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ALMOST AUTOMORPHIC SOLUTIONS OF NEUTRAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract

In this paper, we study the existence of almost automorphic solutions for some neutral first-order functional integrodifferential equations with S^p-almost (Stepanov-like) automorphic coefficients.

1. Introduction

The concept of Stepanov-like almost automorphy was first introduced by Casarino [3] and the historic development of automorphy was discussed by

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Xia and Fan [24]. N'Guerekata and Pankov [22] utilized S^p-almost automorphy to study the existence of weak almost automorphic solutions to some parabolic evolution equations and Diagana and N'Guerekata [4] studied the existence and uniqueness of almost automorphic solutions to the semilinear differential equations

$$u'(t) = Au(t) + F(t, u(t)) \text{ for } t \in R, \tag{1}$$

where $A : D(A) \subset X \to X$ is a densely defined closed linear operator on a Banach space X, which also is the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t \geq 0}$ on X and $F : R \times X \to X$ is S_p-almost automorphic for $p > 1$ and jointly continuous.

In this paper, we investigate the existence and uniqueness of an almost automorphic solutions to the neutral first-order functional integrodifferential equations

$$\frac{d}{dt}\left[u(t) + f\left(t, u_t, \int_0^t h(s, u(s))ds\right)\right]$$
$$= Au(t) + g\left(t, u_t, \int_0^t h(s, u(s))ds\right), \forall t \in R, \tag{2}$$

where $u_t : (-\infty, 0] \times X$ defined by $u_t(\tau) = u(t + \tau)$ belongs to some abstract phase space B, which is defined axiomatically, and the coefficients f, g are S^p-almost automorphic for $p > 1$ and jointly continuous.

The application of our result will be utilized to study the existence of almost automorphic solutions to a slightly modified integrodifferential equation which was considered in Diagana et al. [9] in the pseudo almost periodic case.

The existence of almost automorphic, almost periodic, asymptotically almost periodic and pseudo almost periodic solutions is certainly one of the most attractive topics in qualitative theory of differential equations due to their significance and various applications. The concept of almost automorphy was first introduced by Bochner [1] and is a natural generalization of almost periodicity. It has generated several developments and extensions. For the recent developments, we refer the book by N'Guerekata [20]. Recently, the authors [14, 18] studied the existence of almost periodic and asymptotically almost periodic solutions to ordinary neutral differential equations and abstract partial neutral differential equations. The study of existence of almost automorphic solutions to functional-differential equations with delay was initiated by [5, 10, 11, 16]. The applications of neutral differential equations arise in many areas of applied mathematics. For this reason, those equations have been of a great interest for several mathematicians during the past few decades.

This work is organized as follows: In Section 2, we recall some definitions and notations come from [5] and also we obtain very general results on the existence of S^p-almost automorphic solutions for neutral functional integrodifferential equations. In Section 3, we obtain the existence and uniqueness of S^p-almost automorphic mild solutions for neutral functional integrodifferential equations. Finally, in Section 4, an example is provided.

2. Preliminaries

In this section, we introduce notations, definitions, lemmas and preliminary facts which are used throughout this work. Most of these definitions and notations come from [5].

Let $(Z, \|\cdot\|_Z)$, $(W, \|\cdot\|_W)$ be Banach spaces. The notation $L(Z, W)$ stands for the Banach space of bounded linear operators from Z into W equipped with its natural topology; in particular, this is simply denoted $L(Z)$ when $Z = W$. The spaces $C(R, Z)$ and $BC(R, Z)$ stand, respectively, for the collection of all continuous functions from R into Z and the Banach space of all bounded continuous functions from R into Z equipped with the sup norm defined by

$$\| f \|_\infty = \sup_{t \in R} \| f(t) \|.$$

We have similar definitions as above for both $C(R \times Z, W)$ and $BC(R \times Z, W)$.

In this paper, $(X, \| \cdot \|)$ stands for a Banach space and the linear operator A is the infinitesimal generator of a C_0-semigroup $(T(s))_{s \geq 0}$, which is asymptotically stable. There exists some constants M, $\delta > 0$ such that

$$\| T(t) \| \leq Me^{-\delta t}, \text{ for every } t \geq 0.$$

Definition 2.1 (Bochner)**.** A function $f \in C(R, X)$ is said to be *almost automorphic* if for every sequence of real numbers $(s'_n)_{n \in N}$, there exists a subsequence $(s_n)_{n \in N}$ such that

$$g(t) = \lim_{n \to \infty} f(t + s_n)$$

is well defined for each $t \in R$, and

$$\lim_{n \to \infty} g(t - s_n) = f(t)$$

for each $t \in R$.

Remark 2.2. The function g in Definition 2.1 is measurable, but not necessarily continuous. Moreover, if g is continuous, then f is uniformly continuous [21, Theorem 2.6]. If the convergence above is uniform in $t \in R$, then f is almost periodic. Denote by $AA(X)$ the collection of all almost automorphic functions $R \to X$. Note that $AA(X)$ equipped with the sup norm, $\| \cdot \|_\infty$, turns out to be a Banach space. Among other things, almost automorphic functions satisfy the following properties.

Theorem 2.3 ([2], [19, Theorem 2.1.3])**.** *If* f, f_1, $f_2 \in AA(X)$, *then*

(i) $f_1 + f_2 \in AA(X)$,

(ii) $\lambda f \in AA(X)$ *for any scalar* λ,

(iii) $f_\alpha \in AA(X)$, *where* $f_\alpha : R \to X$ *is defined by* $f_\alpha(\cdot) = f(\cdot + \alpha)$,

(iv) *the range* $R_f = \{f(t) : t \in R\}$ *is relatively compact in X, thus f is bounded in norm*,

(v) *if* $f_n \to f$ *uniformly on R, where each* $f_n \in AA(X)$, *then* $f \in AA(X)$ *too*,

(vi) *if* $g \in L^1(R)$, *then* $f * g \in AA(R)$, *where* $f * g$ *is the convolution of f with g on R*.

For more on almost automorphic functions and related issues, we refer the reader to the following books by N'Guerekata [19, 20].

We will denote by $AA_u(X)$ the closed subspace of all functions $f \in AA(X)$ with $g \in C(R, X)$. Equivalently, $f \in AA_u(X)$ if and only if *f* is almost automorphic and the convergence in Definition 2.1 is uniform on compact intervals, i.e., in the Fréchet space $C(R, X)$. Indeed, if *f* is almost automorphic, then, by Theorem 2.1.3(iv) [19], its range is relatively compact.

Obviously, the following inclusions hold:

$$AP(X) \subset AA_u(X) \subset AA(X) \subset BC(X),$$

where $AP(X)$ stands for the collection of all *X*-valued almost periodic functions.

Definition 2.4. The Bochner transform $f^b(t, s)$, $t \in R$, $s \in [0, 1]$ of a function $f : R \to X$ is defined by

$$f^b(t, s) = f(t + s).$$

Remark 2.5. Note that a function $\varphi(t, s)$, $t \in R$, $s \in [0, 1]$ is the Bochner transform of a certain function $f(t)$,

$$\varphi(t, s) = f^b(t, s),$$

if and only if $\varphi(t+\tau, s-\tau) = \varphi(s, t)$ for all $t \in R$, $s \in [0, 1]$ and $\tau \in [s-1, s]$.

Definition 2.6 [23]. *Let* $p \in [1, \infty)$. The space $BS^p(X)$ of all Stepanov bounded functions, with the exponent *p*, consists of all measurable functions *f* on *R* with values in *X* such that $f^b \in L^\infty(R, L^p(0, 1; X))$. This is a Banach space with the norm

$$\| f \|_{S^p} = \| f^b \|_{L^\infty(R, L^p)} = \sup_{t \in R} \left(\int_t^{t+1} \| f(\tau) \|^p d\tau \right)^{1/p}.$$

Definition 2.7 [22]. The space $AS^p(X)$ of Stepanov-like almost automorphic functions (or S^p-almost automorphic) consists of all $f \in BS^p(X)$ such that $f^b \in AA(L^p(0, 1; X))$.

In other words, a function $f \in L^p_{loc}(R; X)$ is said to be S^p-*almost automorphic* if its Bochner transform $f^b : R \to L^p(0, 1; X)$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_{n \in N}$, there exists a subsequence $(s_n)_{n \in N}$ and a function $g \in L^p_{loc}(R; X)$ such that

$$\left[\int_t^{t+1} \| f(s_n + s) - g(s) \|^p ds \right]^{1/p} \to 0,$$

$$\left[\int_t^{t+1} \| g(s - s_n) - f(s) \|^p ds \right]^{1/p} \to 0$$

as $n \to \infty$ pointwise on *R*.

Remark 2.8. It is clear that if $1 \le p < q < \infty$ and $f \in L^q_{loc}(R; X)$ is S^q-almost automorphic, then f is S^p-almost automorphic. Also, if $f \in AA(X)$, then *f* is S^p-almost automorphic for any $1 \le p < \infty$.

It is also clear that $f \in AA_u(X)$ if and only if $f^b \in AA(L^\infty(0, 1; X))$. Thus, $AA_u(X)$ can be considered as $AS^\infty(X)$.

Example 2.9 [22]**.** Let $x = (x_n)_{n \in Z} \in l^\infty(X)$ be an almost automorphic sequence [21] and let $\varepsilon_0 \in \left(0, \frac{1}{2}\right)$. Let $f(t) = x_n$ if $t \in (n - \varepsilon_0, n + \varepsilon_0)$ and $f(t) = 0$, otherwise. Then $f \in AS^p(X)$ for $p \geq 1$ but $f \notin AA(X)$, as f is discontinuous.

Theorem 2.10 [22]**.** *The following statements are equivalent*:

(i) $f \in AS^p(X)$;

(ii) $f^b \in AA_u(L^p(0, 1; X))$;

(iii) *for every sequence* $(s'_n)_{n \in N}$ *of real numbers*, *there exists a subsequence* (s_n) *such that*

$$g(t) = \lim_{n \to \infty} f(t + s_n) \tag{3}$$

exists in the space $L^p_{loc}(R; X)$ *and*

$$f(t) = \lim_{n \to \infty} g(t - s_n) \tag{4}$$

in the sense of $L^p_{loc}(R; X)$.

In view of the above, the following inclusions hold:

$$AP(X) \subset AA_u(X) \subset AA(X) \subset AS^p(X) \subset BS^p(X).$$

Definition 2.11. A function $F : R \times X \times X \to X$, $(t, u, v) \to F(t, u, v)$ with $F(\cdot, u, v) \in L^p_{loc}(R; X)$ for each $u, v \in X$ is said to be S^p*-almost automorphic* in $t \in R$ uniformly in $u, v \in X$ if $t \to F(t, u, v)$ is S^p-almost automorphic for each $u, v \in X$ that is, for every sequence of real numbers $(s'_n)_{n\in N}$, there exists a subsequence $(s_n)_{n\in N}$ and a function $G(\cdot, u, v) \in L^p_{loc}(R; X)$ such that

$$\left[\int_t^{t+1} \| F(s_n + s, u, v) - G(s, u, v) \|^p ds\right]^{1/p} \to 0,$$

$$\left[\int_t^{t+1} \| G(s - s_n, u, v) - F(s, u, v) \|^p ds\right]^{1/p} \to 0$$

as $n \to \infty$ pointwise on R for each $u \in X$.

The collection of those S^p-almost automorphic functions $F : R \times X \times X \to X$ will be denoted by $AS^p(R \times X \times X, X)$.

The following theorem is the composition theorem which is a slight generalization of [3, Theorem 2.15].

Theorem 2.12. *Let* $F : R \times Z \to W$ *be a* S^p*-almost automorphic. Suppose that there exists a continuous function* $L_F : R \to (0, \infty)$ *satisfying* $L_F = \sup_{t\in R} L_F(t) < \infty$ *and such that*

$$\| F(t, u) - F(t, v) \|_W \leq L_F(t) \| u - v \|_Z \tag{5}$$

for all $t \in R$, $(u, v) \in Z \times Z$.

If $\varphi \in AS^p(Z)$, then $\Gamma : R \to W$ defined by $\Gamma(\cdot) = F(\cdot, \varphi(\cdot))$ belongs to $AS^p(W)$.

We will herein define the phase space B axiomatically, using ideas and notions developed in [17]. More precisely, B will denote the vector space of functions defined from $(-\infty, 0]$ into X endowed with a seminorm $\| \cdot \|_B$ and such that the following axioms hold:

(A) If $u : (-\infty, \sigma + a) \to X$, $a > 0$, $\sigma \in R$, is continuous on $[\sigma, \sigma + a)$ and $u_\sigma \in B$, then for every $t \in [\sigma, \sigma + a)$, the following hold:

(i) u_t is in B;

(ii) $\| u(t) \| \leq H \| u_t \|_B$;

(iii) $\| u_t \|_B \leq K(t - \sigma) \sup\{\| u(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| u_\sigma \|_B$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $u(\cdot)$.

(B) For the function $u(\cdot)$ appearing in (A), its corresponding history $t \to u_t$ is continuous from $[\sigma, \sigma + a)$ into B.

(C) The space B is complete.

(D) If $(v_n)_{n \in N}$ is a uniformly bounded sequence in $C((-\infty, 0], X)$ given by functions with compact support and $v_n \to \varphi$ the compact-open topology, then $v \in B$ and $\| v_n - v \|_B \to 0$ as $n \to \infty$.

In what follows, we let $B_0 = \{v \in B : v(0) = 0\}$.

Definition 2.13. Let $S(t) : B \to B$ be the C_0-semigroup defined by $S(t)v(\theta) = v(\theta)$ on $[-t, 0]$ and $S(t)v(\theta) = v(t + \theta)$ on $(-\infty, -t]$. The phase space B is called a *fading memory* if $\| S(t)v \|_B \to 0$ as $t \to \infty$ for every $v \in B_0$. Now, B is called *uniform fading memory* whenever $\| S(t) \|_{L(B_0)} \to 0$ as $t \to \infty$.

Remark 2.14 (See [16, Proposition 7.1.1])**.** In this paper, we suppose $Q > 0$ is such that $\| v \|_B \leq Q \sup_{\theta \leq 0} \| v(\theta) \|$ for each $v \in B$ bounded continuous. Moreover, if B is a fading memory, then we assume that $\max\{K(t), M(t)\} \leq Q'$ for $t \geq 0$.

Remark 2.15. It is worth mentioning that in [16, p. 190], it is shown that the phase B is a uniform fading memory space if and only if axiom (D) holds, the function $K(\cdot)$ is then bounded and $\lim_{t \to \infty} M(t) = 0$.

Example 2.16 (The phase space $C_r \times L^p(\rho, X)$)**.** Let $r \geq 0$, $1 \leq p < \infty$ and let $\rho : (-\infty, -r] \to R$ be a nonnegative measurable function which satisfies the conditions [16, (g_5)-(g_6)]. Basically, this means that ρ is locally integrable and there exists a nonnegative locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)$ for all $\xi \leq 0$ with $\theta \in (-\infty, -r)/N_\xi$, where $N_\xi \subset (-\infty, -r)$ is a subset whose Lebesgue measure is zero. The space $B = C_r \times L^p(\rho, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\rho\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\rho, X)$ is defined as follows:

$$\|\varphi\|_B = \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\|^p d\theta\right)^{1/p}.$$

The space $B = C_r \times L^p(\rho, X)$ satisfies axioms (A), (B) and (C). Moreover, when $r = 0$ and $p = 2$, one can then take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^{0} \rho(\theta)d\theta\right)^{1/2}$ for $t \geq 0$ (see [16, Theorem 1.3.8] for details).

It is worth mentioning that if the conditions [16, (g_5)-(g_7)]. hold, then B is a uniform fading memory.

3. Existence of Almost Automorphic Solutions

This section is devoted to the search of an almost automorphic solution to the neutral functional integrodifferential equation (2).

Definition 3.1. A continuous function $u : [\sigma, \sigma + a) \to X$ for $a > 0$ is said to be a *mild solution* to the neutral functional integrodifferential equation (2) on $[\sigma, \sigma + a)$ whenever the function

$$s \to AT(s) f\left(s, u_s, \int_0^s h(\gamma, u(\gamma))d\gamma\right)$$

is integrable $[\sigma, t)$ for every $\sigma < t < \sigma + a$, and

$$u(t) = T(t-\sigma)(\varphi(0) + f(\sigma, \varphi)) - f\left(t, u_t, \int_0^t h(s, u(s))ds\right)$$

$$-\int_\sigma^t AT(t-s)f\left(s, u_s, \int_0^s h(\gamma, u(\gamma))d\gamma\right)ds$$

$$+\int_\sigma^t T(t-s)g\left(s, u_s, \int_0^s h(\gamma, u(\gamma))d\gamma\right)ds, \quad t \in [\sigma, \sigma + a).$$

Let $p > 1$ and $q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. From now on, let $(Y, \|\cdot\|_Y)$ denote a Banach space continuously embedded into X and we make the following assumptions:

(H_1) The function $s \to AT(t-s)$ defined from $(-\infty, t)$ into $L(Y, X)$ is strongly measurable and there exist a non-increasing function $H : [0, \infty) \to [0, \infty)$ and $\gamma > 0$ with $s \to e^{-\gamma s}H(s) \in L^1[0, \infty) \cap L^q[0, \infty)$ such that

$$\| AT(s) \|_{L(Y, X)} \leq e^{-\gamma s}H(s), \; s > 0$$

and $T(t)$ is asymptotically stable.

(H_2) The function $f, g \in AS^p(R \times X \times X, X)$, f is Y-valued, $f : R \times X \times X \to Y$ is continuous and there is a constant $L_f \in (0, 1)$ and a continuous function $L_g : R \to (0, \infty)$ satisfying $L_g = \sup_{t \in R} L_g(t) < \infty$ and such that

$$\| f(t, x_1, y_1) - f(t, x_2, y_2) \|_Y \leq L_f[\| x_1 - x_2 \| + \| y_1 - y_2 \|],$$

$$\| g(t, x_1, y_1) - g(t, x_2, y_2) \|_Y \leq L_g(t)[\| x_1 - x_2 \| + \| y_1 - y_2 \|], \; t \in R$$

for each $x_i, y_i \in X$, $i = 1, 2$.

(H_3) The function $h : R \times X \to X$ is an almost automorphic function in *t* uniformly in $x \in X$ and satisfies

$$\| h(t, x) - h(t, y) \| \leq L_h \| x - y \|, \text{ for each } x, y \in X.$$

Remark 3.2. Note that the assumption on *f* and (H_1) are linked to the integrability of the function $s \to ST(t-s) f(s, u_s)$ over $[0, t)$. Observe, for instance, that except trivial cases, the operator function $s \to AT(s)$ is not integrable over $[0, a]$. If we assume that $AT(\cdot) \in L^1([0, t])$, then from the relation

$$T(t)x - x = A \int_0^t T(s) ds = \int_0^t AT(s) ds,$$

it follows that the semigroup is uniformly continuous and as a consequence that *A* is a bounded linear operator on *X*, which is not interesting, especially for applications. On the other hand, if we assume that (H_1) is valid, then from the Bochner's criterion for integrable functions and the estimate

$$\left\| AT(t-s) f\left(s, u_s, \int_0^s h(\gamma, u(\gamma)) d\gamma\right) \right\|$$

$$\leq H(s) e^{-\gamma(t-s)} \left\| f\left(s, u_s, \int_0^s h(\gamma, u(\gamma)) d\gamma\right) \right\|_Y,$$

it follows that the function $s \to AT(t-s) f\left(s, u_s, \int_0^s h(\gamma, u(\gamma)) d\gamma\right)$ is integrable over $(-\infty, t)$ for each $t > 0$.

Lemma 3.3 [5]. *Let* $u = AA_u(X) \subset AS^p(X)$. *Then the function* $t \to u_t$ *belongs to* $AA_u(B) \subset AS^p(B)$.

Proof. For a given sequence $(s'_n)_{n \in N}$ of real numbers, fix a subsequence $(s_n)_{n \in N}$ of $(s'_n)_{n \in N}$ and a function $v \in BC(R, X)$ such that $u(s + s_n)$

$\to v(s)$ uniformly on compact subsets of R. Since B satisfies axiom C_2, from [17, Proposition 7.1.1], we infer that $u_{s+s_n} \to v_s$ in B for each $s \in R$. Let $\Omega \subset R$ be an arbitrary compact and let $L > 0$ such that $\Omega \subset [-L, L]$. For $\varepsilon > 0$, fix $N_{\varepsilon, L} \in N$ such that

$$\| u(s + s_n) - v(s) \| \leq \varepsilon, \quad s \in [-L, L]$$

$$\| u_{-L+s_n} - v_{-L} \| \leq \varepsilon$$

whenever $n \geq N_{\varepsilon, L}$.

In view of the above, for $t \in \Omega$ and $n \geq N_{\varepsilon, L}$, we get

$$\begin{aligned} \| u_{t+s_n} - v_t \|_B &\leq M(L+t) \| u_{L+s_n} - v_{-L} \|_B \\ &\quad + K(L+t) \sup_{\theta \in [-L, L]} \| u(\theta + s_n) - v(\theta) \| \\ &\leq 2Q'\varepsilon, \end{aligned}$$

where Q' is the constant appearing in Remark 2.14.

In view of the above, u_{t+s_n} converges to v_t uniformly on Ω. Similarly, one can prove that v_{t-s_n} converges to u_t uniformly on Ω. Thus, the function $s \to u_s$ belongs to $AA_c(B)$.

Lemma 3.4. *Under assumption (H_1), define the function ϕ, for $u \in AS^p(Y)$, by*

$$\phi(t) = \int_{-\infty}^{t} AT(t-s)u(s)ds$$

for each $t \in R$ and suppose

$$h_q^{\gamma, H} = \sum_{n=1}^{\infty} \left[\int_{n-1}^{n} e^{-q\gamma r} H^q(r)\, dr \right]^{1/q} < \infty.$$

Then $\phi \in AA(X)$.

Remark 3.5. Note that there are several functions H for which the assumption $h_q^{\gamma, H} < \infty$ appearing in Lemma 3.4. is achieved. For instance when $H_0(s) = e^{-\beta s}$ for all $\beta > 0$, then $h_q^{\gamma, H_0} < \infty$.

Proof of Lemma 3.4. Define for all $n = 1, 2, 3, \ldots,$ the sequence of integral operators

$$\phi_n(t) = \int_{n-1}^{n} AT(s)u(t-s)ds, \text{ for each } t \in R.$$

Now, let $r = t - s$, $dr = -ds$.

To find the limits of r: when $s = n - 1$, $r = t - n + 1$, and $s = n$, $r = t - n$.

It follows that

$$\begin{aligned}
\phi_n(t) &= \int_{n-1}^{n} AT(s)u(t-s)ds \\
&= \int_{t-n+1}^{t-n} AT(t-r)u(r)(-dr) \\
&= -\int_{t-n+1}^{t-n} AT(t-r)u(r)dr \\
&= \int_{t-n}^{t-n+1} AT(t-r)u(r)dr, \text{ for all } t \in R.
\end{aligned}$$

From the Bochner's criterion on integrable functions and the estimate

$$\begin{aligned}
\| AT(t-r)u(r) \| &\leq \| AT(t-r) \|_{L(Y,X)} \cdot \| u(r) \|_Y \\
&\leq e^{-\gamma(t-r)} H(t-r) \| u(r) \|_Y, \qquad (6)
\end{aligned}$$

it follows that the function $s \to AT(t-r)u(r)$ is integrable over $(-\infty, t)$ for each $t \in R$, by assumption (H_1).

Using Holder's inequality, it follows that

$$\| \phi_n(t) \| = \left\| \int_{t-n}^{t-n+1} AT(t-r)u(r)dr \right\|$$

$$\leq \int_{t-n}^{t-n+1} \| AT(t-r) \| \| u(r) \| dr$$

$$\leq \int_{t-n}^{t-n+1} \| AT(t-r) \|_{L(Y,\, X)} \| u(r) \|_Y dr$$

$$\leq \left(\int_{t-n}^{t-n+1} e^{-q\gamma(t-r)} H^q(t-r)dr \right)^{1/q} \cdot \left(\int_{t-n}^{t-n+1} \| u(r) \|_Y^p dr \right)^{1/q}$$

$$\leq \left(\int_{t-n}^{t-n+1} e^{-q\gamma(t-r)} H^q(t-r)dr \right)^{1/q} \| u \|_{S^p}$$

$$\leq \left(\int_{n-1}^{n} e^{-q\gamma s} H^q(r)(dr) \right)^{1/q} \| u \|_{S^p}.$$

Using the assumption $h_q^{\gamma, H} = \sum_{n=1}^{\infty} \left(\int_{n-1}^{n} e^{-q\gamma r} H^q(r)(dr) \right)^{1/q} < \infty,$ we then deduce from the well-known Weierstrass theorem that series $\sum_{n=0}^{\infty} \phi_n(t)$ is uniformly convergent on R. Furthermore,

$$\phi(t) = \sum_{\infty}^{n+1} \phi_n(t), \ \phi \in C(R,\, Y) \ \text{and}$$

$$\| \phi(t) \| = \left\| \sum_{n=1}^{\infty} \phi_n(t) \right\|$$

$$\leq \sum_{n=1}^{\infty} \| \phi_n(t) \|$$

$$\leq \sum_{n=1}^{\infty} \left(\int_{n-1}^{n} e^{-q\gamma s} H^q(s) ds \right)^{1/q} \| u \|_{S^p}$$

$$\leq h_q^{\gamma, H} \| u \|_{S^p}, \text{ for each } t \in R.$$

The next step consists of showing that $\phi_n(t) \in AA(X)$. Indeed, let $(s_m)_{m \in N}$ be a sequence of real numbers. Since $u \in AS^p(Y)$, there exist a subsequence $(s_{m_k})_{k \in N}$ of $(s_m)_{m \in N}$ and a function $v \in AS^p(Y)$ such that

$$\left[\int_t^{t+1} \| u(s_{m_k} + \sigma) - v(\sigma) \|_Y^p d\sigma \right]^{1/p} \to 0 \text{ as } k \to \infty.$$

Define

$$\psi_n(t) = \int_{n-1}^{n} AT(\xi) v(t - \xi) d\xi.$$

Here we use

$$\phi_n(t) = \int_{n-1}^{n} AT(s) u(t - s) ds \text{ and}$$

$$\phi_n(t + s_{m_k}) = \int_{n-1}^{n} AT(\xi) u(t + s_{m_k} - \xi) d\xi,$$

$$\psi_n(t) = \int_{n-1}^{n} AT(\xi) v(t - \xi) d\xi.$$

Here $t = t + s_{m_k}$ and $s = \xi$.

Then using the Holder's inequality, we get

$$\| \phi_n(t + s_{m_k}) - \psi_n(t) \|$$

$$= \left\| \int_{n-1}^{n} AT(\xi) u(t + s_{m_k} - \xi) d\xi - \int_{n-1}^{n} AT(\xi) v(t - \xi) d\xi \right\|$$

$$\leq \int_{n-1}^{n} \| AT(\xi) \| \| u(t + s_{m_k} - \xi) - v(t - \xi) \| d\xi$$

$$\leq \int_{n-1}^{n} e^{-\gamma\xi} H(\xi) \| u(t + s_{m_k} - \xi) - v(t - \xi) \|_Y d\xi$$

$$\leq \left(\int_{n-1}^{n} e^{-q\gamma\xi} H^q(\xi) d\xi \right)^{1/q} \left(\int_{n-1}^{n} \| u(t + s_{m_k} - \xi) - v(t - \xi) \|_Y^p d\xi \right)^{1/p}$$

$$\leq g_q^{\gamma, H} \left[\int_{n-1}^{n} \| u(t + s_{m_k} - \xi) - v(t - \xi) \|_Y^p d\xi \right]^{1/p},$$

where $g_q^{\gamma, H} = \sup_n \left[\int_{n-1}^{n} e^{-q\gamma s} H^q(s) ds \right]^{1/q} < \infty,$ as $h_q^{\gamma, H} < \infty$. Obviously, $\| \phi_n(t + s_{m_k}) - \psi_n(t) \| \to 0$ as $k \to \infty$.

Similarly, we can prove

$$\| \psi_n(t + s_{m_k}) - \phi_n(t) \|$$

$$= \left\| \int_{n-1}^{n} AT(\xi) v(t + s_{m_k} - \xi) d\xi - \int_{n-1}^{n} AT(\xi) u(t - \xi) d\xi \right\|$$

$$\leq \int_{n-1}^{n} \| AT(\xi) \| \| v(t + s_{m_k} - \xi) - u(t - \xi) \| d\xi$$

$$\leq \int_{n-1}^{n} e^{-\gamma\xi} H(\xi) \| v(t + s_{m_k} - \xi) - u(t - \xi) \|_Y d\xi$$

$$\leq \left(\int_{n-1}^{n} e^{-q\gamma\xi} H^q(\xi) d\xi \right)^{1/q} \left(\int_{n-1}^{n} \| v(t + s_{m_k} - \xi) - u(t - \xi) \|_Y^p d\xi \right)^{1/p}$$

$$\leq g_q^{\gamma, H} \left[\int_{n-1}^{n} \| v(t + s_{m_k} - \xi) - u(t - \xi) \|_Y^p d\xi \right]^{1/p}.$$

Obviously,

$$\| \psi_n(t + s_{m_k}) - \phi_n(t) \| \to 0 \text{ as } k \to \infty.$$

Therefore, each $\phi_n(t) \in AA(X)$ for each n and hence their uniform limit $\phi \in AA(X)$, by using [19, Theorem 2.1.10].

Lemma 3.6. *If* $u \in AS^p(X)$ *and if* Ξ *is the function defined by*

$$\Xi(t) = \int_{-\infty}^{t} T(t-s)u(s)ds$$

for each $t \in R$, *then* $\Xi \in AA(X)$.

Proof. Define the sequence of operators

$$\Xi_n(t) = \int_{n-1}^{n} T(s)u(t-s)ds \text{ for each } t \in R.$$

Let $r = t - s$, $dr = -ds$ when $s = n - 1$, $r = t - n + 1$ and $s = n$, $r = t - n$,

$$\Xi_n(t) = \int_{t-n+1}^{t-n} T(t-r)u(r)(-dr)$$

$$= \int_{t-n}^{t-n+1} T(t-r)u(r)dr \text{ for each } t \in R.$$

From the asymptotic stability of $T(t)$, it follows that the function $s \to T(t-r)u(r)$ is integrable over $(-\infty, t)$ for each $t \in R$. Furthermore, using the Holder's inequality, it follows that

$$\| \Xi_n(t) \| = \left\| \int_{t-n}^{t-n+1} T(t-r)u(r)dr \right\|$$

$$\leq M \int_{t-n}^{t-n+1} e^{-\delta(t-r)} \| u(r) \| dr$$

$$\leq M\left(\int_{t-n}^{t-n+1} e^{-q\delta(t-r)}dr\right)^{1/q} \| u \|_{S^p}$$

$$\leq M\left(\int_{n-1}^{n} e^{-q\delta s}ds\right)^{1/q} \| u \|_{S^p}$$

$$\leq M\left\{\left[\frac{e^{-q\delta s}}{-q\delta}\right]_{n-1}^{n}\right\}^{1/q} \| u \|_{S^p}$$

$$\leq M\left[\frac{e^{-q\delta n}}{-q\delta} + \frac{e^{-q\delta(n-1)}}{q\delta}\right]^{1/q} \| u \|_{S^p}$$

$$\leq Me^{-\delta n}\sqrt[q]{\frac{e^{q\delta}-1}{q\delta}}\| u \|_{S^p}.$$

Now since $M\sqrt[q]{\frac{e^{q\delta}-1}{q\delta}}\sum_{n=1}^{\infty} e^{-\delta n} < \infty$, we deduce from the well-known Weierstrass theorem that the series $\sum_{n=1}^{\infty} \Xi_n(t)$ is uniformly convergent on R. Furthermore,

$$\Xi(t) = \sum_{n=1}^{\infty} \Xi_n(t), \quad \Xi \in C(R, Y), \text{ and}$$

$$\| \Xi(t) \| \leq \sum_{n=1}^{\infty} \| \Xi_n(t) \|$$

$$\leq k_q^{\delta, M} \| u \|_{S^p},$$

where $k_q^{\delta, M} = M\sqrt[q]{\frac{e^{q\delta}-1}{q\delta}}\sum_{n=1}^{\infty} e^{-\delta n} > 0$ is a constant, which depends on the parameters q, δ and M only.

The next step consists of showing that $\Xi_n(t) \in AA(X)$. Indeed, let $(s_m)_{m\in N}$ be a sequence of real numbers. Since $u \in AS^p(X)$, there exists a subsequence $(s_{m_k})_{k\in N}$ of $(s_M)_{m\in N}$ and a function $v \in AS^p(X)$ such that

$$\left[\int_t^{t+1} \| u(s_{m_k} + \sigma) - v(\sigma) \|^p d\sigma\right]^{1/p} \to 0 \text{ as } k \to \infty.$$

Define

$$\Omega_n(t) = \int_{n-1}^{n} T(\xi)v(t-\xi)d\xi.$$

Then using the Holder's inequality, we get

$$\begin{aligned}\| \Xi_n(t + s_{m_k}) - \Omega_n(t) \| &= \left\| \int_{n-1}^{n} T(\xi)[u(t + s_{m_k} - \xi) - v(t-\xi)]d\xi \right\| \\ &\le M\int_{n-1}^{n} e^{-\delta\xi}\| u(t + s_{m_k} - \xi) - v(t-\xi) \| d\xi \\ &\le M_q^{\delta, M}\left[\int_{n-1}^{n} \| u(t + s_{m_k} - \xi) - v(t-\xi) \|^p d\xi\right]^{1/p},\end{aligned}$$

where $m_q^{\delta, M} = M\sqrt[q]{\frac{e^{q\delta} - 1}{q\delta}}$. Obviously,

$$\| \Xi_n(t + s_{m_k}) - \Omega_n(t) \| \to 0 \text{ as } k \to \infty.$$

Similarly, we can prove that

$$\| \Omega_n(t + s_{m_k}) - \Xi_n(t) \| \to 0 \text{ as } k \to \infty.$$

Therefore, each $\Xi_n \in AA(X)$ for each n and hence their uniform limit $\Xi(t) \in AA(X)$, by using [19, Theorem 2.1.10].

Definition 3.7. A function $u \in AA(X)$ is a mild solution to the neutral system (2) provided that the function $s \to AT(t-s)f\left(s, u_s, \int_0^t h(s, u(s))ds\right)$

is integrable on $(-\infty, t)$ for each $t \in R$ and

$$u(t) = -f\left(t, u_t, \int_0^s h(t, u(t))dt\right)$$

$$-\int_{-\infty}^t AT(t-s)f\left(s, u_s, \int_0^t h(s, u(s))ds\right)ds$$

$$+\int_{-\infty}^t T(t-s)g\left(s, u_s, \int_0^t h(s, u(s))ds\right)ds$$

for each $t \in R$.

Theorem 3.8. *Under previous assumptions and if* (H_1)-(H_3) *hold, then there exists a unique almost automorphic solution to* (2) *whenever*

$$C = \left[L_f(1 + L_h H) + L_f \sup_{t \in R} \int_{-\infty}^t e^{-\gamma(t-s)} H(t-s)(1 + L_h H)ds\right.$$

$$\left. + M \sup_{t \in R} \int_{-\infty}^t e^{-\delta(t-s)} L_g(s)(1 + L_h H)ds\right] Q < 1$$

whenever Q is the constant appearing in Remark 2.14.

Proof. In $AS^p(X)$, define the operator $\Gamma : AS^p(X) \to C(R, X)$ by setting

$$\Gamma u(t) = -f\left(t, u_t, \int_0^s h(t, u(t))dt\right)$$

$$-\int_{-\infty}^t AT(t-s)f\left(s, u_s, \int_0^t h(s, u(s))ds\right)ds$$

$$+\int_{-\infty}^t T(t-s)g\left(s, u_s, \int_0^t h(s, u(s))ds\right)ds, \text{ for each } t \in R$$

since Γu is well-defined and continuous. Moreover, from Lemmas 3.3, 3.4 and 3.6, we infer that Γ maps $AS^p(X)$ into $AA(X)$. In particular, Γ maps $AA(X) \subset AS^p(X)$ into $AA(X)$. Next we prove that Γ is a strict contraction on $AA(X)$. Indeed, if Q is the constant appearing in Remark 2.14 for $u, v \in AA(X)$, then we get

$$\| \Gamma u(t) - \Gamma v(t) \|$$

$$= \Bigg\| -f\left(t, u_t, \int_0^s h(t, u(t))dt \right) - \int_{-\infty}^t AT(t-s) f\left(s, u_s, \int_0^t h(s, u(s))ds \right) ds$$

$$+ \int_{-\infty}^t T(t-s) g\left(s, u_s, \int_0^t h(s, u(s))ds \right) ds + f\left(t, v_t, \int_0^s h(t, v(t))dt \right)$$

$$+ \int_{-\infty}^t AT(t-s) f\left(s, v_s, \int_0^t h(s, v(s))ds \right) ds$$

$$- \int_{-\infty}^t T(t-s) g\left(s, v_s, \int_0^t h(s, v(s))ds \right) ds \Bigg\|$$

$$\leq \left\| f\left(t, v_t, \int_0^s h(t, v(t))dt \right) - f\left(t, u_t, \int_0^s h(t, u(t))dt \right) \right\|$$

$$+ \int_{-\infty}^t \| AT(t-s) \| \Bigg\| f\left(s, v_s, \int_0^t h(s, v(s))ds \right) ds$$

$$- f\left(s, u_s, \int_0^t h(s, u(s))ds \right) \Bigg\| ds$$

$$+ \int_{-\infty}^t \| T(t-s) \| \left\| g\left(s, u_s, \int_0^t h(s, u(s))ds \right) ds - g\left(s, v_s, \int_0^t h(s, v(s))ds \right) \right\| ds$$

$$\leq L_f [\| v_t - u_t \| + L_h \| v(t) - u(t) \|]$$

$$+ \int_{-\infty}^t e^{-\gamma(t-s)} H(t-s) L_f [\| v_s - u_s \| + L_h \| v(s) - u(s) \|] ds$$

$$+ \int_{-\infty}^t M e^{-\delta(t-s)} L_g(s) [\| u_s - v_s \| + L_h \| u(s) - v(s) \|] ds$$

$$\leq L_f \| u_t - v_t \|_B + L_f L_h H \| u_t - v_t \|_B$$

$$+\int_{-\infty}^{t} e^{-\gamma(t-s)} H(t-s)[L_f \| u_s - v_s \|_B + L_f L_h H \| u_s - v_s \|_B] ds$$

$$+ M \int_{-\infty}^{t} e^{-\delta(t-s)} L_g(s) [\| u_s - v_s \|_B + HL_h \| u_s - v_s \|_B] ds$$

$$\leq L_f \left[1 + HL_h + \sup_{t \in R} \int_{-\infty}^{t} e^{-\gamma(t-s)} H(t-s)[1 + L_h H] ds \right] Q \| u - v \|_\infty$$

$$+ \left[M \sup_{t \in R} \int_{-\infty}^{t} e^{-\delta(t-s)} L_g(s)[1 + HL_h] ds \right] Q \| u - v \|_\infty$$

$$\leq C \| u - v \|_\infty.$$

The assertion is now a consequence of the classical Banach fixed-point principle.

4. Example

In this section, we provide with an example to illustrate our main result. We study the existence of almost automorphic solutions to a nonautonomous integrodifferential equation which was considered in Diagana et al. [9] in the pseudo almost periodic case. Consider

$$\frac{\partial}{\partial t} \left[\varphi(t, x) + \int_{-\infty}^{t} \int_{0}^{\pi} b(t-s, \eta, x) \varphi(s, \eta) d\eta ds \right]$$

$$= \frac{\partial^2}{\partial x^2} \varphi(t, x) + V\varphi(t, x) + \int_{-\infty}^{t} a_1(t-s) \varphi(s, x) ds + a_2(t, x), \tag{7}$$

$$\varphi(t, 0) = \varphi(t, \pi) = 0, \tag{8}$$

for $t \in R$ and $x \in I = [0, \pi]$.

It is worth mentioning that systems of the type (7)-(8) arise in control systems described by abstract retarded functional differential equations with feedback control governed by proportional integrodifferential law [12].

The existence and qualitative properties of the solutions to (7)-(8) was recently described in [12, 13] for the existence and regularity of mild solutions, [12] for the existence of periodic solutions, [14] for the existence of almost periodic and asymptotically almost periodic solutions, [9] for pseudo almost periodic solutions, and [5] for the existence of almost automorphic solutions. For similar works, we refer the reader to Hernandez [13] and Diagana and N'Guerekata [7, 8].

To establish the existence of almost automorphic solutions to equations (7)-(8), we need to introduce the required technical tools.

Let $X = L^2[0, \pi]$ and $B = C_0 \times L^2(\rho, X)$ (see Example 2.16). Define the linear operator by

$$D(A) = \{\varphi \in L^2[0, \pi] : \varphi'' \in L^2[0, \pi], \varphi(0) = \varphi(\pi) = 0\},$$

$$A\varphi = \varphi'' + V\varphi \text{ for all } \varphi \in D(A),$$

where V is a constant satisfying $V < 1$.

The operator A is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ on $L^2[0, \pi]$ satisfying

$$\| T(s) \| \leq e^{-(1-V)s} \text{ for every } s \geq 0.$$

For the rest of the paper, we assume that the following conditions hold:

(i) The functions $b(\cdot)$, $\dfrac{\partial^i}{\partial \zeta^i}$, $b(\tau, \eta, \zeta)$ for $i = 1, 2,$ are Lebesgue measurable, $b(\tau, \eta, \pi) = 0$, $b(\tau, \eta, 0) = 0$ for every (τ, η), and

$$L_f = \max\left\{\int_0^\pi \int_{-\infty}^0 \int_0^\pi \left(\frac{\partial^i}{\partial \zeta^i} b(\tau, \eta, \zeta)\right)^2 d\eta d\tau d\zeta;\ i = 0, 1, 2\right\} < \infty.$$

(ii) The functions a_1, a_2, b are continuous, S^p-almost automorphic and

$$L_g = \left(\int_{-\infty}^0 \frac{a_1^2(-\theta)}{\rho(\theta)} d\theta\right)^{1/2} < \infty.$$

Additionally, we define the operators $f, g : B \to L^2[0, \pi]$ by setting

$$f(t, \psi)(x) = \int_{-\infty}^{0} \int_{0}^{\pi} b(s, \eta, x)\psi(s, \eta)d\eta ds, \tag{9}$$

$$g(t, \psi)(x) = \int_{-\infty}^{0} a_1(s)\psi(s, x)ds + a_2(t, x), \tag{10}$$

which enable us to transform the system (7)-(8) into an equation of the form (2). Obviously, f, g are continuous. Moreover, using a straightforward estimation, which can be obtained with the help of both (i) and (ii), it is then easy to see that f has values in $Y = (D(A), \|\cdot\|_1)$, where the norm $\|\cdot\|_1$ is defined by: $\|\varphi\|_1 = \|A\varphi\|$ for each $\varphi \in D(A)$. Furthermore, f is a Y-valued bounded linear operator with $\|f\|_{L(B, Y)} \le L_f$. Note also that g is Lipschitz with respect to the second variable ψ whose Lipschitz constant is L_g.

The next result is a direct consequence of Theorem 3.8.

Theorem 4.1. *Under the previous assumptions*, *the system* (7)-(8) *has a unique almost automorphic solution whenever*

$$Q\left[L_f\left(1 + \frac{1}{1 - V}\right) + L_g\right] < 1.$$

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