Far East Journal of Mathematical Sciences (FJMS)

Published Online: September 2014
Available online at http://pphmj.com/journals/fjms.htm
Volume 91, Number 1, 2014, Pages 123-131

NOTES OF FUZZY DIFUNCTIONAL RELATIONS

Yeoul Ouk Sung

Department of Applied Mathematics
Kongju National University
182 Sinkwandong
Kongju-city, 314-701, Korea

Abstract

The images and preimages of fuzzy conformable relations and fuzzy difunctional relations are studied under the so-called semibalanced maps.

1. Introduction

Ounalli and Jaoua [4] extended difunctional relations in the framework of fuzzy relations with max-min composition for the purpose of gaining a better understanding of their properties and their structure. Seo et al. [5] characterized fuzzy difunctional relations on a set and proved there exists a relationship between fuzzy equivalence relations and fuzzy difunctional relations. Kuroki [2] first introduced the concept of fuzzy congruences on a groupoid and characterized fuzzy congruences on a group in terms of fuzzy normal subgroups. In the present note, we apply the idea of difunctional relation in the setting of semibalanced mappings and studies images and preimages of fuzzy conformable relations and fuzzy difunctional relations

Received: May 29, 2014; Accepted: July 16, 2014
2010 Mathematics Subject Classification: 83A03, 93C04, 90B02.
Keywords and phrases: fuzzy conformable relations, fuzzy difunctional relations, fuzzy congruence relation.

under semibalanced maps. In Section 2, we briefly define fuzzy conformable relations and fuzzy congruence relations. Section 3 describes our main results.

2. Preliminaries

For details we refer to [4-6].

Definition 2.1. Let *f* be a mapping from the set *X* into the set *Y*.

If *R* is a fuzzy subset of *X*, then the image $f(R)$ of *R* is the fuzzy subset of *Y* defined by

$$f(R)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} R(x), & \text{if } f^{-1}(y) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

If *R* is a fuzzy subset of *Y*, then the preimage $f^{-1}(R)$ of *R* is the fuzzy subset of *X* defined by $f^{-1}(R)(x) = R(f(x)),\ x \in X$.

Definition 2.2. Let *X* and *Y* be two nonempty sets.

A mapping $f : X \times X \to Y \times Y$ is called a *semibalanced mapping* if

(i) given $a \in X$, there exists $u \in Y$ such that $f(a, a) = (u, u)$.

(ii) $f(a, a) = (u, u)$ and $f(b, b) = (v, v)$, where $a, b \in X$, $u, v \in Y$, implies that $f(a, b) = (u, v)$.

Definition 2.3. Let *f* be a map from $X \times X$ into $Y \times Y$.

A fuzzy relation R on X is *f*-invariant, if $f(a, b) = f(a_1, b)$ implies $R(a, b) = R(a_1, b)$.

A fuzzy relation R on X is weakly *f*-invariant, if $f(a, b) = f(a_1, b_1)$ implies that $R(a, b) = R(a_1, b_1)$.

Theorem 2.4 [4, Proposition 7]. *If a fuzzy relation R is fuzzy symmetric and fuzzy transitive, then R is fuzzy difunctional*.

Definition 2.5. Let R be a fuzzy relation on a groupoid X. R is fuzzy left conformable if $R(c, c) \geq R(a, b)$ implies $R(ca, cb) \geq R(a, b)$ for all $a, b, c \in X$, and R is fuzzy right conformable. If $R(c, c) \geq R(a, b)$ implies $R(ac, bc) \geq R(a, b)$ for all $a, b, c \in X$.

Definition 2.6. Let R be a fuzzy relation on a groupoid X. R is fuzzy left compatible if $R(ca, cb) \geq R(a, b)$ for all $a, b, c \in X$, and R is fuzzy right compatible if $R(ac, bc) \geq R(a, b)$ for all $a, b, c \in X$. R is fuzzy compatible if $R(ac, bd) \geq R(a, b) \cap R(c, d)$ for all a, b, c and $d \in X$.

Definition 2.7. A fuzzy relation R on a set X if fuzzy difunction if and only if. It satisfies condition $RR^{-1}R \subseteq R$, which is equivalent to $RR^{-1}R = R$.

3. Main Results

In this section, we deal with the images and preimages of fuzzy difunctional relations and fuzzy conformable relations under semibalanced maps.

Theorem 3.1. *If R is a right (left) conformable fuzzy relation on a group Y and f is a group homomorphism from* $X \times X$ *into* $Y \times Y$, *which is a semibalanced map, then* $f^{-1}(R)$ *is a right (left) conformable fuzzy relation on X.*

Proof. Let $f^{-1}(R)(c, c) \geq f^{-1}(R)(a, b)$ for $a, b, c \in X$. Then $R(f(c, c)) \geq R(f(a, b))$. And we have,

$$\begin{aligned} f^{-1}(R)(ac, bc) &= R(f(ac, bc)) \\ &= R(f(a, b) \cdot f(c, c)) \\ &\geq R(f(a, b)), \text{ as } R \text{ is a right conformable} \\ &= f^{-1}(R)(a, b). \end{aligned}$$

Which yields $f^{-1}(R)$ is a right conformable. Also, we can prove in the same manner that $f^{-1}(R)$ is a left conformable fuzzy relation on *X*.

Theorem 3.2. *If R is a right* (*left*) *conformable fuzzy relation on a group X and f is a group homomorphism from* $X \times X$ *into* $Y \times Y$, *which is a semibalanced map, then* $f(R)$ *is a right* (*left*) *conformable fuzzy relation on Y*.

Proof. Let $f(R)(w, w) \geq f(R)(u, v), u, v, w \in Y$. Then we show that $f(R)(uw, vw) \geq f(R)(u, v)$. In the case when $f^{-1}(u, v)$ is empty. This means $f(R)(u, v) = 0$. Then we have $f(R)(uw, vw) \geq f(R)(u, v)$. Next, we consider the case when $f^{-1}(u, v)$ is nonempty. Then $f^{-1}(w, w)$ is nonempty, and so $f^{-1}(uw, vw)$ is nonempty. We then have

$$
\begin{aligned}
f(R)(uw, vw) &= \bigvee_{(x, x') \in f^{-1}(uw, vw)} R(x, x') \\
&\geq \bigvee_{f(ac, bc) = (uw, vw)} R(ac, bc) \\
&= \bigvee_{f(a, b) \cdot f(c, c) = (u, v) \cdot (w, w)} R((a, b) \cdot (c, c)) \\
&\geq \bigvee_{f(a, b) = (u, v)} R(a, b) \text{ as } R \text{ is conformable} \\
&= f(R)(u, v).
\end{aligned}
$$

Which yields $f(R)$ is a right conformable relation on *Y*. Also, we can prove in the same manner that $f(R)$ is a left conformable on *X*.

Theorem 3.3. *Let f be a semibalanced map from* $X \times X$ *into* $Y \times Y$. *If R is a fuzzy difunctional relation on Y*, *which is fuzzy reflexive*, *then* $f^{-1}(R)$ *is a fuzzy difunctional relation on X*.

Proof. It suffices to show that $f^{-1}(R)$ is fuzzy symmetric and fuzzy transitive. First, now let $f(a, b) = (u, v)$. Then

$$\begin{aligned}
f^{-1}(R)(a, b) &= R(f(a, b)) \\
&= R(u, v) \\
&= R(v, u), \text{ as } R \text{ is fuzzy symmetric} \\
&= R(f(b, a)), \text{ as } f \text{ is semibalanced map} \\
&= f^{-1}(R)(b, a) \text{ for all } a, b \in X.
\end{aligned}$$

Which yields $f^{-1}(R)$ is fuzzy symmetric.

Further, for $f(a, b) = (u, v)$. We have

$$\begin{aligned}
(f^{-1}(R) f^{-1}(R))(a, b) &= \bigvee_{x \in X} (f^{-1}(R)(a, x) \wedge f^{-1}(R)(x, b)) \\
&= \bigvee_{x \in X} (R(f(a, x)) \wedge R(f(x, b))) \\
&= \bigvee_{x \in X} (R(u, t_x) \wedge R(t_x, v)) \\
&\leq \bigvee_{w \in Y} (R(u, w) \wedge R(w, v)) \\
&= (R \circ R)(u, v) \\
&\leq R(u, v) \\
&= f^{-1}(R)(a, b) \text{ for all } a, b \in X.
\end{aligned}$$

Which yields $f^{-1}(R)$ is fuzzy transitive. This completes the proof.

Theorem 3.4. *Let f be a semibalanced map from $X \times X$ into $Y \times Y$. If R is a fuzzy difunctional relation on X, which is fuzzy reflexive and weakly f-invariant, then R is f-invariant.*

Proof. Let $f(a, b) = f(a_1, b_1) = (u, v)$, where $a, a_1, b, b_1 \in X$ and $u, v \in Y$.

Given $x \in X$, since f is a semibalanced map, there exists unique $t_x \in Y$ such that $f(x, x) = (t_x, t_x)$, and so $f(a, x) = (u, t_x)$, $f(a_1, x) = (u, t_x)$, $f(x, b) = (t_x, v)$ and $f(x, b_1) = (t_x, v)$. Hence, we have $f(a, x) = f(a_1, x)$ and $f(x, b) = f(x, b_1)$.

Since R is weakly f-invariant, we get $R(a, x) = R(a_1, x)$ and $R(x, b) = R(x, b_1)$. We must prove that $R(a, b) = R(a_1, b_1)$. By earlier result, we note that R is a fuzzy equivalence relation on X. And we have

$$\begin{aligned} R(a, b) &= RR(a, b) \\ &= \bigvee_{x \in X} (R(a, x) \wedge R(x, b)) \\ &= \bigvee_{x \in X} (R(a_1, x) \wedge R(x, b_1)) \\ &= RR(a_1, b_1) \\ &= R(a_1, b_1) \text{ for all } a, b \in X. \end{aligned}$$

Which yields R is f-invariant.

Theorem 3.5. *Let f be a semibalanced map from $X \times X$ onto $Y \times Y$. If R is a fuzzy difunctional relation on X, which is fuzzy reflexive and weakly f-invariant, then $f(R)$ is a fuzzy difunctional relation on Y.*

Proof. It suffices to show that $f(R)$ is fuzzy symmetric and fuzzy transitive.

Let $u, v \in Y$. Since f is an onto semibalanced map, there exist $a, b \in X$, such that $f(a, b) = (u, v)$. Hence, we have

$$\begin{aligned} f(R)(u, v) &= \bigvee_{(x, y) \in f^{-1}(u, v)} R(x, y) \\ &= R(a, b), \text{ as } R \text{ is } f\text{-invariant} \end{aligned}$$

$$= R(b, a), \text{ as } R \text{ is fuzzy symmetric}$$

$$= f(R)(v, u) \text{ for all } u, v \in Y.$$

Which yields $f(R)$ is symmetric.

Further, let $x \in X$ be given. Then there exists a unique $t_x \in Y$ such that $f(a, x) = (u, t_x)$, $f(x, b) = (t_x, v)$, $f(x, x) = (t_x, t_x)$.

And we have

$$(f(R)f(R))(u, v) = \bigvee_{w \in Y}(f(R)(u, w) \wedge f(R)(w, v))$$

$$= \bigvee_{x \in X}(f(R)(u, t_x) \wedge f(R)(t_x, v))$$

$$= \bigvee_{x \in X}(R(a, x) \wedge R(x, b))$$

$$= RR(a, b)$$

$$\leq R(a, b)$$

$$= f(R)(u, v) \text{ for all } u, v \in Y.$$

Which yields $f(R)$ is fuzzy transitive. This completes the proof.

Theorem 3.6 [7, Theorem 3.1]**.** *Let R be a fuzzy relation on a group X such that R is fuzzy reflexive and fuzzy transitive. Then R is compatible iff R is right and left compatible.*

Theorem 3.7 [7, Theorem 3.3]**.** *Let R be a fuzzy relation on a group X such that R is fuzzy reflexive. Then R is right* (*left*) *compatible iff R is right* (*left*) *conformable.*

Theorem 3.8. *Let f be a semibalanced map and a group homomorphism from* $X \times X$ *into* $Y \times Y$. *If R is a fuzzy difunctional on Y such that R is fuzzy reflexive and right and left compatible, then* $f^{-1}(R)$ *is a fuzzy congruence on X.*

Proof. Let $x \in X$ be given. Since *f* is a semibalanced map, there exists unique $t_x \in Y$ such that $f(x, x) = (t_x, t_x)$ and $f^{-1}(R)(x, x) = R(f(x, x))$ $= R(t_x, t_x) = 1.$ This means $f^{-1}(R)$ is fuzzy reflexive. Due to Theorem 3.3, $f^{-1}(R)$ is a fuzzy difunctional relation on *X*.

Further, we show that $f^{-1}(R)$ is compatible. Now let $a, b, c \in X$. Then $f^{-1}(R)(ac, bc) = R(f(ac, bc)) \geq R(f(a, b)) = f^{-1}(R)(a, b)$ and

$$f^{-1}(R)(ca, cb) = R(f(ca, cb)) \geq R(f(a, b)) = f^{-1}(R)(a, b).$$

This completes the proof.

Theorem 3.9. *Let f be a semibalanced map and a group homomorphism from* $X \times X$ *onto* $Y \times Y$. *If R is a fuzzy difunctional relation on X such that R is fuzzy reflexive and right and left conformable*, *which is weakly f-invariant*, *then* $f(R)$ *is a fuzzy congruence on Y.*

Proof. Let $u \in Y$ be given, but *f* being an onto semibalanced, there exist $a, a^1 \in X$ such that $f(a, a') = (u, u)$. And so $f(a, a) = (u, u)$. This leads to $f(a, a') = f(a, a)$. Since *R* is *f*-invariant, we then have

$$\begin{aligned} f(R)(u, u) &= \bigvee_{(x, x') \in f^{-1}(u, u)} R(x, x') \\ &= R(a, a) \\ &= 1. \quad \text{Hence, } f(R) \text{ is fuzzy reflexive.} \end{aligned}$$

By earlier result, we have $f(R)$ is a fuzzy difunctional relation on *Y*. Therefore, $f(R)$ is a fuzzy equivalence relation on *Y*. Due to Theorem 3.6 and Theorem 3.7, we see that *R* is compatible.

This completes the proof.

References

[1] K. C. Gupta and T. P. Singh, Fuzzy *G*-equivalences and *G*-congruences on a groupoid under semibalanced maps, Fuzzy Sets and Systems 108 (1999), 111-116.

[2] N. Kuroki, Fuzzy congruences and fuzzy normal subgroups, Inform. Sci. 60 (1992), 247-259.

[3] V. Mrali, Fuzzy equivalence relations, Fuzzy Sets and System 30 (1989), 155-163.

[4] H. Ounalli and A. Jaoua, On fuzzy difunctional relations, Inform. Sci. 95 (1996), 219-232.

[5] C. H. Seo, K. H. Han, Y. O. Sung and H. C. Eun, On the relationships between fuzzy equivalence relations and fuzzy difunctional relations, and their properties, Fuzzy Sets and Systems 109 (2000), 459-462.

[6] Y.-O. Sung, Balanced mappings, Far East J. Math. Sci. (FJMS) 76(1) (2013), 89-95.

[7] Y. O. Sung, Fuzzy congruences on groups, Far East J. Math. Sci. (FJMS) 89(2) (2014), 227-237.