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MORE APPLICATIONS OF δ-FINE TAGGED PARTITIONS IN REAL ANALYSIS

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Abstract

Thomson introduced the notion of full covers and proved that every full cover of a closed interval $[a, b]$ contains a partition of $[a, b]$. The partitions extracted from full covers played an important role in the work of Botsko, Klaimon, Zangara and Marafino to simplify and unify proofs of several theorems in real analysis.

In this research, we continue our project and develop the techniques given by Zangara and Marafino to show how the concept of δ-fine tagged partitions can be used in place of partitions extracted from full covers.

1. Introduction

Botsko [2, 3], Klaimon [5] as well as Zangara and Marafino [9] showed

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how the concept of full covers could be used to simplify and unify the proofs of many theorems in real analysis. First formulated by Pierre Cousin, Thomson's Lemma ensures that any full cover $\mathcal{C}$ of a closed interval $[a, b]$ contains a partition of $[a, b]$, i.e., there is a partition of $[a, b]$ all of whose closed subintervals belong to $\mathcal{C}$. It seems to us that partitions extracted from full covers are the keys to the results of Botsko, Klaimon, Zangara and Marafino, prompting us to seek easy-to-get partitions to replace partitions coming from full covers. It is δ-fine tagged partitions that we shall use to reprove almost all of the theorems discussed in [9]. We also discuss functions of bounded variation.

Indeed, this paper is a continuation of the authors' project [7] to use δ-fine tagged partitions in real analysis.

Henceforth, we assume that $a, b \in \mathbb{R}$ with $a < b$.

By a *gauge* on $[a, b]$, we mean a strictly positive real-valued function $\delta : [a, b] \to \mathbb{R}^+$. A *tagged partition* $\mathcal{P}$ of the closed interval $[a, b]$ is a finite collection of ordered pairs

$$\{(t_i, [x_{i-1}, x_i]) : i = 1, ..., m\},$$

where $a = x_0 < x_1 < \cdots < x_m = b$ and each t_i is in $[x_{i-1}, x_i]$.

Given a gauge δ on $[a, b]$, a tagged partition $\mathcal{P}$ of $[a, b]$ is said to be δ*-fine* if

$$[x_{i-1}, x_i] \subseteq \,]\, t_i - \delta(t_i), t_i + \delta(t_i)\,[$$

for all $i \in \{1, ..., m\}$.

The following lemma ensures the existence of a δ-fine tagged partition of $[a, b]$ for a given gauge δ on $[a, b]$.

Lemma 1.1 (Cousin's Lemma). [1, Theorem 5.5.5]. *For every gauge* δ *on* $[a, b]$, *there exists a* δ*-fine tagged partition of* $[a, b]$.

2. More Use of δ-fine Tagged Partitions

We will now apply δ-fine tagged partitions to prove some well-known theorems in a fashion similar to Zangara and Marafino's proofs. Furthermore, we also show that if a function f is "of locally bounded variation" on $[a, b]$, then f is of bounded variation on the whole interval $[a, b]$.

Theorem 2.1. *Let* $\underline{D}f(x)$ (*respectively*, $\overline{D}f(x)$) *denote the lower derivative* (*respectively*, *the upper derivative*) *of* f *at* $x \in [a, b]$.

(1) *If* $\underline{D}f(x) \geq 0$ *everywhere on* $[a, b]$, *then* f *is increasing on* $[a, b]$.

(2) *If* $\overline{D}f(x) \leq 0$ *everywhere on* $[a, b]$, *then* f *is decreasing on* $[a, b]$.

Proof. (1) Assume that $\underline{D}f(x) \geq 0$ everywhere on $[a, b]$. Let $c, d \in [a, b]$ with $c < d$ and let $\varepsilon > 0$ be given. Define a gauge δ on $[c, d]$ as follows: Let $x \in [c, d]$. Since $\underline{D}f(x) > -\varepsilon$, there exists a $\delta_x > 0$ such that

$$\frac{f(s) - f(x)}{s - x} > -\varepsilon$$

provided that $s \in \,]x - \delta_x, x + \delta_x[\, \cap [c, d]$ and $s \neq x$. We, thus, obtain that

$$f(v) - f(u) \geq -\varepsilon(v - u)$$

provided that $u, v \in \,]x - \delta_x, x + \delta_x[\, \cap [c, d]$ and $u \leq x \leq v$. Define $\delta(x) := \delta_x$.

Now, let $\mathcal{P} := \{(t_i, [x_{i-1}, x_i]) : i = 1, ..., m\}$ be a δ-fine tagged partition of $[c, d]$. Since

$$f(x_i) - f(x_{i-1}) \geq -\varepsilon(x_i - x_{i-1})$$

for each $i \in \{1, ..., m\}$, it follows that

$$\begin{aligned} f(d) - f(c) &= \sum_{i=1}^{m} (f(x_i) - f(x_{i-1})) \\ &\geq -\varepsilon \sum_{i=1}^{m} (x_i - x_{i-1}) \\ &= -\varepsilon(d - c). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $f(d) - f(c) \geq 0$.

(2) Similar to the proof of (1). $\square$

Theorem 2.2 (Lebesgue Number Lemma)**.** *Let* $\mathcal{J}$ *be an open cover of* $[a, b]$. *There exists a number* $\eta > 0$ *such that if B is any nonempty subset of* $[a, b]$ *with diameter less than* η, *then there exists a set* $J \in \mathcal{J}$ *such that* $J \supseteq B$.

Proof. For each $x \in [a, b]$, there exists an open set J_x in $\mathcal{J}$ containing x; hence there must be a $\delta_x > 0$ such that $]x - \delta_x, x + \delta_x[\subseteq J_x$. Define a gauge δ on $[a, b]$ by $\delta(x) = \frac{1}{2}\delta_x$ for all $x \in [a, b]$.

Let $\mathcal{P} := \{(t_i, [x_{i-1}, x_i]) : i = 1, ..., m\}$ be a δ-fine tagged partition of $[a, b]$. We choose $\eta = \min\{\delta(t_i) : i = 1, ..., m\}$. Let B be any nonempty subset of $[a, b]$ with diameter less than η. Then B intersects $[x_{j-1}, x_j]$ for some $j \in \{1, ..., m\}$. We choose $J := J_{t_j}$; thus, $B \subseteq]t_j - \delta_{t_j}, t_j + \delta_{t_j}[\subseteq J_{t_j} = J$, as required. $\square$

The Intermediate Value Theorem [4, Theorem 2] and Rolle's Theorem [7, Theorem 2.4] were proved by using the notion of δ-fine tagged partitions. Because the Mean Value Theorem can be proved via Rolle's Theorem, we deduce that the Mean Value Theorem is a result of the use of δ-fine tagged partitions. In 2004, Olsen [6] gave a proof of Darboux's Theorem or the Intermediate Value Theorem for Derivatives, by applying the Intermediate Value Theorem and the Mean Value Theorem to avoid the (ε, δ)-argument, which is fairly accessible to many beginning undergraduate students in real analysis. That is, Darboux's Theorem is a consequence of applications of δ-fine tagged partitions. We state Darboux's Theorem here and one can find the proof of Darboux's Theorem in [6, p. 714].

Theorem 2.3 (Darboux's Theorem)**.** *If* f *is the derivative of some function g on an open interval containing* $[a, b]$, *and if* y_0 *lies between* $f(a)$ *and* $f(b)$, *then there exists an* x_0 *in* $]a, b[$ *such that* $f(x_0) = y_0$.

Recently, Zangara and Marafino [9, p. 300] proved Darboux's Theorem via the notion of full covers, but we prefer Olsen's advantageous strategy.

In order to present the proof of Baire's Theorem, we need the following definition.

Definition 2.4. A function $f : [a, b] \to \mathbb{R}$ is said to be *upper semicontinuous* (respectively, *lower semicontinuous*) at $x \in [a, b]$ if $\limsup_{s \to x} f(s) \le f(x)$ (respectively, $\liminf_{s \to x} f(s) \ge f(x)$).

The function *f* is said to be *upper semicontinuous* on $[a, b]$ (respectively, *lower semicontinuous* on $[a, b]$) if *f* is upper semicontinuous (respectively, lower semicontinuous) at every $x \in [a, b]$.

Theorem 2.5 (Baire's Theorem)**.** (1) *Suppose that f is upper semicontinuous and bounded above on* $[a, b]$. *Then there exists a sequence of continuous functions* $\langle H_n \rangle$ *such that* $\langle H_n(x) \rangle \downarrow f(x)$ *for all* $x \in [a, b]$.

(2) *Suppose that f is lower semicontinuous and bounded below on* $[a, b]$. *Then there exists a sequence of continuous functions* $\langle h_n \rangle$ *such that* $\langle h_n(x) \rangle \uparrow f(x)$ *for all* $x \in [a, b]$.

Proof. (1) Let *M* be an upper bound of *f* on $[a, b]$.

For each $\varepsilon > 0$, we will construct a continuous function $H_\varepsilon^* : [a, b] \to \mathbb{R}$ that approximates *f* in a sense to be clarified below. Let $\varepsilon > 0$ be given. Define a gauge δ on $[a, b]$ as follows: Let $x \in [a, b]$ be fixed. Since *f* is upper semicontinuous at *x*, there is a $\delta_x \in \left]0, \frac{\varepsilon}{2}\right[$ such that for each $s \in [a, b]$, if $s \in \left]x - \delta_x, x + \delta_x\right[$, then $f(s) < f(x) + \varepsilon$. Define $\delta(x) = \delta_x$.

Let $\mathcal{P} := \{(t_i, [x_{i-1}, x_i]) : i = 1, ..., m\}$ be a δ-fine tagged partition of $[a, b]$. For each $i \in \{1, ..., m\}$, let $\Phi_i \equiv f(t_i) + \varepsilon$ be a constant function defined on $I_i := [x_{i-1}, x_i]$. Note that, for each $i \in \{1, ..., m\}$, $f(s) < \Phi_i(s)$

for all $s \in I_i$. If $\Phi_1 > \Phi_2$, then connect the horizontal graph of Φ_1 to that of Φ_2 by the line segment $\overline{P_1P_2}$, where P_1 coincides with the end point of the graph of Φ_1 and P_2 lies $1/3$ of the distance on the graph of Φ_2. If $\Phi_1 < \Phi_2$, then connect the horizontal graph of Φ_1 to that of Φ_2 by the line segment $\overline{P_1P_2}$, where P_1 is $2/3$ of the distance on the graph of Φ_1 and P_2 coincides with the initial point of the graph of Φ_2. If $\Phi_1 = \Phi_2$, then we have nothing to do. Doing this for each $i \in \{1, 2, ..., m\}$, we can construct a continuous function H_ε^* on $[a, b]$ with the property $f(x) < H_\varepsilon^*(x)$ for all $x \in [a, b]$. (See Figure 1.)

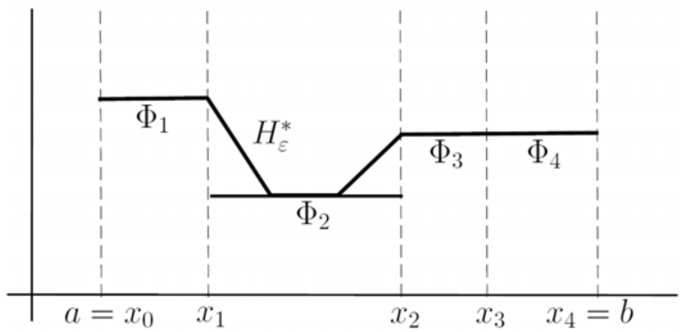

Figure 1. Construction of H_ε^*.

Explicitly, H_ε^* approximates f in the following sense: Given any $s_0 \in [a, b]$; then $s_0 \in I_i$ for some i, and either the point $(s_0, H_\varepsilon^*(s_0))$ will be on a horizontal step of H_ε^* or it will be on a line segment connecting two consecutive horizontal steps. In either case, it is evident that

$$H_\varepsilon^*(s_0) \leq \sup_{|s-s_0|<2\varepsilon} f(s) + \varepsilon,$$

since all I_i are of length less than ε.

We now construct the desired sequence $\langle H_n \rangle$. Define $H_1 := \inf\{H_1^*, M\}$, $H_2 := \inf\{H_{2^{-1}}^*, H_1\}$, $H_3 = \inf\{H_{3^{-1}}^*, H_2\}$, and so on. Let $x \in [a, b]$ be given. The sequence $\langle H_n(x)\rangle$ converges, since it is decreasing and bounded below by $f(x)$. Moreover, $\lim_{n\to+\infty} H_n(x) \geq f(x)$. Since the inequality

$$H_n(x) \leq \sup_{|s-x|<\frac{2}{n}} f(s) + \frac{1}{n}$$

holds for all $n \in \mathbb{N}$, it follows that $\lim_{n\to\infty} H_n(x) \leq \limsup_{s\to x} f(s) \leq f(x)$. Hence $\langle H_n(x)\rangle \downarrow f(x)$.

(2) Similar to the proof of (1). □

Definition 2.6. A family Ω of functions is said to be *uniformly bounded* on $[a, b]$ if there is a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$ and all $f \in \Omega$.

Definition 2.7. A family Ω of functions is said to be *equicontinuous* at $x \in [a, b]$ if for each $\varepsilon > 0$, there exists a $\delta(\varepsilon, x) > 0$ such that for all $s \in [a, b]$, if $|s - x| < \delta(\varepsilon, x)$, then $|f(s) - f(x)| < \varepsilon$ for all $f \in \Omega$.

Theorem 2.8 (Ascoli's Theorem)**.** *Let Ω be a family of functions such that Ω is uniformly bounded on $[a, b]$ and equicontinuous at every point of $[a, b]$. Then every sequence $\langle f_n\rangle$ in Ω has a subsequence that converges uniformly on $[a, b]$.*

Proof. We shall divide the proof into 3 steps as follows:

Step 1. For any $\varepsilon > 0$ and any $x \in [a, b]$, there is a $\delta(\varepsilon, x) > 0$ such that for all $c, d \in [a, b]$, if $x \in [c, d] \subseteq\,]x - \delta(\varepsilon, x), x + \delta(\varepsilon, x)[$, then every sequence $\langle h_n\rangle$ in Ω has a subsequence $\langle h_{n(k)}\rangle_{k=1}^{+\infty}$ such that

$$|h_{n(i)}(s) - h_{n(j)}(s)| < \varepsilon \tag{1}$$

for all $s \in [c, d]$ and all $i, j \in \mathbb{N}$.

Proof of Step 1. Let $\varepsilon > 0$ be arbitrary and let $x \in [a, b]$ be fixed. Since Ω is equicontinuous at *x*, there exists a $\delta(\varepsilon, x) > 0$ such that for all $s \in [a, b]$, if $|s - x| < \delta(\varepsilon, x)$, then

$$| f(s) - f(x) | < \frac{\varepsilon}{3}$$

for all $f \in \Omega$. Let $c, d \in [a, b]$ be such that $x \in [c, d] \subseteq]x - \delta(\varepsilon, x), x + \delta(\varepsilon, x)[$ and let $\langle h_n \rangle$ be any sequence in Ω. Since the sequence $\langle h_n(x) \rangle$ is bounded, it has a Cauchy subsequence $\langle h_{n(k)}(x) \rangle$ such that

$$| h_{n(i)}(x) - h_{n(j)}(x) | < \frac{\varepsilon}{3}$$

for all $i, j \in \mathbb{N}$. Hence, if $s \in [c, d]$ and $i, j \in \mathbb{N}$, then

$$\begin{aligned} &| h_{n(i)}(s) - h_{n(j)}(s) | \\ &\leq | h_{n(i)}(s) - h_{n(i)}(x) | + | h_{n(i)}(x) - h_{n(j)}(x) | + | h_{n(j)}(x) - h_{n(j)}(s) | \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon, \end{aligned}$$

as required. □

Step 2. For any $\varepsilon > 0$, every sequence $\langle g_n \rangle$ in Ω has a subsequence $\langle g_{n(\varepsilon, k)} \rangle_{k=1}^{+\infty}$ such that

$$| g_{n(\varepsilon, i)}(s) - g_{n(\varepsilon, j)}(s) | < \varepsilon \tag{2}$$

for all $s \in [a, b]$ and all $i, j \in \mathbb{N}$.

Proof of Step 2. Let $\varepsilon > 0$ be given. Define a gauge δ on $[a, b]$ as follows: Let $x \in [a, b]$ be fixed. By Step 1, there is a $\delta(\varepsilon, x) > 0$ such that the condition (1) holds. Define $\delta(x) = \delta(\varepsilon, x) > 0$. Now, let $\mathcal{P} := \{(t_i, [x_{i-1}, x_i]) : i = 1, ..., m\}$ be a δ-fine tagged partition of $[a, b]$ and let $\langle g_n \rangle$ be any sequence in Ω. Since

$$t_1 \in [x_0, x_1] \subseteq]t_1 - \delta(\varepsilon, t_1), t_1 + \delta(\varepsilon, t_1)[,$$

it follows from Step 1 that there is a subsequence $\langle g_{n_1(k)} \rangle$ of $\langle g_n \rangle$ such that

$$| g_{n_1(i)}(s) - g_{n_1(j)}(s) | < \varepsilon \tag{3}$$

for all $s \in [x_0, x_1]$ and all $i, j \in \mathbb{N}$. Again, since

$$t_2 \in [x_1, x_2] \subseteq]t_2 - \delta(\varepsilon, t_2), t_2 + \delta(\varepsilon, t_2)[,$$

it follows from Step 1 that the sequence $\langle g_{n_1(k)} \rangle$ has a subsequence $\langle g_{n_2(k)} \rangle$ such that

$$| g_{n_2(i)}(s) - g_{n_2(j)}(s) | < \varepsilon \tag{4}$$

for all $s \in [x_1, x_2]$ and all $i, j \in \mathbb{N}$. It is readily seen that $\langle g_{n_2(k)} \rangle$ is a subsequence of $\langle g_n \rangle$ such that

$$| g_{n_2(i)}(s) - g_{n_2(j)}(s) | < \varepsilon \tag{5}$$

for all $s \in [x_0, x_1] \cup [x_1, x_2]$ and all $i, j \in \mathbb{N}$. Continuing this process for m steps, we eventually obtain the required subsequence $\langle g_{n(\varepsilon, k)} \rangle := \langle g_{n_m(k)} \rangle$ of $\langle g_n \rangle$. $\square$

Step 3. Every sequence $\langle f_n \rangle$ in Ω has a subsequence that converges uniformly on $[a, b]$.

Proof of Step 3. Let $\langle f_n \rangle$ be any sequence in Ω. Applying Step 2 with $\langle g_n \rangle = \langle f_n \rangle$ and $\varepsilon = 1$, we obtain a subsequence $\langle f_{n(1,k)} \rangle$ of $\langle f_n \rangle$ such that

$$| f_{n(1,i)}(s) - f_{n(1,j)}(s) | < 1 \tag{6}$$

for all $s \in [a, b]$ and all $i, j \in \mathbb{N}$. Applying Step 2 again with $\langle g_n \rangle = \langle f_{n(1,k+1)} \rangle$ and $\varepsilon = 2^{-1}$, we obtain a subsequence $\langle f_{n(2^{-1},k)} \rangle$ of $\langle f_{n(1,k+1)} \rangle$ such that

$$| f_{n(2^{-1},i)}(s) - f_{n(2^{-1},j)}(s) | < 2^{-1} \tag{7}$$

for all $s \in [a, b]$ and all $i, j \in \mathbb{N}$. Observe that $\langle f_{n(2^{-1}, k)} \rangle$ is also a subsequence of $\langle f_n \rangle$ and that $n(2^{-1}, 1) \geq n(1, 2) > n(1, 1)$. Continuing this process inductively, we conclude that there is a family $\{\langle f_{n(\ell^{-1}, k)} \rangle_{k=1}^{+\infty} : \ell \in \mathbb{N}\}$ of subsequences of $\langle f_n \rangle$ such that

(i) $n((\ell + 1)^{-1}, 1) > n(\ell^{-1}, 1)$ for all $\ell \in \mathbb{N}$, and

(ii) $| f_{n(\ell^{-1}, i)}(s) - f_{n(\ell^{-1}, j)}(s) | < \ell^{-1}$ for all $s \in [a, b]$ and all $\ell, i, j \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrary. Choose $m \in \mathbb{N}$ so that $m^{-1} < \varepsilon$. The subsequence $\langle f_{n(m^{-1}, k)} \rangle_{k=1}^{+\infty}$ of $\langle f_n \rangle$ satisfies the property that

$$| f_{n(m^{-1}, i)}(s) - f_{n(m^{-1}, j)}(s) | < m^{-1} < \varepsilon \tag{8}$$

for all $s \in [a, b]$ and all $i, j \in \mathbb{N}$. Therefore, $\langle f_{n(m^{-1}, k)} \rangle_{k=1}^{+\infty}$ converges uniformly on $[a, b]$.

This completes the proof of Ascoli's Theorem. $\square$

Definition 2.9. Let $f : [a, b] \to \mathbb{R}$ be a function. For each partition $P = \{x_0, ..., x_m\}$ of $[a, b]$ with $a = x_0 < x_1 < \cdots < x_m = b$, let

$$V(f, P) := \sum_{i=1}^{m} | f(x_i) - f(x_{i-1}) |.$$

The *variation* of *f* on $[a, b]$ is defined by

$$Var(f) := \sup\{V(f, P) : P \text{ is a partition of } [a, b]\} \in [0, +\infty].$$

Definition 2.10. Let $f : [a, b] \to \mathbb{R}$. Then *f* is said to be of *bounded variation* on $[a, b]$ if $Var(f) < +\infty$.

In general, the function f is said to be of *bounded variation* on a nondegenerate closed subinterval $[c, d]$ of $[a, b]$ if $f|_{[c, d]}$ is of *bounded variation* on $[c, d]$.

Lemma 2.11. *If f is of bounded variation on* $[a, b]$, *then f is of bounded variation on any nondegenerate closed subinterval of* $[a, b]$.

Proof. Assume that f is of bounded variation on $[a, b]$. Let $[c, d]$ be any nondegenerate closed subinterval of $[a, b]$ and let $P_{cd} = \{x_0, ..., x_m\}$ be any partition of $[c, d]$. Put $P_{ab} = P_{cd} \cup \{a, b\}$; hence P_{ab} is a partition of $[a, b]$. We, thus, have

$$
\begin{aligned}
V(f|_{[c, d]}, P_{cd}) &= \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| \\
&\leq |f(x_0) - f(a)| + \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| + |f(b) - f(x_m)| \\
&= V(f, P_{ab}) \\
&\leq Var(f).
\end{aligned}
$$

As a result, $Var(f|_{[c, d]}) \leq Var(f) < +\infty$. Therefore, f is of bounded variation on $[c, d]$. □

Definition 2.12. A function $f : [a, b] \to \mathbb{R}$ is said to be of *locally bounded variation at a point* $x \in [a, b]$ if there exists a number $\delta_x > 0$ such that f is of bounded variation on $[x - \delta_x, x + \delta_x] \cap [a, b]$.

Theorem 2.13. *If f is of locally bounded variation at every point on* $[a, b]$, *then f is of bounded variation on the whole interval* $[a, b]$.

Proof. Define a gauge δ on $[a, b]$ as follows: Let $x \in [a, b]$ be given. Then there exists a $\delta_x > 0$ such that f is of bounded variation on $[x - \delta_x, x + \delta_x] \cap [a, b]$. We define $\delta(x) = \delta_x$.

Let $\mathcal{P} := \{(t_i, [x_{i-1}, x_i]) : i = 1, ..., m\}$ be a δ-fine tagged partition of $[a, b]$. By Lemma 2.11, f is of bounded variation on $[x_{i-1}, x_i]$ for all $i \in \{1, ..., m\}$.

To show f is of bounded variation on $[a, b]$, let $P = \{y_0, ..., y_n\}$ be any partition of $[a, b]$. Then put $\acute{P} := \{x_0, ..., x_m\} \cup \{y_0, ..., y_n\}$, so $\acute{P}$ is a partition of $[a, b]$ containing all partition points $x_0, x_1, ..., x_m$ of the δ-fine tagged partition $\mathcal{P}$. For each $i \in \{1, ..., m\}$, let $\acute{P}_i := \acute{P} \cap [x_{i-1}, x_i]$. It follows that, for each $i \in \{1, ..., m\}$, $\acute{P}_i$ is a partition of $[x_{i-1}, x_i]$, and that

$$\begin{aligned} V(f, P) &\leq V(f, \acute{P}) \\ &= \sum_{i=1}^{m} V(f|_{[x_{i-1}, x_i]}, \acute{P}_i) \\ &\leq \sum_{i=1}^{m} Var(f|_{[x_{i-1}, x_i]}), \end{aligned}$$

so $Var(f) \leq \sum_{i=1}^{m} Var(f|_{[x_{i-1}, x_i]}) < +\infty$. This completes the proof. □

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References

[1] R. G. Bartle and D. R. Sherbert, Introduction to Real Analysis, 3rd ed., John Wiley & Sons, New York, 2000.

[2] M. W. Botsko, A unified treatment of various theorems in elementary analysis, Amer. Math. Monthly 94 (1987), 450-452.

[3] M. W. Botsko, The use of full covers in real analysis, Amer. Math. Monthly 96 (1989), 328-333.

[4] R. A. Gordon, The use of tagged partitions in elementary real analysis, Amer. Math. Monthly 105 (1998), 107-117 and 886.

[5] K. Klaimon, More applications of full covering, Pi Mu Epsilon J. 9(3) (1990), 156-161.

[6] L. Olsen, A new proof of Darboux's theorem, Amer. Math. Monthly 111 (2004), 713-715.

[7] S. Prongjit and W. Sodsiri, Applications of δ-fine tagged partitions in real analysis, Far East J. Math. Sci. (FJMS) 91(1) (2014), 97-109.

[8] H. L. Royden, Real Analysis, 3rd ed., Macmillan, New York, 1988.

[9] K. Zangara and J. Marafino, Applications of full covers in real analysis, Involve 2(3) (2009), 297-304.