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PARTICULAR SOLUTIONS TO SIMULTANEOUS LINEAR EQUATIONS

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Abstract

In this approach, orthogonal vectors are a particular solution of simultaneous linear equations. A plane orthogonal vector is a solution to one linear equation in two unknowns while a spatial orthogonal vector is a particular solution to two linear equations in three unknowns. Also, parallelism is related with the generic solutions and the null vector: the latter is parallel to any vector, while orthogonality is related to particular solutions to simultaneous linear equations.

Introduction

Solving simultaneous linear equations is involved with either the case of unique solution or generic solution. These cases are determined by a number called the *determinant* of the given set of linear equations where the number of equations equals to that of the unknowns, which tell us if it actually has less equations than unknowns so that it is necessary to handle it with the Gauss' method.

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The general solution, also called generic solution to simultaneous linear equations arises in the case that the number of simultaneous linear equations is less than the number of unknowns after reduction, which happens when the determinant of the original simultaneous equations is zero.

The generic solution consists of all scalar multiples of a particular solution of those simultaneous linear equations as the Gauss' method leads to it.

The general solution to one or two simultaneous linear equations in two or three unknowns consists of all scalar multiples of a particular solution, i.e., a linear combination of particular solutions of the given simultaneous linear equations. Our aim is to describe these particular solutions to the simultaneous linear equations with null determinant, such that the work of finding them, may not be so hard or could be practical.

The use of Gauss' method to do the above work is not a practical way for finding the generic solution of simultaneous linear equations of two unknowns or three unknowns. However, the Gauss method leads to the formula of vector product or cross product. The vector product is an attempt of getting a particular solution for a simultaneous of two equations with three unknowns, which arises from the problem of finding a third force which is orthogonal to the two forces given, or the plane generated by them.

Orthogonal Vectors and Particular Solutions

An eigenvector is a particular solution where the determinant of certain simultaneous equations is zero. Taking this into account, we shall call to any particular solution of a simultaneous linear equation an *orthogonal vector*.

A solution to only one linear equation in two unknowns will be called a *planar orthogonal vector* while one to two simultaneous linear equations in three unknowns will be called a *spatial orthogonal vector*. These definitions, of course, are equivalent to those given in any standard vector approach.

A planar orthogonal vector is constructed with the pair of coefficients of the linear equation in two unknowns by reversing the order and changing the sign to the first number of the resulting pair. In this building, the pair of coefficients and the solution are orthogonal pairs, i.e., the product of their slopes turns to be minus one.

A spatial orthogonal vector is a solution of two simultaneous linear equations in three unknowns. A particular solution for this set of equations is the vector product of the rows of coefficient of the pair of the linear equations. Each number of the triad is computed with a two order determinant such that the third order resulting determinant with the triads given and the constructed triad is zero, proving that the vector product or cross product is a solution for the given simultaneous linear equations, since it is orthogonal to the two given triads. This orthogonal vector is orthogonal by definition to the triads obtained with the coefficients of each equation given, respectively, i.e., the sum of the products with their coefficient wise of each one of them, respectively. Thus, the triads with the coefficients of the given equation are solutions to the linear equations whose coefficients are those of the orthogonal vectors, therefore, these triads can be also considered orthogonal vectors.

The concept of parallelism among orthogonal vectors is useful to understand, and practically handle them. The generic solution to one linear equation in two unknowns consists of the collection of multiple scalar of the solution particular, i.e., any multiple scalar of a planar orthogonal vector is also a planar orthogonal vector as well as the sum of two spatial orthogonal vectors is a spatial orthogonal vector. Note that each orthogonal vector is perpendicular to the fixed orthogonal vector whose entries are the coefficients of the given linear equation. Hence two orthogonal vectors are parallel to each other. On the other hand, two planar orthogonal vectors are parallel if and only if they are scalar multiples of each other because they are solutions to the same linear equation. This can be extended to spatial vectors because they also are orthogonal vectors.

Vector Sum and the Null Vector

The principle of superposition is also recovered from this approach.

Indeed, clearly, the sum of two parallel planar orthogonal vectors is also a solution, i.e., any combination of two planar orthogonal vectors is a solution or planar orthogonal vector, in particular, the sum of two planar orthogonal vectors is a planar orthogonal vector (the coefficients of the linear combination are equal to one in this case). This property defines practically what a vector is. Indeed, a vector is one that can be summed to another vector to obtain again other vector (closure of the vector sum). However, this discourse has not touched yet other key properties that determine a vector, which shows that it cannot be practically defined individually. It is a part of general solution to simultaneous linear equations, the trivial solution, i.e., the pair with both of its numbers equal to zero, thus, in our case, the multiple zero of an orthogonal vector is an orthogonal vector. Hence the trivial solution is called the *null vector*. This vector is common to any general solution of one linear equation. When the null vector is added to any vector, the resulting vector is the itself vector (solutions or equations). The only vector with this property is the null vector itself, since the only number that added to any number is the same number is the null number. In terms of parallelism, the null vector is parallel to any vector (solution or equation) even it is parallel to itself.

Since the general solution is the set of scalar multiples of one orthogonal vector, it can be considered as a collection of solutions starting in the null vector and ending in the orthogonal vector itself. The orthogonal vector can be considered as the collection of the vectors that are scalar multiple of it with scalar between zero and one, i.e., the segment with extremes the null vector and the vector given itself. Hence a vector is member of other vectors with an extreme in common. This is compatible with the physics mechanics where a point mass is the common point of all the acting forces on it.

To determine that an orthogonal vector is parallel to other one given, means that the latter is a particular solution to the same linear equation. This is equivalent to check that there is a common factor between numbers, or coordinates of the orthogonal vector, or particular solution, of each orthogonal vector, i.e., that the quotients obtained coordinate wise of the orthogonal vectors are the same, which is equivalent to find the solutions to suitable simultaneous linear equations whose determinant has the vectors given as its rows or columns. Thus, two orthogonal vectors are parallel if and only if the determinant defined by them is zero (the cycle is closed since simultaneous linear equations with null determinant was the start of this work). By the linear property of determinants, the latter happens if and only if the vectors have a linear combination equal to zero with some of coefficients of the combination considered different from zero. The latter is called *linear dependence* so that a pair of vectors is called *linear independent* if and only if there is not a linear combination of them equal to zero with some of its coefficients different from zero. Thus, two vectors are linear independent if and only if they are not parallel, if and only if they are not solutions to the same linear homogeneous equation because in this case they will be a multiple of each other.

That a sum of two given vectors, a scalar multiple of a given vector and the null solution are vectors all of them as well as that two given vectors are independent if and only if there is no linear combination of them equals to zero with at least one of its coefficients different from zero are valid issues for plane orthogonal vectors but also now for spatial orthogonal vectors because an orthogonal vector is a solution of a simultaneous linear equations given.

Dimension of a Vector

So far, it has been described the properties of orthogonal vectors that are not useful to distinguish between plane and spatial orthogonal vectors.

To calculate a planar orthogonal vector is necessary one equation but to calculate a spatial orthogonal vector is necessary two equations: the cross product is a common orthogonal vector to two linear equations.

A particular solution, i.e., an orthogonal vector, for one linear equation in three unknowns can be constructed by setting the third number of the solution triad equal to zero, then it proceeds with the first and second coefficients of the given equations as in the case of two unknowns. A second orthogonal vector can be constructed by setting the second number of the solution triad equal to zero and proceeding as before with the rest of the coefficients. In this way, it can be constructed a third orthogonal vector, however, there is a linear combination of the former that equals the latter. If it is considered a second equation whose coefficients do not form a triad that is a scalar multiple of the triad for the first given equation, then the solutions given by the Gauss' method is not a combination of the solutions previously founded. The cross product is a solution, which is neither of those obtained as in the planar case nor a linear combination of them. Observe that any other common orthogonal vector must be a scalar multiple of it as in the case of planar orthogonal vector, since they are solutions to the same two simultaneous linear equations whose generic solution consists of all scalar multiples of the cross product determined by the given equations.

The two first orthogonal vectors for a linear equation in three unknowns constructed as in the case of two unknowns are not scalar multiple of each other by construction, and also the cross product is not a scalar multiple of them. In fact, there is not a null linear combination of them, with some of its coefficients different from the null number. In contrast, for the planar case, there is a null combination of three pairs with some of its coefficients that are not null. The problem of finding a common orthogonal vector to two linear equations cannot be solved using the set of all pairs of numbers, since two planar orthogonal vectors are scalar multiples of each other.

Two planar orthogonal vectors are parallel if and only if they are solutions to the same reduced linear equation, i.e., two orthogonal vectors are linear independent if and only if they are solutions of different reduced linear equations. The pair of coefficients of a linear expression is an orthogonal vector, since it is a solution of the linear equation with coefficients obtained by reversing the order of the coefficients of the linear expression given and change the sign to the second number of the resulting pair after reversing the original order of the given pair. Note that the orthogonal vector has the property that its slope is reciprocal with opposite sign to that of the pair of the linear expression. A planar vector is a pair whose slope is reciprocal with a minus sign to the slope defined by the pair of coefficients of the linear equation, i.e., a planar orthogonal vector is any pair that is orthogonal to the pair of the coefficients of the linear equation. The planar orthogonal vectors that are orthogonal to each other are example of those that are not solutions to the same linear equation, i.e., they are linear independent. Hence any planar orthogonal vector is the linear combination of two orthogonal vectors.

For an orthogonal vector for two simultaneous linear equations, there are two triads such that there is not a null linear combination of them and the former with some coefficients that are no null, i.e., such that they and the former are linear independent. In the case of the planar orthogonal vector, there only exists one pair that is linear independent with the orthogonal vector.

The generic solution for two simultaneous linear equations in three unknowns is generated by one three-dimensional element while the generic solution for one equation in two unknowns is generated by one two-dimensional element.

Some Comments

A given pair of numbers is a planar orthogonal vector, because it is solution to a suitable linear equation in two unknowns as well as a given triad of numbers, is a spatial orthogonal vector to the suitable two simultaneous equations in three unknowns.

None fact of other sciences, for example of physics is needed. This explanation could be break the ill circle about that vectors are forces or velocities or oriented segments (arrows), and conversely. They just are particular solutions to simultaneous linear equations with less equations than unknowns, which can be interpreted as geometrical and physical items.

The response given by most of the beginner undergraduate students to the concrete problem of finding the generic solution of simultaneous equations with null determinant, might be: there is an infinity of solutions. Its concrete solution results to be basic to calculate the kernel and the eigenvalues of a matrix given. With this approach, the previous concrete problem has a concrete and practical answer: all of the orthogonal vectors. The set of all of them is a one-dimensional set that is immersed in a two-dimensional object for the one in two unknowns and in a three-dimensional one for the two ones in three unknowns, respectively.

Vectors are items that can be added (therefore changed their scales) among them and have one of their extremes in common, i.e., they are a system of vectors so that they cannot be naturally considered one by one.

Constructing particular solutions of linear equations leads naturally to parallelism, since any solution is parallel to an orthogonal vector.

Studying a linear combination of the three pairs leads to the simultaneous linear non-homogeneous equations in the coefficients of the linear combination. If two of these pairs are linear independent, then they can be solved by the Cramer's rule so that any pair turns to be a linear combination of those two linear independent pairs. A third pair is then a linear combination of two pairs, i.e., there are not three linear independent pairs. In the same way, there are not four linear independent triads.

This particular point of view for lecturing the topic of the vectors and their properties depends entirely on the handling of the solutions to simultaneous linear equations. The current style is also compatible with the handling in most of the books where vectors are an independent topic of linear equations, e.g., [1].

After a course of plane and spatial vector, it is expected that an out coming student can handle simultaneous linear equations with determinant equal to zero in a practical way, however, most of them that are taking an introductory course of linear algebra might not be able to calculate the generic solution. In fact, most of the students that take an undergraduate course to introduce the concept of vector for first time, might not be able to find the solutions to one linear equation in one unknown, which is a student obstacle to getting a practice with finding the generic solution of the simultaneous linear equations with null determinant.

With this approach, it is expected to help students to solve simultaneous linear equations with null determinant in a practical way, and to give them an alternative definition of vector into the mathematics area.

A necessary requirement to take advantage of this approach is to have practice with solving a linear equation in one unknown in any case but, in particular, where the coefficient of the given equation is null, which might not be sufficiently practiced in the corresponding courses.

Any undergraduate book of your preference, e.g., [1], dedicated to linear algebra can serve as a reference of the current work.

Reference

[1] H. Anton, Elementary Linear Algebra, 10th ed., John Wiley and Sons, 2010.