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A TYPE OF INTEGRABLE COUPLING MODEL OF KN HIERARCHY

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Abstract

Integrable coupling of the well-known Kaup Newell hierarchy is obtained by constructing a higher loop algebra and using a relation of direct sum between two subalgebras. As a reduction case, an integrable coupling of a generalized MKdV equation is presented.

1. Introduction

Integrable couplings are an interesting aspect in the field of soliton theory. A central and very important topic in the study of integrable system is to search for integrable models as many as possible and such that they be associated with certain evolution equations with physical meaning. It originates from an investigation on centerless Virasoro symmetry algebras of integrable systems or soliton equations [1]. Let

$$u_t = K(u) \tag{1}$$

be a known integrable system. Then the following system

$$\begin{cases} u_t = K(u) \\ v_t = S(u, v) \end{cases} \tag{2}$$

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is called *integrable coupling* from (1), if (2) is also integrable, and contains explicitly derivatives. The reference [3] consists of a theory for constructing integrable coupling of soliton equations by using various perturbations around solutions of perturbed soliton equations. Actually, two approaches have been developed for searching integrable couplings [1-4]: The common characteristics of above methods are that isospectral problem gives rise to the integrable couplings for only one equation. Recently, Tu [5] proposed a new method which is based on the analysis of loop algebra. It is called as the *Tu model*. His key idea is to construct a loop algebra. In this paper, a direct method is proposed by considering an isospectral problem. Integrable couplings for the corresponding equation hierarchy can be obtained by constructing a suitable loop algebra $\widetilde{G}$. In what follows, integrable coupling for Kaup Newell hierarchy will be established by means of an example illustrating our method.

2. Integrable Coupling for Kaup Newell Hierarchy

Consider the basis of the loop algebra $\widetilde{A}_1$ as follows:

$$\begin{cases} h(n) = \begin{pmatrix} \lambda^n & 0 \\ 0 & -\lambda^n \end{pmatrix}, \ e(n) = \begin{pmatrix} 0 & \lambda^n \\ 0 & 0 \end{pmatrix}, \ f(n) = \begin{pmatrix} 0 & 0 \\ \lambda^n & 0 \end{pmatrix} \\ [h(m), e(n)] = 2e(m+n), [h(m), f(n)] = -2f(m+n), [e(m), f(n)] = h(m+n), \\ deg(h(n)) = deg(e(n)) = deg(f(n)) = n \end{cases}$$

and let

$$\begin{cases} \psi_x = U\psi, \ \lambda_t = 0, \ \psi = (\psi_1, \psi_2)^T, \\ U = -ih(2) + qe(1) + rf(1) = \begin{pmatrix} -i\lambda_2 & \lambda q \\ \lambda r & i\lambda^2 \end{pmatrix}. \end{cases}$$

Then the Kaup Newell hierarchy is derived from it as

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = JL^{n-1}\begin{pmatrix} ir \\ iq \end{pmatrix}, \ J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix},$$

$$L = \frac{1}{2}\begin{pmatrix} i\partial - r\partial^{-1}q\partial & -r\partial^{-1}r\partial \\ -q\partial^{-1}q\partial & -i\partial - q\partial^{-1}r\partial \end{pmatrix}. \tag{3}$$

Let the loop algebra $\widetilde{G}$ possess two subalgebras $\widetilde{G}_1$ and $\widetilde{G}_2$ which satisfy the following:

$\widetilde{G}_1$ is isomorphic to $\widetilde{A}_1$ and

$$[\widetilde{G}_1, \widetilde{G}_2] \subset \widetilde{G}_2 \tag{4}$$

from which a corresponding isospectral problem is established and the derived integrable equation hierarchy becomes integrable coupling for Kaup Newell hierarchy. Set G to be a linear space with basis $\{e_1, e_2, e_3, e_4, e_5\}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The commutation relations are defined as

$$[e_1, e_2] = 2e_2, [e_1, e_3] = -2e_3, [e_2, e_3] = e_1, [e_1, e_4] = e_4,$$

$$[e_1, e_5] = -e_5, [e_2, e_4] = 0, [e_2, e_5] = e_4,$$

$$[e_3, e_4] = e_5, [e_3, e_5] = 0, [e_4, e_5] = 0.$$

Put $a = \sum_{i=1}^5 a_i e_i$, $b = \sum_{i=1}^5 b_i e_i$, $c = \sum_{i=1}^5 c_i e_i$, where a_i, b_i and c_i are constants (or functions). Then we have the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

Thus, G is a Lie algebra with basis

$$\begin{cases} e_i(n) = e_i \lambda^n, \ i = 1, 2, 3, 4, 5; \\ [e_i(m), e_j(n)] = [e_i, e_j]\lambda^{m+n} \\ deg(e_i(n)) = n, \ i = 1, 2, 3, 4, 5 \end{cases}$$

which consists of loop algebra $\widetilde{G}$. Let the subalgebras $\widetilde{G}_1$ and $\widetilde{G}_2$ of the loop algebra $\widetilde{G}$ have for bases $\{e_1(n), e_2(n), e_3(n)\}$ and $\{e_4(n), e_5(n)\}$, respectively. It is easy to find that $\widetilde{G}_1$ and $\widetilde{G}_2$ satisfy (4).

Consider a linear problem as follows:

$$\begin{cases} \Psi_x = [U, \Psi], \lambda_t = 0, \\ \Psi_t = [V, \Psi], \end{cases} \tag{5}$$

where $\Psi = \sum_{i=1}^{5} \Psi_i e_i$, Ψ_i is arbitrary, $U = U(u, \lambda) \in \widetilde{G}$, $V = V(u, \lambda) \in \widetilde{G}$, $u = (u_1, u_2, \ldots, u_p)^T$, λ is a spectral parameter. Then the compatibility condition in (5) reads as

$$\Psi_{xt} = [U_t, \Psi] + [U, \Psi_t] = [U_t, \Psi] + [U, [V, \Psi]]$$

$$= \Psi_{tx} = [V_x, \Psi] + [V, \Psi_x]$$

$$= [V_x, \Psi] + [V, [U, \Psi]],$$

$$[U_t, \Psi] + [U, [V, \Psi]] - [V_x, \Psi] - [V, [U, \Psi]] = 0.$$

In terms of Jacobi identity, the compatibility condition of (12) reduces to the zero curvature equation

$$U_t - V_x + [U, V] = 0.$$

Consider the following isospectral problem

$$\begin{cases} \varphi_x = [U, \varphi], \lambda_t = 0, \\ U = -ie_1(2) + u_1 e_2(1) + u_2 e_3(1) + u_3 e_4(0) + u_4 e_5(0), \end{cases} \tag{6}$$

and set

$$V = \sum_{m \geq 0} (a_m e_1(-m) + b_m e_2(-m) + c_m e_3(-m)$$

$$+ d_m e_4(-m) + f_m e_5(-m)).$$

Solving the stationary zero curvature equation $V_x = [U, V]$, gives rise to the recursion relations as follows:

$$\begin{cases} a_{nx} = u_1 c_{n+1} - u_2 b_{n+1} = -\frac{i}{2}(u_1 c_{n+1x} + u_2 b_{n-1x}), \\ b_{nx} = -2ib_{n+2} - 2u_1 a_{n+1},\ c_{nx} = 2ic_{n+2} + 2u_2 a_{n+1}, \\ d_{nx} = -id_{n+2} + u_1 f_{n+1} - u_3 a_n - u_4 b_n, \\ f_{nx} = if_{n+2} + u_2 d_{n+1} - u_3 c_n + u_4 a_n, \\ b_0 = c_0 = d_0 = f_0 = 0,\ a_0 = 1, \\ a_{2k+1} = b_{2k} = c_{2k} = d_{2k+1} = f_{2k+1} = 0,\ k = 0, 1, 2, \ldots. \end{cases} \tag{7}$$

Denote

$$V_+^{(n)} = \sum_{m=0}^{n} [a_m e_1(n-m) + b_m e_2(n-m) + c_m e_3(n-m) + d_m e_4(n-m) + f_m e_5(n-m)]$$

$$V_-^{(n)} = \lambda^n V - V_+^{(n)}.$$

Then zero curvature equation can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}]. \tag{8}$$

A direct calculation gives that the terms on the left-hand side in (8) are of degree $(deg) \geq 0$, while the terms on the right-hand side are of degree $(deg) \leq 1$. Therefore, both sides of (8) are of degrees 0 and 1. It is easy to see that for $V^n = (\lambda^n V)_+ + \Delta_n$, taking $n = 2m$, $\Delta_n = 0$, a direct calculation provides that

$$-V_{+x}^{(n)} + [U, V^{(n)}] = -b_{2n-1x} e_2(1) - c_{2n-1x} e_3(1) - d_{2nx} e_4(0) - f_{2nx} e_5(0).$$

Thus, the zero curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{9}$$

leads to the following Lax integrable system

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = \begin{pmatrix} b_{2n-1x} \\ c_{2n-1x} \\ d_{2nx} \\ f_{2nx} \end{pmatrix} = \begin{pmatrix} 0 & \partial & 0 & 0 \\ \partial & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & \partial \end{pmatrix} \begin{pmatrix} b_{2n-1} \\ c_{2n-1} \\ d_{2n} \\ f_{2n} \end{pmatrix} = J \begin{pmatrix} b_{2n-1} \\ c_{2n-1} \\ d_{2n} \\ f_{2n} \end{pmatrix}, \tag{10}$$

where J is the Hamiltonian operator.

From (7), a recurrence operator is given by

$$L = \frac{1}{2} \begin{pmatrix} u_2\partial^{-1}u_1\partial - i\partial & u_2\partial^{-1}u_2\partial & 0 & 0 \\ u_1\partial^{-1}u_1\partial & u_1\partial^{-1}u_2\partial + i\partial & 0 & 0 \\ u_3\partial^{-1}u_1\partial & u_3\partial^{-1}u_2\partial & 2i\partial & 0 \\ u_4\partial^{-1}u_1\partial & u_4\partial^{-1}u_2\partial & 0 & -2i\partial \end{pmatrix}.$$

Therefore (10) can be rewritten as

$$u_t = JL^{n-1} \begin{pmatrix} iu_2 \\ iu_1 \\ iu_3 \\ iu_4 \end{pmatrix}. \tag{11}$$

Since the hierarchy (11) is derived from the zero-curvature equation (9), it is integrable. From the comparison of the construction of J and L with that in (3), we find that (11) is integrable coupling for Kaup Newell hierarchy (3). Of course, (11) is also a type of expanding integrable model of (3).

In particular, taking $n = 3$, (8) and (10) lead to the reduction of (11) as follows:

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = \begin{pmatrix} b_{5x} \\ c_{5x} \\ d_{6x} \\ f_{6x} \end{pmatrix}, \tag{12}$$

where

$$
\begin{aligned}
b_{5x} &= -\frac{i}{4} u_{1xxx} - \frac{3}{4} u_{1x}^2 u_2 - \frac{3}{4} u_1 u_{1x} u_{2x} - \frac{3}{4} u_1 u_2 u_{1xx} \\
&\quad + \frac{9i}{8} u_1^2 u_2^2 u_{1x} + \frac{3i}{4} u_1^3 u_2 u_{2x} \\
c_{5x} &= -\frac{i}{4} u_{2xxx} + \frac{3}{4} u_{2x}^2 u_1 + \frac{3}{4} u_1 u_{1x} u_{2x} + \frac{3}{4} u_1 u_2 u_{2xx} \\
&\quad + \frac{9i}{8} u_2^2 u_1^2 u_{2x} + \frac{3i}{4} u_2^3 u_1 u_{1x} \\
d_{6x} &= -iu_{3xxx} - \frac{1}{4} u_1 u_{2xx} u_3 - u_{1x} u_{3x} u_3 - \frac{5}{4} u_2 u_{1x} u_{2x} \\
&\quad - \frac{3}{4} u_1 u_{2x} u_{3x} - \frac{3}{4} u_2 u_3 u_{1xx} - \frac{1}{2} u_1 u_2 u_{3xx} \\
&\quad + \frac{3i}{4} u_1^2 u_{2x} u_2 u_3 + \frac{3i}{4} u_2^2 u_{1x} u_1 u_3 + \frac{3i}{8} u_1^2 u_2^2 u_{3x} \\
f_{6x} &= -iu_{4xxx} + \frac{1}{4} u_2 u_{1xx} u_4 + u_{1x} u_{2x} u_4 + \frac{3}{4} u_2 u_{1x} u_{4x} \\
&\quad + \frac{3}{4} u_1 u_{2xx} u_4 + \frac{5}{4} u_1 u_{2x} u_{4x} + \frac{1}{2} u_1 u_2 u_{4xx} \\
&\quad + \frac{3i}{4} u_1 u_{1x} u_2^2 u_4 + \frac{3i}{4} u_1^2 u_{2x} u_2 u_4 + \frac{3i}{8} u_1^2 u_2^2 u_{4x}.
\end{aligned}
$$

It is easy to find that (12) is reduced to the integrable coupling for the generalized MKdV equation when $u_1 = u$, $u_2 = 1$,

$$
\begin{cases}
u_t = -\frac{i}{4} u_{xxx} + \frac{9i}{8} u_2^2 u_x - \frac{3}{4} u u_{xx} - \frac{3}{4} u^2, \\
u_{3t} = -iu_{3xxx} - \frac{5}{4} u_x u_{3x} - \frac{3}{4} u_3 u_{xx} + \frac{1}{2} u u_{3xx} + \frac{3i}{4} u u_3 u_x + \frac{3i}{8} u^2 u_{3x}, \\
u_{4t} = -iu_{4xxx} + \frac{1}{4} u_4 u_{xx} + \frac{3}{4} u_x u_{4x} + \frac{1}{2} u u_{4xx} + \frac{3i}{4} u u_x u_4 + \frac{3i}{8} u^2 u_{4x}.
\end{cases}
\tag{13}
$$

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