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EXPLICIT FORMULAS OF AVERAGE RUN LENGTH FOR ARMA(1, 1) PROCESS OF CUSUM CONTROL CHART

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Abstract

The cumulative sum (CUSUM) control chart is widely used for monitoring industrial process. In real applications, there are many situations in which the process data are serially correlation such as in chemical process. It is important to be able to evaluate the average run length (ARL) of CUSUM control chart when observations are correlations. In this article, we use a Fredholm integral equation technique to derive the explicit formulas for the ARL of CUSUM control chart for autoregressive and moving average: ARMA(1, 1) process. We prove that the ARL is the unique solution to the integral equation under some weak regularity conditions. The precision of new explicit formulas was verified by using numerical integral equation techniques for several parameter values. The results show that the explicit formulas for ARL have high accuracy and take time much less than the numerical integral equation techniques.

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1. Introduction

The cumulative sum (CUSUM) control chart is widely used in statistical control and it was proposed by Page [3]. Its properties have been throughly studies in the literature (see, e.g., Hawkins and Olwell [2]). A numerical comparison of the EWMA and CUSUM control charts was given by Lucas and Saccucci [7] and Yashchin [4]. Srivastava and Wu [9] and Wu [15] considered design of the optimal EWMA control chart and compared it with the CUSUM and Shiryaev-Roberts control charts. The statistical control charts such as Shewhart [13], CUSUM and EWMA control charts have widespread applications in improving the quality for manufacturing process. Control charts are usually designed under the assumption that the observations from a process are independent and identically distributed (i.i.d.). There are many situations in which the processes data are autocorrelated such as in chemical process, so its need to be monitored by appropriate control charts. Consequently, the aim of the paper is to derive the explicit formulas of average run length (ARL) of CUSUM control chart for autoregressive and moving average: ARMA(1, 1) process with exponential white noise.

The ARL is a traditional measurement of control chart's performance, the expected number of observations taken from an in-control process until the control chart falsely signals out-of-control is denoted by ARL_0. An ARL_0 will be regarded as acceptable if it is large enough to keep the level of false alarms at an acceptable level. A second common characteristic is the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control is denoted by ARL_1. There are several methods that can be utilized to find the ARL such as Markov chain approach (MCA), Integral equation approach (IE) and Monte Carlo simulations (MC). Several researchers to evaluate the ARL when the process observations are serially correlation, Mastrangelo and Montgomery [1] have been evaluated the performance of EWMA control charts for serially-correlated process based on Monte Carlo simulation technique. VanBrackle and Reynolds [8] were estimated the ARL by using an integral equation and Markov chain approach to evaluate EWMA and CUSUM control charts in case of AR(1) process with additional random error. The limitations of the MCA, IE and MC methods provide the motivation for finding explicit analytical formulas for evaluating the ARL. Sukparungsee and Novikov [10] have used the Martingale approach to derive approximate analytical formulas for ARL and AD in the case of Gaussian distribution and some non-Gaussian distribution. Later, Areepong [14] derived the explicit formulas of average run length (ARL) and average delay (AD) for EWMA control chart for the case of exponential distribution. Recently, Mititelu et al. [5] presented the explicit formulas for ARL by Fredholm integral equation for one-sided EWMA control chart with Laplace distribution and CUSUM control chart with hyperexponential distribution. Later, Busaba et al. [6] were analyzed the explicit formulas of ARL for CUSUM control chart for AR(1) process with exponential white noise.

The objective of this paper is to study analytical and numerical methods for the derivation of formulas of ARL for CUSUM control chart for ARMA(1, 1) process with exponential white noise by using integral equation technique. The paper is organized as follows: in Section 1, we present an introduction of statistical process control charts. In Section 2, describe the characteristics of CUSUM control chart for ARMA(1, 1) process. In Section 3, we prove that the ARL is the unique solution to the integral equation. In Section 4, we derive the explicit formulas of ARL for CUSUM control charts for ARMA(1, 1) process with exponential white noise. The numerical method for solving integral equations to obtain approximation for ARL for ARMA(1, 1) process is presented in Section 5. In Section 6, we compare the results. Finally, we show our conclusions.

2. CUSUM Control Chart for ARMA(1, 1) Process

In this section we derive the explicit formulas of ARL of CUSUM control chart for ARMA(1, 1) process. The CUSUM control chart is designed to detect mean shift of independent and identical distribution (i.i.d) observed sequence of random variables $\zeta_1, \zeta_2, \ldots.$ The general, CUSUM statistics can be written as a sequence with

$$Z_t = \max(Z_{t-1} + \zeta_t - a, 0), \quad t = 1, 2, \ldots. \tag{1}$$

In this paper, we define that $\zeta_t, \ t = 1, 2, \ldots$ are ARMA(1, 1) processes with the exponential white noise. The ARMA(1, 1) process described by the following recursion:

$$X_t = \mu + \phi X_{t-1} + \zeta_t - \theta\zeta_{t-1}; \quad t = 1, 2, \ldots, \tag{2}$$

where ζ_t are independent and identically distributed observed sequences of exponential distribution. The initial value $\zeta_0 = 1,$ an autoregressive coefficient $0 \le \phi \le 1$ and a moving average coefficient $0 \le \theta \le 1.$ We assume the initial value of ARMA(1, 1) process $X_0 = 1.$

The recursive CUSUM based on ARMA(1, 1) process is defined as the following form:

$$Z_t = \max(Z_{t-1} + X_t - a, 0), \quad t = 1, 2, \ldots, \tag{3}$$

where Z_t is CUSUM statistics, X_t is a sequence of an ARMA(1, 1) process, $Z_0 = u$ is an initial value, *a* is a constant recall reference value of CUSUM control chart.

The first passage time of CUSUM control chart is given by:

$$\tau_b = \inf\{t > 0 : Z_t > b\},$$

where *b* is a constant parameter known as the *upper control limit*.

In this paper, notations $\mathbb{P}_z$ denote the probability measure and $\mathbb{E}_z$ denote the expectation corresponding to the initial value $Z_0 = u.$ Let $L(u) = \mathbb{E}(\tau_b) < \infty$ be the ARL of CUSUM control chart after it is reset at $u \in [0, b].$ The solution of integral equation as follows:

$$L(u) = 1 + \mathbb{E}_z[I\{0 < Z_1 < b\}L(Z_1)] + \mathbb{P}_z\{Z_1 = 0\}L(0). \tag{4}$$

The ARL of CUSUM control chart is:

$$L(u) = 1 + \lambda e^{\lambda(u-a+\mu+\varphi X_0-\theta\zeta_0)}\int_0^b L(y)e^{-\lambda y}dy$$

$$+ (1 - e^{-\lambda(a-u-\mu-\varphi X_0+\theta\zeta_0)})L(0); \quad u \in [0, a].$$

3. Unique Solution to the ARL Integral Equation

In this section, we establish that the ARL for CUSUM control chart is the unique solution to the integral equation under some regularity conditions. On the metric space of all continuous functions $X = (C(I), \| \|)$, where I is a compact interval, and the norm defined as $\| L \| = \sup_{u \in I} | L(u) |$, the operator T is named on contraction, if there exists a number $0 \le q < 1$ such that

$$\| T(L_1) - T(L_2) \| \le q \| L_1 - L_2 \| \text{ for all } L_1, L_2 \in X.$$

Let $C(I)$ be the class of all continuous functions defined on a compact interval $I = [0, b]$ and define the operator T by

$$T(L(u)) = 1 + \lambda e^{\lambda(u-a+\mu+\varphi X_0-\theta\zeta_0)}\int_0^b L(y)e^{-\lambda y}dy$$

$$+ (1 - e^{-\lambda(a-u-\mu-\varphi X_0+\theta\zeta_0)})L(0). \tag{5}$$

Thus, the integral equation in equation (4) can be written as $T(L(u)) = L(u)$. According to Banach’s fixed point theorem, if the operator T is a contraction, then the fixed point equations $T(L(u)) = L(u)$ have unique solutions.

Proposition 3.1. *The operator T is a contraction on the metric space $X = (C(I), \| \|)$ with the norm $\| L \| = \sup_{u \in I} | L(u) |$.*

Proof. To prove that the operator T is a contraction. For any $u \in I$ and $L_1, L_2 \in C(I)$, we have the inequality $\| T(L_1) - T(L_2) \| \le q \| L_1 - L_2 \|$, where $q < 1$ is a positive constant. According to equation (5), we have that:

$$\| T(L_1) - T(L_2) \|_1 = \sup_{u \in I} \Big| L_1(0) - L_2(0)(1 - e^{-\lambda(a-u-\mu-\varphi X_0+\theta\zeta_0)})$$

$$+ \lambda e^{\lambda(u-a+\mu+\varphi X_0-\theta\zeta_0)} \int_0^b (L_1(y) - L_2(y)) e^{-\lambda y} dy \Big|$$

$$\leq \sup_{u \in I} \Big| \| L_1(0) - L_2(0) \|_1 (1 - e^{-\lambda(a-u-\mu-\varphi X_0-\theta\zeta_0)})$$

$$+ \| L_1 - L_2 \|_1 \lambda e^{\lambda(u-a+\mu+\varphi X_0-\theta\zeta_0)} \int_0^b e^{-\lambda y} dy \Big|$$

$$= \| L_1 - L_2 \|_1 \sup_{u \in I} [1 - e^{-\lambda(a-u-\mu-\varphi X_0+\theta\zeta_0)-\lambda b}]$$

$$= \| L_1 - L_2 \|_1 [1 - e^{-\lambda(-\mu-\varphi X_0+\theta\zeta_0)-\lambda b}]$$

$$= q_1 \| L_1 - L_2 \|_1, \text{ where } q_1 = [1 - e^{-\lambda(-\mu-\varphi X_0+\theta\zeta_0+b)}] < 1.$$

Triangular inequality has been used and the fact is

$$| L_1(0) - L_2(0) | \leq \sup_{x \in [0, a]} | L_1(u) - L_2(u) | = \| L_1 - L_2 \|.$$

So we have $\| T(L_1) - T(L_2) \| \leq q \| L_1 - L_2 \|$, thus the operator T is a contraction. According to Banach's fixed point theorem, if the operator T is a contraction, then the fixed point equation $T(L(u)) = L(u)$ has a unique solution.

4. Explicit Formulas of ARL for ARMA(1, 1) Process of CUSUM Chart

We derive explicit solution of Fredholm integral equation of the second kind, which is called *explicit formulas* of ARL for ARMA(1, 1) process based on CUSUM control chart.

Theorem 4.1. *The explicit formulas of ARL for ARMA*(1, 1) *process is*

$$L(u) = e^{\lambda b}(1 + e^{\lambda(a-\mu-\varphi X_0+\theta\zeta_0)} - \lambda b) - e^{\lambda u}; \quad u \geq 0. \tag{6}$$

Proof.

$$L(u) = 1 + \lambda e^{\lambda(u-a+\mu+\varphi X_0-\theta\zeta_0)}\int_0^b L(y)e^{-\lambda y}dy$$

$$+ (1 - e^{-\lambda(a-u-\mu-\varphi X_0+\theta\zeta_0)})L(0); \quad u \in [0, a].$$

Let d be a constant as $d = \int_0^b L(y)e^{-\lambda y}dy.$ Thus, the function $L(u)$ can be written as

$$L(u) = 1 + \lambda e^{\lambda(u-a+\mu+\varphi X_0-\theta\zeta_0)}d$$

$$+ (1 - e^{-\lambda(a-u-\mu-\varphi X_0+\theta\zeta_0)})L(0); \quad u \in [0, a]. \tag{7}$$

For the case $u = 0,$ thus we have the function $L(0)$ as the following form:

$$L(0) = 1 + \lambda e^{\lambda(-a+\mu+\varphi X_0-\theta\zeta_0)}d + (1 - e^{-\lambda(a-\mu-\varphi X_0+\theta\zeta_0)})L(0)$$

$$= e^{\lambda(a-\mu-\varphi X_0+\theta\zeta_0)} + \lambda d.$$

Hence, substituting $L(0)$ into equation (7) as the following form:

$$L(u) = 1 + \lambda e^{\lambda(u-a+\mu+\varphi X_0-\theta\zeta_0)}d$$

$$+ (1 - e^{-\lambda(a-u-\mu-\varphi X_0+\theta\zeta_0)})e^{\lambda(a-\mu-\varphi X_0+\theta\zeta_0)} + \lambda d$$

$$= 1 + e^{\lambda(a-\mu-\varphi X_0+\theta\zeta_0)} + \lambda d - e^{\lambda u}. \tag{8}$$

The constant d can be solved as:

$$d = \int_0^b L(y)e^{-\lambda y}dy$$

$$= \int_0^b (1 + \lambda d + e^{\lambda(a-\mu-\varphi X_0+\theta\zeta_0)} - e^{\lambda y}) \cdot e^{-\lambda y}dy$$

$$= \int_0^b (1 + \lambda d + e^{\lambda(a-\mu-\varphi X_0+\theta\zeta_0)} - e^{\lambda y})dy - \int_0^b e^{\lambda y-\lambda y}dy$$

$$= \frac{e^{\lambda b}}{\lambda}(1 - e^{-\lambda b})(1 + e^{\lambda(a-\mu-\varphi X_0+\theta\zeta_0)}) - be^{\lambda b}.$$

Consequently, the explicit formulas obtained by substituting the d constant into equation (8) as the following form:

$$L(u) = e^{\lambda b}(1 + e^{\lambda(a-\mu-\varphi X_0+\theta\zeta_0)} - \lambda b) - e^{\lambda u}; \quad u \geq 0. \tag{9}$$

As mentioned above, the value of the parameter λ is equal to λ_0 when the process is in-control. Therefore, we obtain the explicit formula for ARL_0 as follows:

$$ARL_0 = e^{\lambda_0 b}(1 + e^{\lambda_0(a-\mu-\varphi X_0+\theta\zeta_0)} - \lambda_0 b) - e^{\lambda_0 u}; \quad u \geq 0. \tag{10}$$

On the other hand, the process is out-of-control, the value of the parameter λ is equal to λ_1; where $\lambda_1 = \lambda_0(1+\delta)$. The explicit formula for ARL_1 can be written as follows:

$$ARL_1 = e^{\lambda_1 b}(1 + e^{\lambda_1(a-\mu-\varphi X_0+\theta\zeta_0)} - \lambda_1 b) - e^{\lambda_1 u}; \quad u \geq 0, \tag{11}$$

where $0 \leq \varphi \leq 1$ is an autoregressive coefficient and $0 \leq \theta \leq 1$ is the moving average coefficient and λ is a parameter of the exponential distribution, and b is an upper control limit and X_0, ζ_0 are the initial values.

5. Numerical Results

In this section, we compare the results of ARL_0 and ARL_1 for ARMA(1, 1) process, which obtained from the explicit formulas with numerical solution of integral equation method for the number of division points $m = 500$. A numerical scheme to evaluate the solution of the integral equations is given by: (Phanyaem et al. [11])

$$\tilde{L}(u) = 1 + \tilde{L}(a_1)F(a - u - \mu - \varphi X_0 + \theta\zeta_0)$$

$$+ \sum_{j=1}^{m} w_j \tilde{L}(a_j) f(a_j + a - u - \mu - \varphi X_0 + \theta\zeta_0), \tag{12}$$

where

$$a_j = \frac{b}{m}\left(j - \frac{1}{2}\right) \text{ and } w_j = \frac{b}{m}; \; j = 1, 2, ..., m.$$

The results of ARL are presented in Tables 1-4. The parameter values for CUSUM control chart were chosen by given desired $ARL_0 = 370$ and 500, in-control parameter $\lambda_0 = 1$ and shift sizes $\delta = 0.1, 0.2, 0.3, 0.4$ and 0.5, respectively. We consider the performance of the explicit formulas by the computational times and the absolute percentage difference. The absolute percentage difference can be computed as follows:

$$Diff\,(\%) = \frac{|\, ARL_{Explicit\ Formulas} - ARL_{Numerical\ IE}\,|}{ARL_{Explicit\ Formulas}} \times 100.$$

Table 1. Comparison of ARL_0 and ARL_1 using explicit formulas against numerical integral equation (IE) approximation for initial value $u = 1$, $a = 2.5$, $b = 3.67$, $\varphi = 0.10$ and $\theta = 0.10$ for $ARL_0 = 370$

Shift size δ	Explicit formulas	Numerical IE	Diff. (%)
0.0	370.000	370.000 (**33.63**)[a]	
0.1	204.723	204.125 (**33.58**)	0.00292
0.2	124.873	124.553 (**33.52**)	0.00256
0.3	82.303	82.116 (**33.43**)	0.00227
0.4	57.689	57.573 (**33.40**)	0.00201
0.5	42.494	42.418 (**33.67**)	0.00179

[a]The values in parentheses is time used in IE methods (minutes)

Table 2. Comparison of ARL_0 and ARL_1 using explicit formulas against numerical integral equation (IE) approximation for initial value $u = 1$, $a = 2.5$, $b = 3.53$, $\varphi = 0.20$ and $\theta = 0.30$ for $ARL_0 = 370$

Shift size δ	Explicit formulas	Numerical IE	Diff. (%)
0.0	370.000	370.000 (**33.2**)[a]	
0.1	205.979	205.390 (**32.5**)	0.00286
0.2	126.304	125.985 (**35.6**)	0.00253
0.3	83.574	83.386 (**36.3**)	0.00225
0.4	58.745	58.628 (**34.3**)	0.00199
0.5	43.356	43.278 (**34.1**)	0.00180

[a]The values in parentheses is time used in IE methods (minutes)

Table 3. Comparison of ARL_0 and ARL_1 using explicit formulas against numerical integral equation (IE) approximation for initial value $u = 1$, $a = 2.5$, $b = 4.005$, $\varphi = 0.10$ and $\theta = 0.10$ for $ARL_0 = 500$

Shift size δ	Explicit formulas	Numerical IE	Diff. (%)
0.0	500.000	500.000 (**31.86**)[a]	
0.1	266.879	266.025 (**31.28**)	0.00320
0.2	157.969	157.528 (**31.03**)	0.00279
0.3	101.518	101.269 (**31.68**)	0.00245
0.4	69.655	69.505 (**32.43**)	0.00215
0.5	50.390	50.293 (**31.65**)	0.00192

[a]The values in parentheses is time used in IE methods (minutes)

Table 4. Comparison of ARL_0 and ARL_1 using explicit formulas against numerical integral equation (IE) approximation for initial value $u = 1$, $a = 2.5$, $b = 3.86$, $\varphi = 0.20$ and $\theta = 0.30$ for $ARL_0 = 500$

Shift size δ	Explicit formulas	Numerical IE	Diff. (%)
0.0	500.000	500.000 (**31.02**)[a]	
0.1	268.886	268.043 (**31.29**)	0.00314
0.2	160.152	159.711 (**31.29**)	0.00275
0.3	103.403	103.152 (**31.72**)	0.00243
0.4	71.1896	71.036 (**31.62**)	0.00216
0.5	51.6195	51.520 (**31.15**)	0.00193

[a]The values in parentheses is time used in IE methods (minutes)

The results from Tables 1-4 show the absolute percentage difference around 0.2% by numerical integral equation (IE) for the case $m = 500$ division points, and computational times of approximately 31-34 minutes. The computational times from the proposed explicit formulas are less than 1 second.

6. Performance Comparison of CUSUM and EWMA Control Charts

In this paper, we compare the efficiency of control charts between the CUSUM and EWMA control charts. We first consider the case in which the observations are autoregressive and moving average, ARMA(1, 1) process with exponential distribution white noise. We use the explicit formulas obtained previously to evaluate ARL_0 and ARL_1 for CUSUM control chart, which obtained the solution of integral equation in equations (10) and (11), respectively, and we compare the performance of control charts with the explicit formulas for EWMA control chart which have been found by Phanyaem et al. [12].

Comparing our results from the CUSUM and EWMA control charts shows that for the case of a one-sided shift, it has been shown that the EWMA control chart is the best control chart in the sense that it has minimize the supremum of the conditional average run length (ARL_1) when the process has a small shift $(\delta < 0.50)$.

Table 5. Comparison ARL for ARMA(1, 1) between CUSUM and EWMA control charts, given $ARL_0 = 370$, $u = 0$, $\varphi = 0.10$ and $\theta = 0.10$

Shift size δ	CUSUM chart $a = 2.5$, $b = 3.665$	EWMA chart $\lambda = 0.20$, $b = 0.222689$
0.00	370.000	370.000
0.01	347.618	63.707
0.02	326.370	35.367
0.03	306.797	24.718
0.04	288.741	19.134
0.05	272.064	15.696
0.10	205.393	8.609
0.20	125.734	4.948
0.30	83.201	3.703
0.40	58.567	3.072
0.50	43.332	2.689

Table 6. Comparison ARL for ARMA(1, 1) between CUSUM and EWMA control charts, given $ARL_0 = 370$, $u = 0$, $\varphi = 0.20$ and $\theta = 0.30$

Shift size δ	CUSUM chart $a = 2.5$, $b = 3.525$	EWMA chart $\lambda = 0.20$, $b = 0.249293$
0.00	370.000	370.000
0.01	347.597	68.740
0.02	326.594	38.403
0.03	307.230	26.887
0.04	289.353	20.824
0.05	272.828	17.081
0.10	206.628	9.345
0.20	127.149	5.337
0.30	84.460	3.972
0.40	59.615	3.281
0.50	44.187	2.861

7. Conclusion

In this paper, we have used the integral equations methods to obtain the closed form analytical expressions for the ARL of the CUSUM control chart when observations are autoregressive and moving average, ARMA(1, 1) processes with exponential distribution white noise. We compare our analytical results with the numerical integral equation. The methods are consistent with a high level of accuracy up to 99%. In addition, the explicit formulas take computational time much less than the numerical integral equation.

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