm p, then $(E,\ p)$ is called a *seminormed space*. If p is a norm, then $(E,\ p)$ is a normed space.

Remarks [5, 7]**.** (1) Without loss of generality (by taking the equivalent system of seminorms $\dot{p}_n(x) = \max_{i=1}^{n} p_i(x)),$ one can assume that the sequence of seminorms $\{p_n;\ n = 1, 2, ...\}$ is increasing, i.e., $p_1(x) \leq p_2(x) \leq \cdots \leq p_n(n) \leq \cdots,\ \forall x \in E.$

(2) Any countably seminormed space is metrizable and its metric d can be defined by $d(x,\ y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p_i(x-y)}{1+p_i(x-y)}.$

Definition 1.2 (Compatible norms) [7]**.** Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in a linear space E are said to be *compatible* if, whenever a sequence $\{x_n\}$ in E is Cauchy with respect to both norms and converges to a limit $x \in E$ with respect to one of them, it also converges to the same limit x with respect to the other norm.

Remark 1.3 [7, 3]**.** If only one of the seminorms, say p_{n_0}, is a norm, then by adding this norm to each of the seminorms, we will get an equivalent system of increasing norms and if these norms are pairwise compatible, then E is, in fact, a countably normed space.

The following examples are Fréchet spaces [3].

(1) The space $\mathcal{R}^\infty$ of all sequences $\{a_i\}$ of real numbers is equipped with

$$p_n(\{a_i\}) = \sum_{i=1}^{i=n} |\, a_i \,| \text{ for } n \in \mathcal{N}.$$

(2) The space $\mathcal{C}^{\infty}[a, b]$ of smooth functions (infinitely differentiable on $]a, b[$ and continuous at the ends of the interval) is equipped with

$$p_n(f) = \sum_{i=0}^{i=n} \sup_x |D^i f(x)| \text{ for } n \in \mathcal{N}.$$

(3) X is a compact manifold and V is a vector bundle over X. Let $\mathfrak{C}(X, V)$ be the vector space of smooth sections of the bundle over X. Choose Riemannian metrics and connections on the bundles TX and V and let $D^i f$ denote the ith covariant derivative of a section f of V. $\mathfrak{C}(X, V)$ is equipped with

$$p_n(f) = \sum_{i=0}^{i=n} \sup_x |D^i f(x)| \text{ for } n \in \mathcal{N}.$$

Example 1.4 [4]**.** For $1 < p < \infty$, the space $\ell^{p+0} := \bigcap_{q>p} \ell^q$ is a countably normed space. In fact, one can easily see that $\ell^{p+0} = \bigcap_n \ell^{p_n}$ for any choice of a monotonic decreasing sequence $\{p_n\}$ converging to p. ℓ^{p_n} is Banach for every n, it is clear now that the countably normed space ℓ^{p+0} is complete.

Notation. Let L be a subspace of a topological linear space E. We may write $x + L$ to denote the equivalence class $\hat{x}$ belonging to the factor space E/L. So we can write $x \in \hat{x} = x + L \in E/L$.

Definition 1.5 (Normed space associated with seminormed space) [5]**.** For a seminormed space (E, p), there is a normed space $E/\ker p$ with the norm $\| x + \ker p \|_p = p(x)$ called the *associated normed space* with the seminormed space (E, p).

Definition 1.6 (Uniformly convex normed space) [8]**.** A normed linear space E is called *uniformly convex* if for any $\varepsilon \in (0, 2]$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in E$ with $\| x \| = 1$, $\| y \| = 1$ and $\| x - y \| \geq \varepsilon$, then $\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta$.

Definition 1.7 (Modulus of convexity) [8]**.** Let E be a normed linear space with $\dim E \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) := \inf\left\{1 - \left\| \frac{x + y}{2} \right\| : \| x \| \leq 1, \| y \| \leq 1; \| x - y \| \geq \varepsilon\right\}.$$

Definition 1.8 (Uniformly smooth normed space) [8]**.** A normed linear space E is said to be *uniformly smooth* if whenever given $\varepsilon > 0$ there exists $\delta > 0$ such that if $\| x \| = 1$ and $\| y \| \leq \delta$, then

$$\| x + y \| + \| x - y \| < 2 + \varepsilon \| y \|.$$

Definition 1.9 (Modulus of smoothness) [8]**.** Let E be a normed linear space with $\dim E \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\| x + y \| - \| x - y \|}{2} - 1 : \| x \| = 1; \| y \| = \tau\right\}$$

$$= \sup\left\{\frac{\| x + \tau y \| + \| x - \tau y \|}{2} - 1 : \| x \| = 1 = \| y \|\right\}.$$

Definition 1.10 [5]**.** A linear functional f on a countably seminormed space E is continuous if there is a seminorm p_{n_0} on E and a constant $c > 0$ such that for all $x \in E$,

$$| f(x) | \leq c p_{n_0}(x).$$

Definition 1.11 [5]**.** The space of all linear continuous functionals on a countably seminormed space E is called the *dual* of E and is denoted by E^*.

We suggest the following definitions:

Definition 1.12 (Uniformly convex seminormed space)**.** We call a seminormed linear space (E, p) *uniformly convex* if for any $\varepsilon \in (0, 2]$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in E$ with $p(x) = 1$, $p(y) = 1$ and $p(x - y) \geq \varepsilon$, then $p\left(\frac{1}{2}(x + y)\right) \leq 1 - \delta$.

Definition 1.13 (Uniformly smooth seminormed space)**.** We call a seminormed linear space (E, p) *uniformly smooth* if whenever given $\varepsilon > 0$ there exists $\delta > 0$ such that if $p(x) = 1$ and $p(y) \leq \delta$, then

$$p(x + y) + p(x - y) < 2 + \varepsilon p(y).$$

2. Technical Lemmas

Since the norm in the associated normed space is defined by the seminorm, so the following proposition is easy to prove.

Proposition 2.1. (1) *A seminormed space* (E, p) *is uniformly convex if and only if its associated normed space is uniformly convex*. *This gives an equivalent definition of a uniformly convex seminormed space*.

(2) *A seminormed space* (E, p) *is uniformly smooth if and only if its associated normed space is uniformly smooth*. *This gives an equivalent definition of a uniformly smooth seminormed space*.

(3) *A seminormed space* (E, p) *is complete if and only if its associated normed space is complete*.

Proof. (1) "$\Rightarrow$" Let (E, p) be a uniformly convex seminormed space. Then we prove that $(E/\ker p, \|\ \|_p)$ is a uniformly convex normed space. Let $\varepsilon > 0$ be given and $\hat{x}, \hat{y} \in E/\ker p$, $\|\hat{x}\| = \|\hat{y}\| = 1$ such that $\|\hat{x} - \hat{y}\| \geq \varepsilon$. Then there exist $x, y \in E$ such that $\hat{x} = x + \ker p$, $\hat{y} = y + \ker p$, $p(x) = p(y) = 1$ and $p(x - y) \geq \varepsilon$. Since (E, p) is uniformly convex according to

Definition 1.12, there exists $\delta(\varepsilon) > 0$ such that $p\left(\frac{1}{2}(x + y)\right) \leq 1 - \delta.$ So $\left\| \frac{1}{2}(\hat{x} + \hat{y}) \right\| \leq 1 - \delta.$

"$\Leftarrow$" Let $(E/\ker p, \| \, \|_p)$ be a uniformly convex normed space. Then we prove that (E, p) is a uniformly convex seminormed space. Let $\varepsilon > 0$ be given and $x, y \in E, \ p(x) = p(y) = 1$ such that $p(x - y) \geq \varepsilon.$ Then there exist $\hat{x} = x + \ker p, \ \hat{y} = y + \ker p \in E/\ker p, \ \| \hat{x} \| = \| \hat{y} \| = 1$ and $\| \hat{x} - \hat{y} \| \geq \varepsilon.$

(2) "$\Rightarrow$" Let (E, p) a be uniformly smooth seminormed space. Then we prove that $(E/\ker p, \| \, \|_p)$ is a smooth convex normed space. Let $\varepsilon > 0$ be given and $\hat{x}, \hat{y} \in E/\ker p, \| \hat{x} \| = 1$ such that $\| \hat{x} + \hat{y} \| + \| \hat{x} - \hat{y} \| < 2 + \varepsilon \| \hat{y} \|.$ Then there exist $x, y \in E$ such that $\hat{x} = x + \ker p, \ \hat{y} = y + \ker p, \ p(x) = 1$ and $p(x + y) + p(x - y) < 2 + \varepsilon p(y).$ Since (E, p) is uniformly convex according to Definition 1.13, there exists $\delta(\varepsilon) > 0$ such that $p(y) \leq \delta.$ So $\| \hat{y} \| \leq \delta.$

"$\Leftarrow$" Let $(E/\ker p, \| \, \|_p)$ be a uniformly smooth normed space. Then we prove that (E, p) is a uniformly smooth seminormed space. Let $\varepsilon > 0$ be given and $x, y \in E, \ p(x) = 1$ such that $p(x + y) + p(x - y) < 2 + \varepsilon p(y).$ Then there exist $\hat{x} = x + \ker p, \ \hat{y} = y + \ker p \in E/\ker p, \ \| \hat{x} \| = 1$ and $\| \hat{x} + \hat{y} \| + \| \hat{x} - \hat{y} \| < 2 + \varepsilon \| \hat{y} \|.$

(3) "$\Rightarrow$" Let (E, p) be complete. Then we prove that $(E/\ker p, \| \, \|_p)$ is also complete. In fact, let $\{\hat{x}_n\}$ be a Cauchy sequence in $(E/\ker p, \| \, \|_p),$ where $\{\hat{x}_n = x_n + \ker p\}.$ Then $\| \hat{x}_n - \hat{x}_m \|_p = P(x_n - x_m) \to 0$ as $n, m \to \infty.$ Hence, $\{x_n\}$ is a Cauchy sequence in $(E, p).$ Since (E, p) is complete, there exists an element $x_0 \in E$ such that $p(x_n - x_0) \to 0$ as $n \to \infty.$ Hence, $\hat{x}_0 = x_0 + \ker p \in E/\ker p$ is the limit of the Cauchy sequence $\{\hat{x}_n\}.$

"$\Leftarrow$" by the same previous way. $\square$

Example 2.2. Let $\mathbb{Q}$ be the set of all rational numbers. $\mathbb{Q}^2$ with the following norm $\| z = (x, y) \| = \sqrt{x^2 + y^2}$ is uniformly convex.

Define $\overline{\mathbb{Q}}^2 = \{(z_n) : (z_n) \text{ is a Cauchy sequence in } \mathbb{Q}^2\}$ and a seminorm on $\overline{\mathbb{Q}}^2$ by $p((z_n)) = \lim_{n \to \infty} \| z_n \|$. Since the associated normed space of $(\overline{\mathbb{Q}}^2, p)$ is $\mathbb{R}^2$ with the norm $\| (x, y) \| = \sqrt{x^2 + y^2}$ and it is uniformly convex (smooth), $(\overline{\mathbb{Q}}^2, p)$ is a uniformly convex (smooth) seminormed space.

Remark 2.3. The dual of a seminormed space (E, p) is a normed space.

In fact, for every $f \in (E, p)^*$, define $\| f \| = \sup_{x \notin \ker p} \frac{| f(x) |}{p(x)}$, which is norm. Since if $\| f \| = 0$, then $| f(x) | = 0$ for all $x \notin \ker p$ and $| f(x) | = 0$ for all $x \in \ker p$ because $| f(x) | \leq cp(x)$, so $f(x) = 0$ for all $x \in E$, f is the zero functional.

It is easy to prove $\| \lambda f \| = | \lambda | \| f \|$ for all $\lambda \in \mathcal{R}$ and $\| f + g \| \leq \| f \| + \| g \|$.

An interesting fact is the following:

Proposition 2.4. *The dual of a seminormed space* (E, p) *is isomorphically isometric to the dual of its associated normed space* $(E/\ker p, \| \|_p)$.

Proof. For any linear continuous functional $f \in (E, p)^*$, we define $\hat{f} \in (E/\ker p, \| \|_p)^*$ by $\hat{f}(x + \ker p) = f(x)$, then $\hat{f}$ is well defined, linear and continuous. In fact, if $x + \ker p = y + \ker p$, then $x - y \in \ker p$, so $| f(x) - f(y) | \leq cp(x - y) = 0$.

On the other hand, for $\hat{f} \in (E/\ker p, \|\ \|_p)^*$, we define $f \in (E, p)^*$ by $f(x) = \hat{f}(x + \ker p)$, then f is well defined, linear and continuous.

To show the isometry, we have for any $f \in (E, p)^*$ and $\hat{f} \in (E/\ker p, \|\ \|_p)^*$,

$$\|\hat{f}\| = \sup_{\hat{x} \neq 0} \frac{|\hat{f}(\hat{x})|}{\|\hat{x}\|_p} = \sup_{x \notin \ker p} \frac{|f(x)|}{p(x)} = \|f\|. \qquad \square$$

Corollary 2.5. *A seminormed space (E, p) is uniformly convex* (*smooth*) *if and only if its dual $(E, p)^*$ is uniformly smooth* (*convex*).

3. Completion of Countably Seminormed Space

In 1998, Merkle [2] introduced the completion of countably seminormed space. We quoted his work and changed some notations to be suitable to prove our work.

For fixed seminorm p_n, one can define a seminorm $\overline{p}_n$ on the space

$$E_{p_n} = \{\{x_i\} : \{x_i\} \text{ is } p_n \text{ Cauchy sequence}\}$$

as follows:

$$\overline{p}_n(\{x_i\}_{i \in \mathcal{N}}) = \lim_{i \to \infty} p_n(x_i).$$

This limit exists, because $\{p_n(x_i)\}_{i \in \mathcal{N}}$ is a Cauchy sequence in $\mathcal{R}$.

By standard arguments, one can prove that $(E_{p_n}, \overline{p}_n)$ is the completion of the countably seminormed space E if equipped with only the seminorm p_n.

In fact, defining a map $\pi : E \to E_{p_n}$ by $\pi(x) = (x, x, x, \ldots)$ gives a 1-1 isometrical and isomorphical mapping of E (as a seminormed space with p_n) onto a dense linear subspace of the space $(E_{p_n}, \overline{p}_n)$.

Since for any $x \in E$, $p_n(x) \leq p_{n+1}(x)$, $\forall n \in \mathcal{N}$, every p_{n+1} Cauchy sequence is p_n Cauchy sequence.

So

$$E \subset \cdots \subset E_{p(n+1)} \subset E_{p_n} \subset \cdots \subset E_{p_1}.$$

Therefore, $\bigcap_{n \in \mathcal{N}} E_{p_n}$ is a countably seminormed space with

$$\overline{p}_1(x) \leq \overline{p}_2(x) \leq \cdots \leq \overline{p}_n(x) \leq \cdots, \ \forall x \in \bigcap_{n \in \mathcal{N}} E_{p_n},$$

but it is not, in general, a Hausdorff space because $\overline{p}_n(\{x_i\}_{i \in \mathcal{N}}) = 0$ for all $n \in \mathcal{N}$ does not necessarily imply that $\{x_i\}_{i \in \mathcal{N}}$ is the zero sequence. In fact, $\bigcap_{n \in \mathcal{N}} \ker \overline{p}_n$ may not be the zero sequence.

On the factor space $\bigcap_{n \in \mathcal{N}} E_{p_n} / \bigcap_{i \in \mathcal{N}} \ker \overline{p}_i$, we define

$$\hat{p}_n(\hat{x}) = \hat{p}_n\left(\{x_i\}_{i \in \mathcal{N}} + \bigcap_{i \in \mathcal{N}} \ker \overline{p}_i\right) = \overline{p}_n(\{x_i\}).$$

Assume $\hat{E}_{p_n} = E_{p_n} / \bigcap_{i \in \mathcal{N}} \ker \overline{p}_i$. In this case, $\bigcap_{n \in \mathcal{N}} \hat{E}_{p_n}$ equipped with the seminorms $\hat{p}_1(\hat{x}) \leq \hat{p}_2(\hat{x}) \leq \cdots \leq \hat{p}_n(\hat{x}) \leq \cdots$, $\forall \hat{x} \in \bigcap_{n \in \mathcal{N}} \hat{E}_{p_n}$ is a countably seminormed Hausdorff space.

Defining $\hat{\pi} : E \to \hat{E}_{p_n}$ such that $\hat{\pi}(x) = \{x, x, x, ...\} + \bigcap_{i \in \mathcal{N}} \ker \overline{p}_i$, we see that E is isomorphically isometric to a linear dense subset of $\hat{E}_{p_n}$, i.e.,

$$E \subset \cdots \subset \hat{E}_{p_n+1} \subset \hat{E}_{p_n} \subset \cdots \subset \hat{E}_{p_1}.$$

Proposition 3.1 [2]. *Let E be a countably seminormed space. Then* $\bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}$ *is a complete space. Moreover, there is an isometric and isomorphic*, 1-1 *mapping* $\hat{\pi}$ *of E onto a dense subspace of* $\bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}$. *E is complete if and only if*

$$\hat{\pi}(E) = \bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}.$$

We may write $E = \bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}$.

Proof. Elements of $\bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}$ are equivalence classes of sequences that are p_n-Cauchy for all n.

For any two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ (in the sense of all p_n) in E, let $\hat{x}, \hat{y} \in \bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}$ be the two equivalence classes containing the two sequences $\{x_n\}$ and $\{y_n\}$, respectively.

One can write $\hat{x} = \{x_n\} + \bigcap_{i\in\mathcal{N}} \ker \overline{p}_i$ and $\hat{y} = \{y_n\} + \bigcap_{i\in\mathcal{N}} \ker \overline{p}_i$.

We define the metric

$$\hat{d}(\hat{x}, \hat{y}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\hat{p}_i(\hat{x} - \hat{y})}{1 + \hat{p}_i(\hat{x} - \hat{y})}$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\overline{p}_i(x_n - y_n)}{1 + \overline{p}_i(x_n - y_n)}.$$

In fact, if $\hat{d}(\hat{x}, \hat{y}) = 0$, then $\overline{p}_i(x_n - y_n) = 0$ for all i, hence $\{x_n\} - \{y_n\} \in \bigcap_{i\in\mathcal{N}} \ker \overline{p}_i$ and so the sequences $\{x_n\}$ and $\{y_n\}$ belong to the same equivalence class in $\bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}$, i.e., $\hat{x} = \hat{y}$.

To show that $\bigcap_{n \in \mathcal{N}} \hat{E}_{p_n}$ is complete, let $\{\hat{x}_k\}$ be a Cauchy sequence in $\bigcap_{n \in \mathcal{N}} \hat{E}_{p_n}$. We have

$$\hat{d}(\hat{x}_j, \hat{x}_k) = \lim_{n \to \infty} d(x_j^n, x_k^n) \to 0 \text{ as } j, k \to \infty.$$

For each k, let us choose a Cauchy sequence $\{x_k^n\}_{n \in \mathcal{N}}$ in E that belongs to the equivalence class $\hat{x}_k$. So we choose n_k such that $d(x_k^m, x_k^{n_k}) < \frac{1}{k}$ if $m \geq n_k$.

Now let us define $\hat{x}_0$ to be the equivalence class that contains the sequence

$$(x_1^{n_1}, x_2^{n_2}, ..., x_k^{n_k}, ...).$$

For $m \geq \max(n_j, n_k)$, we get

$$d(x_j^{n_j}, x_k^{n_k}) \leq d(x_j^{n_j}, x_j^m) + d(x_j^m, x_k^m) + d(x_k^m, x_k^{n_k})$$

$$\leq \frac{1}{j} + \frac{1}{k} + d(x_j^m, x_k^m).$$

Letting $j, k \to \infty$ and so n_j, n_k and $m \to \infty$, we get

$$d(x_j^{n_j}, x_k^{n_k}) \to 0 \text{ as } j, k \to \infty.$$

So $(x_1^{n_1}, x_2^{n_2}, ..., x_k^{n_k}, ...)$ is a Cauchy sequence in E (in the sense of all p_n) and $\hat{x}_0$ is in $\bigcap_{n \in \mathcal{N}} \hat{E}_{p_n}$.

For any fixed j, k with $j > n_k$, we have

$$d(x_k^j, x_j^{n_j}) \leq d(x_k^j, x_k^{n_k}) + d(x_k^{n_k}, x_j^{n_j}) \leq \frac{1}{k} + d(x_k^{n_k}, x_j^{n_j}).$$

Letting $j, k \to \infty,$ we get $\lim_{k\to\infty} \hat{d}(\hat{x}_k, \hat{x}_0) = \lim_{k\to\infty} \lim_{j\to\infty} d(x_k^j, x_j^{n_j}) = 0.$ Then $\bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}$ is complete. □

Defining a mapping $\hat{\pi} : E \to \bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}$ such that

$$\hat{\pi}(x) = (x, x, x, \ldots) + \bigcap_{i\in\mathcal{N}} \ker \overline{p}_i,$$

we prove that $\hat{\pi}(E)$ is a dense subspace in $\bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}.$

Since

$$\hat{d}(\hat{\pi}(x), \hat{\pi}(y)) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p_i(x-y)}{1+p_i(x-y)} = d(x, y),$$

so $\hat{\pi}$ is an isometry and a 1-1 mapping.

Let $\{x_n\} + \bigcap_{n\in\mathcal{N}} \ker \overline{p}_i \in \bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}.$ Then $\{x_n\}$ is a d-Cauchy sequence in E, so $\{\hat{\pi}(x_n)\}_{n\in\mathcal{N}}$ is a $\hat{d}$ -Cauchy sequence in $\hat{\pi}(E),$ and as $n \to \infty,$ we get

$$\hat{d}\left(\hat{\pi}(x_n), \{x_n\} + \bigcap_{i\in\mathcal{N}} \ker \overline{p}_i\right) \to 0,$$

hence $\hat{\pi}(E)$ is dense in $\bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}.$

Now we show that $\hat{\pi}(E) = \bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}$ is a necessary and sufficient condition for completeness of E.

In fact, if $\hat{\pi}(E) = \bigcap_{n\in\mathcal{N}} \hat{E}_{p_n},$ then knowing that $\bigcap_{n\in\mathcal{N}} \hat{E}_{p_n}$ is complete, we see that $\hat{\pi}(E)$ also is complete and hence also is E.

On the other hand, if E is complete, then we show that $\bigcap_{n \in \mathcal{N}} \hat{E}_{p_n} \subset \hat{\pi}(E)$. For any $y = \{x_n\} + \bigcap_{i \in \mathcal{N}} \ker \overline{p}_i \in \bigcap_{n \in \mathcal{N}} \hat{E}_{p_n}$, we get $\{\hat{\pi}(x_n)\}_{n \in \mathcal{N}} \to y$ in the metric $\hat{d}$. By completeness of E, there is $z \in E$ such that $x_n \to z$.

Since $\hat{\pi}$ is an isometry, uniqueness of limits implies that $y = \hat{\pi}(z) \in \hat{\pi}(E)$. □

4. The Dual of a Countably Seminormed Space

Each complete seminormed space $\hat{E}_{p_n}$ has a dual, which is a Banach space denoted by $\hat{E}^*_{p_n}$ (by Proposition 2.4, the dual of $\hat{E}_{p_n}$ is the dual of its associated Banach space $(\hat{E}_{p_n}, \| \, \|_{\overline{p}_n})$, where $\overline{E}_{p_n} = E_{p_n} / \ker \overline{p}_n$).

Proposition 4.1. *The dual of a countably seminormed space E is given by* $E^* = \bigcup_{n=1}^{\infty} \hat{E}^*_{p_n} = \bigcup_{n=1}^{\infty} \overline{E}^*_{p_n}$ *and we have the following inclusions*:

$$\hat{E}^*_{p_1} \subset \cdots \subset \hat{E}^*_{p_n} \subset \hat{E}^*_{p_{n+1}} \subset \cdots \subset E^*.$$

Moreover, for $f \in \hat{E}^*_{p_n}$, *we have* $\| f \|_n \geq \| f \|_{n+1}$.

Proof. *First* we prove that $E^* = \bigcup_{n=1}^{\infty} \hat{E}^*_{p_n}$. For any $\hat{f} \in \hat{E}^*_{p_n}$, the functional defined by $f(x) = \hat{f}(\hat{x})$ for any $x \in E$ and $\hat{x} = (x, x, x, \ldots) + \bigcap_{i \in \mathcal{N}} \ker \overline{p}_i \in \hat{E}_{p_n}$ is well defined, linear and continuous with respect to the seminorm p_n. In fact,

$$| f(x) | = | \hat{f}(\hat{x}) | \leq c\hat{p}_n(\hat{x}) = cp_n(x).$$

Since p_n is one of the seminorms generating the topology on E, $f \in E^*$ (Definition 1.11).

On the other hand, for any $f \in E^*$, there exist a seminorm p_{n_0} on E and a constant $c > 0$ such that for all $x \in E$, $|f(x)| \leq cp_{n_0}(x)$. Since (E, p_{n_0}) is isomorphically isometric to

$$\left(\hat{W} = \left\{\hat{x}_w = (x, x, x, ...) + \bigcap_{i \in \mathcal{N}} \ker \overline{p}_i : x \in E\right\}, \hat{p}_{n_0}\right),$$

we can define $\tilde{f} \in \hat{W}^*$ by $\tilde{f}(\hat{x}_w) = f(x)$.

Since $\hat{W}$ is dense in $\hat{E}_{p_{n_0}}$, defining $\hat{f} \in \hat{E}^*_{p_{n_0}}$ by

$$\begin{aligned} \hat{f}\left((x_n)_{n \in \mathcal{N}} + \bigcap_{i \in \mathcal{N}} \ker \overline{p}_i\right) &= \lim_{n \to \infty} \tilde{f}\left((x_n, x_n, x_n, ...) + \bigcap_{i \in \mathcal{N}} \ker \overline{p}_i\right) \\ &= \lim_{n \to \infty} f(x_n), \end{aligned}$$

$\hat{f}$ is well defined, linear and continuous (because (x_n) is a Cauchy sequence and f is continuous functional, so $f(x_n)$ is a Cauchy sequence in $\mathcal{R}$).

Second since $\overline{E}_{p_n}$ is the associated normed space of $\hat{E}_{p_n}$, by Proposition 2.4, we get $\overline{E}^*_{p_n} = \hat{E}^*_{p_n}$. Hence, $\bigcup_{n=1}^{\infty} \hat{E}^*_{p_n} = \bigcup_{n=1}^{\infty} \overline{E}^*_{p_n}$.

Third since $\hat{p}_n \leq \hat{p}_{n+1}$, the continuity of a functional f with respect to $\hat{p}_n$ implies its continuity with respect to $\hat{p}_{n+1}$. Hence, $\hat{E}^*_{p_n} \subset \hat{E}^*_{p_{n+1}}$. □

Definition 4.2 (Uniformly convex countably seminormed space)**.** A countable seminormed space E is said to be *uniformly convex* if $\hat{E}_{p_n}$ is uniformly convex for all n, i.e., if for each n, $\forall \varepsilon > 0$, $\exists \delta_n(\varepsilon) > 0$ such that if $\bar{x}, \bar{y} \in \overline{E}_{p_n}$ with $\| \bar{x} \|_{\overline{p}_n} = 1 = \| \bar{y} \|_{\overline{p}_n}$ and $\| \bar{x} - \bar{y} \|_{\overline{p}_n} \geq \varepsilon$, then $1 - \left\| \frac{\bar{x} + \bar{y}}{2} \right\|_{\overline{p}_n} \geq \delta_n$.

Definition 4.3 (Uniformly smooth countably seminormed space)**.** A countable seminormed space E is said to be *uniformly smooth* if $\hat{E}_{p_n}$ is uniformly smooth for all n, i.e., if for each n whenever given $\varepsilon > 0$, there exists $\delta_n > 0$ such that if $\bar{x}, \bar{y} \in \overline{E}_{p_n}$ with $\| \bar{x} \|_{\overline{p}_n} = 1$ and $\| \bar{y} \|_{\overline{p}_n} \leq \delta_n$, then

$$\| \bar{x} + \bar{y} \|_{\overline{p}_n} + \| \bar{x} - \bar{y} \|_{\overline{p}_n} < 2 + \varepsilon \| \bar{y} \|_{\overline{p}_n}.$$

Example 4.4. Let $X = \mathbb{Q}^2$. Define $X^{\infty} = \mathbb{Q}^2 \times \mathbb{Q}^2 \times \mathbb{Q}^2 \times \cdots$ and seminorms on X^{∞} by $p_i(z) = \| z_i \| = \sqrt{x_i^2 + y_i^2}$, where $z = (z_1, z_2, z_3, ...) \in X^{\infty}$ and $z_n = (x_n, y_n) \in \mathbb{Q}^2$. $(X^{\infty}, \{p_i, i \in \mathbb{N}\})$ is a countably seminormed space.

Define $\overline{X}^{\infty}_{p_i} = \{(z^n) : (z^n)$ is p_i Cauchy sequence$\}$, where $z^n = (z_1^n, z_2^n, z_3^n, ...) \in X^{\infty}$ and a seminorm on $\overline{X}^{\infty}_{p_i}$ by $\overline{p}_i((z^n)) = \lim_{n \to \infty} p_i(z^n)$.

To show that $(X^{\infty}, \{p_i, i \in \mathbb{N}\})$ is a uniformly convex (smooth) seminormed space, we must prove that $\overline{X}^{\infty}_{p_i} / \bigcap_{i \in \mathcal{N}} \ker \overline{p}_i$ with the seminorm $\hat{p}_i\left((z^n) + \bigcap_{i \in \mathcal{N}} \ker \overline{p}_i\right) = \overline{p}_i((z^n))$ is uniformly convex (smooth) or its associated normed space $\overline{X}^{\infty}_{p_i}/\ker \overline{p}_i$ with $\| (z^n) + \ker \overline{p}_i \|_{p_i} = \overline{p}_i((z^n))$ is uniformly convex (smooth). In the associated normed space, two elements (z^n) and $(\dot{z}^n)$, where $z^n = (z_1^n, z_2^n, z_3^n, ...)$ and $(\dot{z}_1^n, \dot{z}_2^n, \dot{z}_3^n, ...)$ belong to the same equivalence class if both (z_i^n) and $(\dot{z}_i^n)$ are p_i Cauchy sequences (i.e., it is a Cauchy sequence in the ith coordinate) and the p_i limit of the difference between (z_i^n) and $(\dot{z}_i^n)$ is convergent to zero. Thus, the associated normed space is $\mathbb{R}^2$ with the norm $\|(x, y)\| = \sqrt{x^2 + y^2}$, which is uniformly convex (smooth).

Example 4.5. For $\mathbb{R}^2$ with two norms $\|(x, y)\|_2 = \sqrt{x^2 + y^2}$ and $\|(x, y)\|_\infty = \max\{|x|, |y|\}$, $(\mathbb{R}^2, \|\ \|_2)$ is uniformly convex but $(\mathbb{R}^2, \|\ \|_\infty)$ is not uniformly convex.

Define $\ell^1(\mathbb{R}^2) = \left\{(z_n) : z_n \in \mathbb{R}^2, \forall n \in \mathbb{N}, \sum_{n=1}^{\infty} \| z_n \|_2 < \infty\right\}$ and $p_0((z_n))$ $= \sum_{n=1}^{\infty} \| z_n \|_2$, $p_1((z_n)) = \| z_1 \|_2$ and $p_i((z_n)) = \| z_i \|_\infty$, $\forall i = 2, 3, \ldots.$

$(\ell^1(\mathbb{R}^2), p_0)$ is not a uniformly convex normed space. In fact, for $\varepsilon = 1$ and $z = ((1, 0), (0, 0), (0, 0), \ldots)$, $\dot{z} = ((0, 0), (-1, 0), (0, 0), \ldots)$.

Clearly, $p_0(z) = 1 = p_0(\dot{z})$, $p_0(z - \dot{z}) = 2 > \varepsilon$. However, $p_0\left(\frac{1}{2}(z + \dot{z})\right)$ $= 1$.

So there is no $\delta > 0$ satisfying $p_0\left(\frac{1}{2}(z + \dot{z})\right) = 1 - \delta$. $(\ell^1(\mathbb{R}^2), \{p_i, i = 0, 1, 2, \ldots\})$ is a countably normed space (see Remark 1.3). $(\ell^1(\mathbb{R}^2), p_1)$ is a uniformly convex seminormed space. $(\ell^1(\mathbb{R}^2), p_i)$ is not a uniformly convex seminormed space, $\forall i = 2, 3, \ldots.$

Therefore, $(\ell^1(\mathbb{R}^2), \{p_i, i = 0, 1, 2, \ldots\})$ is not a uniformly convex seminormed space.

In the following, we extend some theorems in [4] to the case of countably seminormed spaces.

Proposition 4.6. *A countably seminormed linear space E is uniformly convex if and only if for each n, we have* $\delta_{\overline{E}_{p_n}}(\varepsilon) > 0$ *for all* $\varepsilon \in (0, 2]$.

Proof. Assume that $(\overline{E}_{p_n}, \|\ \|_{\overline{p}_n})$ is uniformly convex for all *n*. Then, for each *n*, given $\varepsilon > 0$, there exists $\delta_n > 0$ such that $\delta_n \leq 1 - \left\| \frac{\overline{x} + \overline{y}}{2} \right\|_{\overline{p}_n}$ for every $\overline{x}$ and $\overline{y}$ in $\overline{E}_{p_n}$ such that $\| \overline{x} \|_{\overline{p}_n} = 1 = \| \overline{y} \|_{\overline{p}_n}$ and $\| \overline{x} - \overline{y} \|_{\overline{p}_n} \leq \varepsilon$. Therefore, $\delta_{\overline{E}_{p_n}}(\varepsilon) \geq \delta_n > 0$ for all *n*.

Conversely, assume that for each *n*, $\delta_{\overline{E}_{p_n}}(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Fix $\varepsilon \in (0, 2]$ and take $\overline{x}$, $\overline{y}$ in $\overline{E}_{p_n}$ with $\| \overline{x} \|_{\overline{p}_n} = 1 = \| \overline{y} \|_{\overline{p}_n}$ and $\| \overline{x} - \overline{y} \|_{\overline{p}_n} \geq \varepsilon$, then $0 < \delta_{\overline{E}_{p_n}}(\varepsilon) \leq 1 - \left\| \frac{\overline{x} + \overline{y}}{2} \right\|_{\overline{p}_n}$ and therefore $\left\| \frac{\overline{x} + \overline{y}}{2} \right\|_{\overline{p}_n} \leq 1 - \delta_n$ with $\delta_n = \delta_{\overline{E}_{p_n}}(\varepsilon)$, which does not depend on $\overline{x}$ or $\overline{y}$. Then $(\overline{E}_{p_n}, \| \cdot \|_{\overline{p}_n})$ is uniformly convex for all *n* and hence the countably normed space *E* is uniformly convex. □

Theorem 4.7. *A countably seminormed space E is uniformly smooth if and only if*

$$\lim_{t \to 0^+} \frac{\rho_{\overline{E}_{p_n}}(t)}{t} = 0, \ \forall n.$$

Proof. Assume that $(\overline{E}_{p_n}, \|\ \|_{\overline{p}_n})$ is uniformly smooth for each *n* and if $\varepsilon > 0$, then there exists $\delta_n > 0$ such that $\frac{\| \overline{x} + \overline{y} \|_{\overline{p}_n} + \| \overline{x} - \overline{y} \|_{\overline{p}_n}}{2} - 1 < \frac{\varepsilon}{2} \| \overline{y} \|_{\overline{p}_n}$ for every $\overline{x}$, $\overline{y}$ in $\overline{E}_{p_n}$ with $\| \overline{x} \|_{\overline{p}_n} = 1$ and $\| \overline{y} \|_{\overline{p}_n} = \delta_n$. This implies that for each *n*, we have $\rho_{\overline{E}_{p_n}}(t) < \frac{\varepsilon}{2} t$ for every $t < \delta_n$.

Conversely, for each n, given $\varepsilon > 0$, suppose that there exists $\delta_n > 0$ such that $\frac{\rho_{\overline{E}_{p_n}}(t)}{t} < \frac{\varepsilon}{2}$ for every $t < \delta_n$. Let $\overline{x}, \overline{y}$ be in $\overline{E}_{p_n}$ such that $\| \overline{x} \|_{\overline{p}_n} = 1$ and $\| \overline{y} \|_{\overline{p}_n} = \delta_n$. Then with $t = \| \overline{y} \|_{\overline{p}_n}$, we have $\| \overline{x} + \overline{y} \|_{\overline{p}_n} + \| \overline{x} - \overline{y} \|_{\overline{p}_n} < 2 + \varepsilon \| \overline{y} \|_{\overline{p}_n}$. Then $(\overline{E}_{p_n}, \| \, \|_{\overline{p}_n})$ is uniformly smooth for all n and hence the countably normed space E is uniformly smooth. □

Now, we prove one of the fundamental links between the Lindenstrauss duality formulas.

Proposition 4.8. *Let E be a countably seminormed space. For each Banach space $\overline{E}_{p_n}$, we have: for every $\tau > 0$, $\overline{x} \in \overline{E}_{p_n}$, $\| \overline{x} \|_{\overline{p}_n} = 1$ and $x^* \in \overline{E}^*_{p_n}$ with $\| x^* \|_n = 1$, then*

$$\rho_{\overline{E}_{p_n}}(\tau) = \sup\left\{ \frac{\tau\varepsilon}{2} - \delta_{\overline{E}^*_{p_n}}(\varepsilon) : 0 < \varepsilon \leq 2 \right\}.$$

Proof. Let $\tau > 0$ and let $x^*, y^* \in \overline{E}^*_{p_n}$ with $\| x^* \|_n = \| y^* \|_n = 1$. For any $\eta > 0$, from the definition of $\| \, \|_n$ in $\overline{E}^*_{p_n}$, there exist $\overline{x}_0, \overline{y}_0 \in \overline{E}_{p_n}$ with $\| \overline{x}_0 \|_{\overline{p}_n} = \| \overline{y}_0 \|_{\overline{p}_n} = 1$ such that

$$\| x^* + y^* \|_n - \eta \leq \langle \overline{x}_0, x^* + y^* \rangle_n, \quad \| x^* - y^* \|_n - \eta \leq \langle \overline{y}_0, x^* - y^* \rangle_n.$$

Using these two inequalities together with the fact that in Banach spaces, we have $\| \overline{x} \|_{\overline{p}_n} = \sup\{ | \langle \overline{x}, x^* \rangle_n | : \| x^* \|_n = 1 \}$, then

$$\begin{aligned} & \| x^* + y^* \|_n + \tau \| x^* - y^* \|_n - 2 \\ & \leq \langle \overline{x}_0, x^* + y^* \rangle_i + \tau \langle \overline{y}_0, x^* - y^* \rangle_i - 2 + \eta(1 + \tau) \\ & = \langle \overline{x}_0 + \tau \overline{y}_0, x^* \rangle_n + \langle \overline{x}_0 - \tau \overline{y}_0, y^* \rangle_n - 2 + \eta(1 + \tau) \end{aligned}$$

$$\leq \| \bar{x}_0 + \tau \bar{y}_0 \|_n + \| \bar{x}_0 - \tau \bar{y}_0 \|_n - 2 + \eta(1 + \tau)$$

$$\leq \sup\{\| \bar{x} + \tau \bar{y} \|_{\bar{p}_n} + \| \bar{x} - \tau \bar{y} \|_{\bar{p}_n} - 2 : \| \bar{x} \|_{\bar{p}_n} = \| \bar{y} \|_{\bar{p}_n} = 1\} + \eta(1 + \tau)$$

$$= 2\rho_{\overline{E}_{p_n}}(\tau) + \eta(1 + \tau).$$

If $0 < \varepsilon \leq \| x^* - y^* \|_n \leq 2,$ then we have

$$\frac{\tau\varepsilon}{2} - \rho_{\overline{E}_{p_n}}(\tau) - \frac{\eta}{2}(1 + \tau) \leq 1 - \left\| \frac{x^* + y^*}{2} \right\|_n,$$

which implies that

$$\frac{\tau\varepsilon}{2} - \rho_{\overline{E}_{p_n}}(\tau) - \frac{\eta}{2}(1 + \tau) \leq \delta_{\overline{E}^*_{p_n}}(\varepsilon).$$

Since η is arbitrary, we conclude that

$$\frac{\tau\varepsilon}{2} - \rho_{\overline{E}_{p_n}}(\tau) \leq \delta_{\overline{E}^*_{p_n}}(\varepsilon), \forall \varepsilon \in (0, 2]$$

$$\Rightarrow \sup\left\{\frac{\tau\varepsilon}{2} - \delta_{\overline{E}^*_{p_n}}(\varepsilon) : \varepsilon \in (0, 2]\right\} \leq \rho_{\overline{E}_{p_n}}(\tau).$$

On the other hand, let $\bar{x}, \bar{y}$ be in $\overline{E}_{p_n}$ with $\| \bar{x} \|_{\bar{p}_n} = \| \bar{y} \|_{\bar{p}_n} = 1$ and let $\tau > 0.$ By Hahn-Banach theorem, there exist $x_0^*, y_0^* \in \overline{E}^*_{p_n}$ with $\| x_0^* \|_n = \| y_0^* \|_n = 1$ such that

$$\langle \bar{x} + \tau \bar{y}, x_0^* \rangle_n = \| \bar{x} + \tau \bar{y} \|_{\bar{p}_n}, \quad \langle \bar{x} - \tau \bar{y}, y_0^* \rangle_n = \| \bar{x} - \tau \bar{y} \|_{\bar{p}_n}.$$

Then

$$\| \bar{x} + \tau \bar{y} \|_{\bar{p}_n} + \| \bar{x} - \tau \bar{y} \|_{\bar{p}_n} - 2 = \langle \bar{x} + \tau \bar{y}, x_0^* \rangle_n + \langle \bar{x} - \tau \bar{y}, y_0^* \rangle_n - 2$$

$$= \langle \bar{x}, x_0^* + y_0^* \rangle_n + \tau \langle \bar{y}, x_0^* - y_0^* \rangle_n - 2$$

$$\leq \| x_0^* + y_0^* \|_n + \tau | \langle \bar{y}, x_0^* - y_0^* \rangle_n | - 2.$$

Hence, if we define $\varepsilon_0 := |\langle \bar{y}, x_0^* - y_0^* \rangle_n|$, then $0 < \varepsilon_0 \le \| x_0^* - y_0^* \|_n \le 2$ and

$$\frac{\| \bar{x} + \tau \bar{y} \|_{\bar{p}_n} + \| \bar{x} - \tau \bar{y} \|_{\bar{p}_n}}{2} - 1 \le \frac{\| x_0^* + y_0^* \|_n + \tau |\langle \bar{y}, x_0^* - y_0^* \rangle_n|}{2} - 1$$

$$= \frac{\tau \varepsilon_0}{2} - \left(1 - \frac{\| x_0^* + y_0^* \|_n}{2} \right)$$

$$\le \frac{\tau \varepsilon_0}{2} - \delta_{\overline{E}_{p_n}^*}(\varepsilon_0)$$

$$\le \sup \left\{ \frac{\tau \varepsilon}{2} - \delta_{\overline{E}_{p_n}^*}(\varepsilon) : 0 < \varepsilon \le 2 \right\}.$$

Therefore,

$$\rho_{\overline{E}_{p_n}}(t) \le \sup \left\{ \frac{\tau \varepsilon}{2} - \delta_{\overline{E}_{p_n}^*}(\varepsilon) : 0 < \varepsilon \le 2 \right\}. \qquad \square$$

The following result gives or determines some duality property between uniform convexity and uniform smoothness.

Theorem 4.9. *Let E be a countably seminormed space. Then*

E is uniformly smooth $\Leftrightarrow$ $\overline{E}_{p_n}^*$ *is uniformly convex for all n.*

Proof. We will prove both directions by contradiction.

"$\Rightarrow$" Assume that $(\overline{E}_{p_{n_0}}^*, \| \cdot \|_{n_0})$ is not uniformly convex for some n_0. Therefore, $\delta_{\overline{E}_{p_{n_0}}^*}(\varepsilon_0) = 0$ for some $\varepsilon_0 \in (0, 2]$. Using previous proposition, we get for any $\tau > 0$,

$$0 < \frac{\varepsilon_0}{2} \le \frac{\rho_{\overline{E}_{p_{n_0}}}(\tau)}{\tau} \text{ hence } \lim_{\tau} \frac{\rho_{\overline{E}_{p_{n_0}}}(\tau)}{\tau} \ne 0,$$

which shows that E is not uniformly smooth.

"$\Leftarrow$" Assume that E is not uniformly smooth. Then

$$\exists n_0 : \lim_{t \to 0^+} \frac{\rho_{\overline{E}_{p_{n_0}}}(t)}{t} \neq 0,$$

this means that there exists $\varepsilon > 0$ such that for every $\delta > 0$, we can find t_δ with $0 < t_\delta < \delta$ and $\rho_{\overline{E}_{p_{n_0}}}(t_\delta) \geq t_\delta \varepsilon$. Consequently, one can choose a sequence (τ_n) such that $0 < \tau_n < 1$, $\tau_n \to 0$ and $\rho_{\overline{E}_{p_{n_0}}}(\tau_n) \geq \varepsilon\tau_n < \frac{\varepsilon}{2}\tau_n$. Using previous proposition, for every n, there exists $\varepsilon_n \in (0, 2]$ such that

$$\frac{\varepsilon}{2}\tau_n \leq \frac{\tau_n \varepsilon_n}{2} - \delta_{\overline{E}^*_{p_{n_0}}}(\varepsilon_n),$$

which implies

$$0 < \delta_{\overline{E}^*_{p_{n_0}}}(\varepsilon_n) \leq \frac{\tau_n}{2}(\varepsilon_n - \varepsilon),$$

in particular, $\varepsilon < \varepsilon_n$ and $\delta_{\overline{E}^*_{p_{n_0}}}(\varepsilon_n) \to 0$. Recalling the fact that δ_{E^*} is a nondecreasing function, we get $\delta_{\overline{E}^*_{p_{n_0}}}(\varepsilon) \leq \delta_{\overline{E}^*_{p_{n_0}}}(\varepsilon_n) \to 0$. Therefore, E^* is not uniformly convex. □

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