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NONTRIVIAL SOLUTION FOR A NONLINEAR SECOND-ORDER THREE-POINT BOUNDARY VALUE PROBLEM

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Abstract

In this paper, we study the existence of nontrivial solution for the second-order three-point boundary value problem

$$u'' + f(t, u) = 0, \;\; 0 < t < 1, \;\; u(0) = 0, \;\; u(1) = \alpha u'(\eta),$$

where $\eta \in (0, 1), \alpha \in R, \alpha \neq 1; f \in C([0, 1] \times R, R)$. Under certain growth conditions on the nonlinearity f and by using Leray-Schauder nonlinear alternative, several sufficient conditions for the existence of nontrivial solution are obtained.

1. Introduction

We are interested in the existence of nontrivial solution for the

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following three-point boundary value problem (BVP):

$$\begin{cases} u'' + f(t, u) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \alpha u'(\eta), \end{cases} \tag{1.1}$$

where $\eta \in (0, 1)$, $\alpha \in R$, $\alpha \neq 1$; $f \in C([0, 1] \times R, R)$, $R = (-\infty, +\infty)$.

The study of three-point BVP for certain nonlinear ordinary differential equations was initiated by Gupta [4]. Since then, by applying the Leray-Schauder continuation theorem, nonlinear alternative of Leray-Schauder, coincidence degree theory, or Krasnosel'skii fixed point theorem, many authors studied more general nonlinear three-point or multi-point boundary value problems, for example, (see [2-6, 8-12]), and references therein. But in the existing literature on the BVP (1.1) is few. In a recent paper [7], Infante and Webb investigated the BVP (1.1) by the fixed point index theory. The aim of this paper is to establish some simple criterions of the existence of nontrivial solution for the BVP (1.1). Note that we do not require any monotonicity and nonnegative on f.

2. Some Lemmas

A solution $u(t)$ of BVP (1.1) is called *nontrivial solution* if $u(t) \not\equiv 0$. Let $E = C[0, 1]$, with sup norm $\| y \| = \sup_{t \in [0,1]} | y(t) |$ for any $y \in E$. In arriving at our results, we need to state two preliminary results.

Lemma 2.1. *Let* $\alpha \neq 1$. *Then for* $y \in C[0, 1]$, *the following three-point BVP*:

$$u'' + y(t) = 0, \quad 0 < t < 1,$$

$$u(0) = 0, \quad u(1) = \alpha u'(\eta).$$

has the unique solution

$$u(t) = \frac{t}{1 - \alpha} \int_0^1 (1 - s) y(s) ds - \int_0^t (t - s) y(s) ds - \frac{\alpha t}{1 - \alpha} \int_0^\eta y(s) ds.$$

Proof. The proof of this lemma is easy, and we omit it.

Define an integral operator $T : E \to E$ by

$$Tu(t) = \frac{t}{1-\alpha}\int_0^1 (1-s)f(s, u(s))ds - \int_0^t (t-s)f(s, u(s))ds$$

$$- \frac{\alpha t}{1-\alpha}\int_0^\eta f(s, u(s))ds, \quad t \in [0, 1]. \tag{2.1}$$

By Lemma 2.1, the BVP (1.1) has a solution $u = u(t)$ if and only if u is a fixed point of the operator T defined by (2.1) in E. So we only need to seek a fixed point of T in E. By Ascoli-Arzela Theorem, we can prove that T is a completely continuous operator.

Lemma 2.2 [1]. *Let E be Banach space and Ω be a bounded open subset of E, $0 \in \Omega$, $T : \overline{\Omega} \to E$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

3. Main Results

In this section, we present and prove our main results.

Theorem 3.1. *Suppose $f(t, 0) \not\equiv 0$, $\alpha \neq 1$ and there exist nonnegative functions $k, h \in L^1[0, 1]$ such that*

$$|f(t, x)| \leq k(t)|x| + h(t), \text{ a.e. } (t, x) \in [0, 1] \times R,$$

$$\left(1 + \left|\frac{1}{1-\alpha}\right|\right)\int_0^1 (1-s)k(s)ds + \left|\frac{\alpha}{1-\alpha}\right|\int_0^\eta k(s)ds < 1.$$

Then the BVP (1.1) *has at least one nontrivial solution $u^* \in C[0, 1]$.*

Proof. Let

$$A = \left(1 + \left|\frac{1}{1-\alpha}\right|\right)\int_0^1 (1-s)k(s)ds + \left|\frac{\alpha}{1-\alpha}\right|\int_0^\eta k(s)ds,$$

$$B = \left(1 + \left|\frac{1}{1-\alpha}\right|\right)\int_0^1 (1-s)h(s)ds + \left|\frac{\alpha}{1-\alpha}\right|\int_0^\eta h(s)ds.$$

Then $A < 1$. Since $f(t, 0) \not\equiv 0$, there exists $[\sigma, \tau] \subset [0, 1]$ such that $\min_{\sigma \le t \le \tau} |f(t, 0)| > 0$. On the other hand, from $h(t) \ge |f(t, 0)|$, a.e. $t \in [0, 1]$, we know that $B > 0$. Let $m = B(1 - A)^{-1}$, $\Omega = \{u \in C[0, 1] : \|u\| < m\}$.

Suppose $u \in \partial\Omega$, $\lambda > 1$ such that $Tu = \lambda u$, then

$$\begin{aligned}
\lambda m &= \lambda\|u\| = \|Tu\| = \max_{0 \le t \le 1} |(Tu)(t)| \\
&\le \max_{0 \le t \le 1} \left|\frac{t}{1 - \alpha}\right| \int_0^1 (1 - s)|f(s, u(s))| ds + \max_{0 \le t \le 1} \int_0^t (t - s)|f(s, u(s))| ds \\
&\quad + \max_{0 \le t \le 1} \left|\frac{\alpha t}{1 - \alpha}\right| \int_0^\eta |f(s, u(s))| ds \\
&\le \left(1 + \left|\frac{1}{1 - \alpha}\right|\right) \int_0^1 (1 - s)|f(s, u(s))| ds + \left|\frac{\alpha}{1 - \alpha}\right| \int_0^\eta |f(s, u(s))| ds \\
&\le \left[\left(1 + \left|\frac{1}{1 - \alpha}\right|\right) \int_0^1 (1 - s)k(s)|u(s)| ds + \left|\frac{\alpha}{1 - \alpha}\right| \int_0^\eta k(s)|u(s)| ds\right] \\
&\quad + \left[\left(1 + \left|\frac{1}{1 - \alpha}\right|\right) \int_0^1 (1 - s)h(s) ds + \left|\frac{\alpha}{1 - \alpha}\right| \int_0^\eta h(s) ds\right] \\
&\le A\|u\| + B = Am + B.
\end{aligned}$$

Therefore

$$\lambda \le A + \frac{B}{m} = A + \frac{B}{B(1 - A)^{-1}} = A + (1 - A) = 1,$$

this contradicts $\lambda > 1$. By Lemma 2.2, T has a fixed point $u^* \in \overline{\Omega}$. In view of $f(t, 0) \not\equiv 0$, the BVP (1.1) has a nontrivial solution $u^* \in C[0, 1]$. This completes the proof.

Theorem 3.2. *Suppose* $f(t, 0) \not\equiv 0$, $\alpha < 1$, *and there exist nonnegative functions* $k, h \in L^1[0, 1]$ *such that*

$$|f(t, x)| \le k(t)|x| + h(t), \ a.e. \ (t, x) \in [0, 1] \times R.$$

If one of the following conditions is fulfilled:

(1) *There exists constant* $p > 1$ *such that*

$$\int_0^1 k^p(s)ds < \left[\frac{(1-\alpha)(1+q)^{1/q}}{2-\alpha+|\alpha|[\eta(1+q)]^{1/q}}\right]^p, \quad \frac{1}{p}+\frac{1}{q}=1,$$

(2) *There exists constant* $\mu > -1$ *such that*

$$k(s) \le \frac{(1+\mu)(2+\mu)(1-\alpha)}{2-\alpha+|\alpha|(2+\mu)\eta^{1+\mu}} s^{\mu}, \ a.e.\ s \in [0, 1],$$

$$mes\left\{s \in [0, 1] : k(s) < \frac{(1+\mu)(2+\mu)(1-\alpha)}{2-\alpha+|\alpha|(2+\mu)\eta^{1+\mu}} s^{\mu}\right\} > 0.$$

(3) *There exists constant* $\mu > -1$ *such that*

$$k(s) \le \frac{(1+\mu)(2+\mu)(1-\alpha)}{(2-\alpha)(1+\mu)+|\alpha|(2+\mu)}(1-s)^{\mu}, \ a.e.\ s \in [0, 1].$$

(4) $k(s)$ *satisfies*

$$k(s) \le \frac{2(1-\alpha)}{2-\alpha+2|\alpha|\eta}, \ a.e.\ s \in [0, 1],$$

$$mes\left\{s \in [0, 1] : k(s) < \frac{2(1-\alpha)}{2-\alpha+2|\alpha|\eta}\right\} > 0.$$

(5) $f(t, x)$ *satisfies*

$$\Lambda =: \limsup_{|x|\to\infty} \max_{t\in[0,1]} \left|\frac{f(t, x)}{x}\right| < \frac{2(1-\alpha)}{2-\alpha+2|\alpha|\eta}.$$

Then the BVP (1.1) *has at least one nontrivial solution* $u^* \in C[0, 1]$.

Proof. Let A be given in Theorem 3.1. In view of Theorem 3.1, we only need to prove $A < 1$. Since $\alpha < 1$, we have

$$A = \frac{2-\alpha}{1-\alpha}\int_0^1 (1-s)k(s)ds + \frac{|\alpha|}{1-\alpha}\int_0^{\eta} k(s)ds.$$

(1) By using the Hölder inequality, we have that

$$A \le \left[\int_0^1 k^p(s)ds\right]^{1/p}\left\{\frac{2-\alpha}{1-\alpha}\left[\int_0^1 (1-s)^q ds\right]^{1/q} + \frac{|\alpha|}{1-\alpha}\left[\int_0^{\eta} 1^q ds\right]^{1/q}\right\}$$

$$\leq \left[\int_0^1 k^p(s)ds\right]^{1/p}\left[\frac{2-\alpha}{1-\alpha}\left(\frac{1}{1+q}\right)^{1/q} + \frac{|\alpha|\eta^{1/q}}{1-\alpha}\right]$$

$$< \frac{(1-\alpha)(1+q)^{1/q}}{2-\alpha+|\alpha|[\eta(1+q)]^{1/q}} \cdot \frac{2-\alpha+|\alpha|[\eta(1+q)]^{1/q}}{(1-\alpha)(1+q)^{1/q}} = 1.$$

(2) In this case, we have that

$$A < \frac{(1+\mu)(2+\mu)(1-\alpha)}{2-\alpha+|\alpha|(2+\mu)\eta^{1+\mu}}\left[\frac{2-\alpha}{1-\alpha}\int_0^1 (1-s)s^{\mu}ds + \frac{|\alpha|}{1-\alpha}\int_0^{\eta} s^{\mu}ds\right]$$

$$\leq \frac{(1+\mu)(2+\mu)(1-\alpha)}{2-\alpha+|\alpha|(2+\mu)\eta^{1+\mu}}\left[\frac{2-\alpha}{1-\alpha}\cdot\frac{1}{(1+\mu)(2+\mu)} + \frac{|\alpha|}{1-\alpha}\cdot\frac{\eta^{1+\mu}}{(1+\mu)}\right]$$

$$= \frac{(1+\mu)(2+\mu)(1-\alpha)}{2-\alpha+|\alpha|(2+\mu)\eta^{1+\mu}}\cdot\frac{2-\alpha+|\alpha|(2+\mu)\eta^{1+\mu}}{(1-\alpha)(1+\mu)(2+\mu)} = 1.$$

(3) In this case, we have that

$$A \leq \frac{(1+\mu)(2+\mu)(1-\alpha)}{(2-\alpha)(1+\mu)+|\alpha|(2+\mu)}\left[\frac{2-\alpha}{1-\alpha}\int_0^1 (1-s)^{1+\mu}ds + \frac{|\alpha|}{1-\alpha}\int_0^{\eta}(1-s)^{\mu}ds\right]$$

$$= \frac{(1+\mu)(2+\mu)(1-\alpha)}{(2-\alpha)(1+\mu)+|\alpha|(2+\mu)}\left[\frac{2-\alpha}{1-\alpha}\cdot\frac{1}{2+\mu} + \frac{|\alpha|}{1-\alpha}\cdot\frac{1-(1-\eta)^{(1+\mu)}}{1+\mu}\right]$$

$$< \frac{(1+\mu)(2+\mu)(1-\alpha)}{(2-\alpha)(1+\mu)+|\alpha|(2+\mu)}\left[\frac{2-\alpha}{1-\alpha}\cdot\frac{1}{2+\mu} + \frac{|\alpha|}{1-\alpha}\cdot\frac{1}{1+\mu}\right]$$

$$= \frac{(1+\mu)(2+\mu)(1-\alpha)}{(2-\alpha)(1+\mu)+|\alpha|(2+\mu)}\cdot\frac{(2-\alpha)(1+\mu)+|\alpha|(2+\mu)}{(1-\alpha)(1+\mu)(2+\mu)} = 1.$$

(4) In this case, we have that

$$A < \frac{2(1-\alpha)}{2-\alpha+2|\alpha|\eta}\left[\frac{2-\alpha}{1-\alpha}\int_0^1 (1-s)ds + \frac{|\alpha|}{1-\alpha}\int_0^{\eta} ds\right]$$

$$= \frac{2(1-\alpha)}{2-\alpha+2|\alpha|\eta}\left[\frac{2-\alpha}{2(1-\alpha)} + \frac{|\alpha|\eta}{1-\alpha}\right] = 1.$$

(5) Let $\varepsilon = \frac{1}{2}\left[\frac{2(1-\alpha)}{2-\alpha+2|\alpha|\eta} - \Lambda\right]$. Then there exists $c > 0$ such that

$$|f(t,x)| \leq \left[\frac{2(1-\alpha)}{2-\alpha+2|\alpha|\eta} - \varepsilon\right]|x|, \quad (t,x) \in [0,1]\times R\setminus(-c,c).$$

Set $M = \max\{|f(t, x)| : (t, x) \in [0, 1] \times [-c, c]\}$, then

$$|f(t, x)| \le \left[\frac{2(1-\alpha)}{2-\alpha+2|\alpha|\eta} - \varepsilon\right]|x| + M,\ (t, x) \in [0, 1] \times R.$$

Set $k(s) = \dfrac{2(1-\alpha)}{2-\alpha+2|\alpha|\eta} - \varepsilon$, $h(s) = M$, then (4) holds. This completes the proof.

Corollary 3.1. *Suppose* $f(t, 0) \not\equiv 0$, $\alpha < 1$, *and there exist two nonnegative functions* $k, h \in L^1[0, 1]$ *such that*

$$|f(t, x)| \le k(t)|x| + h(t),\ a.e.\ (t, x) \in [0, 1] \times R.$$

If one of the following conditions holds:

(1) *There exists constant* $p > 1$ *such that*

$$\int_0^1 k^p(s)ds < \left[\frac{(1-\alpha)(1+q)^{1/q}}{2-\alpha+|\alpha|(1+q)^{1/q}}\right]^p,\ \frac{1}{p} + \frac{1}{q} = 1.$$

(2) *There exists constant* $\mu > -1$ *such that*

$$k(s) \le \frac{(1+\mu)(2+\mu)(1-\alpha)}{2-\alpha+|\alpha|(2+\mu)} s^{\mu},\ a.e.\ s \in [0, 1],$$

$$mes\left\{s \in [0, 1] : k(s) < \frac{(1+\mu)(2+\mu)(1-\alpha)}{2-\alpha+|\alpha|(2+\mu)} s^{\mu}\right\} > 0.$$

(3) $k(s)$ *satisfies*

$$k(s) \le \frac{2(1-\alpha)}{2-\alpha+2|\alpha|},\ a.e.\ s \in [0, 1],$$

$$mes\left\{s \in [0, 1] : k(s) < \frac{2(1-\alpha)}{2-\alpha+2|\alpha|}\right\} > 0.$$

(4) $f(t, x)$ *satisfies*

$$\Lambda =: \limsup_{|x| \to \infty} \max_{t \in [0, 1]} \left|\frac{f(t, x)}{x}\right| < \frac{2(1-\alpha)}{2-\alpha+2|\alpha|}.$$

Then the BVP (1.1) *has at least one nontrivial solution* $u^* \in C[0, 1]$.

Proof. In this case, we have that

$$A = \frac{2-\alpha}{1-\alpha}\int_0^1 (1-s)k(s)ds + \frac{|\alpha|}{1-\alpha}\int_0^\eta k(s)ds$$

$$\leq \frac{2-\alpha}{1-\alpha}\int_0^1 (1-s)k(s)ds + \frac{|\alpha|}{1-\alpha}\int_0^1 k(s)ds.$$

The remaining is same as Theorem 3.2. The proof is complete.

Corollary 3.2. *Suppose* $f(t, 0) \not\equiv 0$, $\alpha < 1$, *and there exist two nonnegative functions* $k, h \in L^1[0, 1]$ *such that*

$$|f(t, x)| \leq k(t)|x| + h(t), \ a.e. \ (t, x) \in [0, 1] \times R.$$

If one of the following conditions holds:

(1) *There exists constant* $p > 1$ *such that*

$$\int_0^1 k^p(s)ds < \left[\frac{2-\alpha}{1-\alpha+|\alpha|}\left(\frac{1+q}{2^{1+q}-1}\right)^{1/q}\right]^p, \ \frac{1}{p}+\frac{1}{q} = 1.$$

(2) *There exists constant* $\mu > -2$ *such that*

$$k(s) \leq \frac{(1-\alpha)(2+\mu)}{(2-\alpha+|\alpha|)(2^{2+\mu}-1)}(2-s)^\mu, \ a.e. \ s \in [0, 1],$$

$$mes\left\{s \in [0, 1] : k(s) < \frac{(1-\alpha)(2+\mu)}{(2-\alpha+|\alpha|)(2^{2+\mu}-1)}(2-s)^\mu\right\} > 0.$$

Then the BVP (1.1) *has at least one nontrivial solution* $u^* \in C[0, 1]$.

Proof. In this case, we have that

$$A = \frac{2-\alpha}{1-\alpha}\int_0^1 (1-s)k(s)ds + \frac{|\alpha|}{1-\alpha}\int_0^\eta k(s)ds$$

$$\leq \frac{2-\alpha}{1-\alpha}\int_0^1 (1-s)k(s)ds + \frac{|\alpha|}{1-\alpha}\int_0^1 k(s)ds$$

$$\leq \frac{2-\alpha+|\alpha|}{1-\alpha}\int_0^1 (2-s)k(s)ds.$$

(1) By using the Hölder inequality, we have that

$$A \leq \frac{2-\alpha+|\alpha|}{1-\alpha}\int_0^1 (2-s)k(s)ds$$

$$\leq \frac{2-\alpha+|\alpha|}{1-\alpha}\left[\int_0^1 k^p(s)ds\right]^{1/p}\left[\int_0^1 (2-s)^q ds\right]^{1/q}$$

$$< \frac{2-\alpha+|\alpha|}{1-\alpha}\cdot\frac{2-\alpha}{1-\alpha+|\alpha|}\left(\frac{1+q}{2^{1+q}-1}\right)^{1/q}\cdot\left(\frac{2^{1+q}-1}{1+q}\right)^{1/q} = 1.$$

(2) In this case, we have that

$$A \leq \frac{2-\alpha+|\alpha|}{1-\alpha}\int_0^1 (2-s)k(s)ds$$

$$< \frac{2-\alpha+|\alpha|}{1-\alpha}\cdot\frac{(1-\alpha)(2+\mu)}{(2-\alpha+|\alpha|)(2^{2+\mu}-1)}\int_0^1 (2-s)^{1+\mu}ds$$

$$= \frac{2+\mu}{2^{2+\mu}-1}\cdot\frac{2^{2+\mu}-1}{2+\mu} = 1.$$

The proof is complete.

Theorem 3.3. *Suppose* $f(t, 0) \not\equiv 0$, $\alpha > 1$ *and there exist nonnegative functions* $k, h \in L^1[0, 1]$ *such that*

$$|f(t, x)| \leq k(t)|x| + h(t), \ a.e. \ (t, x) \in [0, 1] \times R.$$

If one of the following conditions holds:

(1) *There exists constant* $p > 1$ *such that*

$$\int_0^1 k^p(s)ds < \left[\frac{(\alpha-1)(1+q)^{1/q}}{\alpha(1+(\eta(1+q))^{1/q})}\right]^p, \ \frac{1}{p}+\frac{1}{q} = 1.$$

(2) *There exists constant* $\mu > -1$ *such that*

$$k(s) \leq \frac{(1+\mu)(2+\mu)(\alpha-1)}{\alpha[1+(2+\mu)\eta^{1+\mu}]}s^{\mu}, \ a.e. \ s \in [0, 1],$$

$$mes\left\{s \in [0, 1] : k(s) < \frac{(1+\mu)(2+\mu)(\alpha-1)}{\alpha[1+(2+\mu)\eta^{1+\mu}]}s^{\mu}\right\} > 0.$$

(3) *There exists constant* $\mu > -1$ *such that*

$$k(s) \le \frac{(1+\mu)(2+\mu)(\alpha-1)}{(3+2\mu)\alpha}(1-s)^{\mu}, \ a.e. \ s \in [0, 1].$$

(4) $k(s)$ *satisfies*

$$k(s) \le \frac{2(\alpha-1)}{\alpha(1+2\eta)}, \ a.e. \ s \in [0, 1],$$

$$mes\left\{s \in [0, 1] : k(s) < \frac{2(\alpha-1)}{\alpha(1+2\eta)}\right\} > 0.$$

(5) $f(t, x)$ *satisfies*

$$\Lambda =: \limsup_{|x|\to\infty} \max_{t\in[0,1]} \left|\frac{f(t, x)}{x}\right| < \frac{2(\alpha-1)}{\alpha(1+2\eta)}.$$

Then the BVP (1.1) *has at least one nontrivial solution* $u^* \in C[0, 1]$.

Proof. Let A be given in Theorem 3.1. In view of Theorem 3.1, we only need to prove $A < 1$. Since $\alpha > 1$,

$$\begin{aligned} A &= \frac{\alpha}{\alpha-1}\int_0^1 (1-s)k(s)ds + \frac{\alpha}{\alpha-1}\int_0^{\eta} k(s)ds \\ &= \frac{\alpha}{\alpha-1}\left[\int_0^1 (1-s)k(s)ds + \int_0^{\eta} k(s)ds\right]. \end{aligned}$$

(1) By using the Hölder inequality, we have that

$$\begin{aligned} A &\le \frac{\alpha}{\alpha-1}\left[\int_0^1 k^p(s)ds\right]^{1/p}\left\{\left[\int_0^1 (1-s)^q ds\right]^{1/q} + \left[\int_0^{\eta} 1^q ds\right]^{1/q}\right\} \\ &= \frac{\alpha}{\alpha-1}\left[\int_0^1 k^p(s)ds\right]^{1/p}\left[\left(\frac{1}{1+q}\right)^{1/q} + \eta^{1/q}\right] \\ &< \frac{\alpha}{\alpha-1}\cdot\frac{(\alpha-1)(1+q)^{1/q}}{\alpha(1+(\eta(1+q))^{1/q})}\cdot\frac{1+(\eta(1+q))^{1/q}}{(1+q)^{1/q}} = 1. \end{aligned}$$

(2) In this case, we have that

$$A < \frac{\alpha}{\alpha-1}\cdot\frac{(1+\mu)(2+\mu)(\alpha-1)}{\alpha[1+(2+\mu)\eta^{1+\mu}]}\left[\int_0^1 (1-s)s^{\mu}ds + \int_0^{\eta} s^{\mu}ds\right]$$

$$= \frac{(1+\mu)(2+\mu)}{1+(2+\mu)\eta^{1+\mu}}\left[\frac{1}{(1+\mu)(2+\mu)} + \frac{\eta^{1+\mu}}{1+\mu}\right]$$

$$= \frac{(1+\mu)(2+\mu)}{1+(2+\mu)\eta^{1+\mu}} \cdot \frac{1+(2+\mu)\eta^{1+\mu}}{(1+\mu)(2+\mu)} = 1.$$

(3) In this case, we have that

$$A \leq \frac{\alpha}{\alpha-1} \cdot \frac{(1+\mu)(2+\mu)(\alpha-1)}{(3+2\mu)\alpha}\left[\int_0^1 (1-s)(1-s)^{\mu} ds + \int_0^{\eta} (1-s)^{\mu} ds\right]$$

$$= \frac{(1+\mu)(2+\mu)}{3+2\mu} \cdot \left[\frac{1}{2+\mu} + \frac{1-(1-\eta)^{1+\mu}}{1+\mu}\right]$$

$$< \frac{(1+\mu)(2+\mu)}{3+2\mu}\left[\frac{1}{2+\mu} + \frac{1}{1+\mu}\right]$$

$$= \frac{(1+\mu)(2+\mu)}{3+2\mu} \cdot \frac{3+2\mu}{(1+\mu)(2+\mu)} = 1.$$

(4) In this case, we have that

$$A < \frac{\alpha}{\alpha-1} \cdot \frac{2(\alpha-1)}{\alpha(1+2\eta)}\left[\int_0^1 (1-s) ds + \int_0^{\eta} ds\right]$$

$$= \frac{2}{1+2\eta}\left[\frac{1}{2} + \eta\right] = 1.$$

(5) Let $\varepsilon = \frac{1}{2}\left[\frac{2(\alpha-1)}{\alpha(1+2\eta)} - \Lambda\right]$. Then there exists $c > 0$ such that

$$| f(t, x) | \leq \left[\frac{2(\alpha-1)}{\alpha(1+2\eta)} - \varepsilon\right] | x |, \ (t, x) \in [0, 1] \times R \setminus (-c, c).$$

Set $M = \max\{| f(t, x) | : (t, x) \in [0, 1] \times [-c, c]\}$, then

$$| f(t, x) | \leq \left[\frac{2(\alpha-1)}{\alpha(1+2\eta)} - \varepsilon\right] | x | + M, \ (t, x) \in [0, 1] \times R.$$

Set $k(s) = \frac{2(\alpha-1)}{\alpha(1+2\eta)} - \varepsilon$, $h(s) = M$, then (4) holds. This completes the proof.

Corollary 3.3. *Suppose* $f(t, 0) \not\equiv 0$, $\alpha > 1$ *and there exist two* *nonnegative functions* $k,\ h \in L^1[0, 1]$ *such that*

$$|f(t, x)| \le k(t)|x| + h(t),\ a.e.\ (t, x) \in [0, 1] \times R.$$

If one of the following conditions holds.

(1) *There exists constant* $p > 1$ *such that*

$$\int_0^1 k^p(s)ds < \left[\frac{(\alpha - 1)(1 + q)^{1/q}}{\alpha(1 + (1 + q))^{1/q}}\right]^p,\quad \frac{1}{p} + \frac{1}{q} = 1.$$

(2) *There exists constant* $\mu > -1$ *such that*

$$k(s) \le \frac{(1 + \mu)(2 + \mu)(\alpha - 1)}{\alpha(3 + \mu)} s^{\mu},\ a.e.\ s \in [0, 1],$$

$$mes\left\{s \in [0, 1] : k(s) < \frac{(1 + \mu)(2 + \mu)(\alpha - 1)}{\alpha(3 + \mu)} s^{\mu}\right\} > 0.$$

(3) $k(s)$ *satisfies*

$$k(s) \le \frac{2(\alpha - 1)}{3\alpha},\ a.e.\ s \in [0, 1],$$

$$mes\left\{s \in [0, 1] : k(s) < \frac{2(\alpha - 1)}{3\alpha}\right\} > 0.$$

(4) $f(t, x)$ *satisfies*

$$\Lambda =: \limsup_{|x| \to \infty} \max_{t \in [0,1]} \left|\frac{f(t, x)}{x}\right| < \frac{2(\alpha - 1)}{3\alpha}.$$

Then the BVP (1.1) *has at least one nontrivial solution* $u^* \in C[0, 1]$.

Proof. In this case, we have that

$$A = \frac{\alpha}{\alpha - 1}\int_0^1 (1 - s)k(s)ds + \frac{\alpha}{\alpha - 1}\int_0^{\eta} k(s)ds$$

$$\le \frac{\alpha}{\alpha - 1}\left[\int_0^1 (1 - s)k(s)ds + \int_0^1 k(s)ds\right].$$

The remaining is same as Theorem 3.3. This completes the proof.

Corollary 3.4. *Suppose* $f(t, 0) \not\equiv 0$, $\alpha > 1$ *and there exist two* *nonnegative functions* $k,\ h \in L^1[0, 1]$ *such that*

$$|f(t, x)| \leq k(t)|x| + h(t),\ a.e.\ (t, x) \in [0, 1] \times R.$$

If one of following conditions holds.

(1) *There exists constant* $p > 1$ *such that*

$$\int_0^1 k^p(s)ds < \left[\frac{\alpha - 1}{\alpha}\left(\frac{1 + q}{2^{1+q} - 1}\right)^{1/q}\right]^p,\quad \frac{1}{p} + \frac{1}{q} = 1.$$

(2) *There exists constant* $\mu > -2$ *such that*

$$k(s) \leq \frac{(2 + \mu)(\alpha - 1)}{\alpha(2^{2+\mu} - 1)}(2 - s)^{\mu},\ a.e.\ s \in [0, 1],$$

$$mes\left\{s \in [0, 1] : k(s) < \frac{(2 + \mu)(\alpha - 1)}{\alpha(2^{2+\mu} - 1)} s^{\mu}\right\} > 0.$$

Then the BVP (1.1) *has at least one nontrivial solution* $u^* \in C[0, 1]$.

Proof. In this case, we have that

$$\begin{aligned} A &= \frac{\alpha}{\alpha - 1}\int_0^1 (1 - s)k(s)ds + \frac{\alpha}{\alpha - 1}\int_0^{\eta} k(s)ds \\ &\leq \frac{\alpha}{\alpha - 1}\left[\int_0^1 (1 - s)k(s)ds + \int_0^1 k(s)ds\right] \\ &= \frac{\alpha}{\alpha - 1}\int_0^1 (2 - s)k(s)ds. \end{aligned}$$

(1) By using the Hölder inequality, we have that

$$\begin{aligned} A &\leq \frac{\alpha}{\alpha - 1}\int_0^1 (2 - s)k(s)ds \\ &\leq \frac{\alpha}{\alpha - 1}\left[\int_0^1 k^p(s)ds\right]^{1/p}\left[\int_0^1 (2 - s)^q ds\right]^{1/q} \\ &< \frac{\alpha}{1} \cdot \frac{\alpha - 1}{\alpha}\left(\frac{1 + q}{2^{1+q} - 1}\right)^{1/q} \cdot \left(\frac{2^{1+q} - 1}{1 + q}\right)^{1/q} = 1. \end{aligned}$$

(2) In this case, we have that

$$A < \frac{\alpha}{\alpha - 1} \cdot \frac{(1+\mu)(2+\mu)(\alpha - 1)}{\alpha[1 + (2+\mu)\eta^{1+\mu}]} \left[\int_0^1 (1-s)s^{\mu} ds + \int_0^{\eta} s^{\mu} ds \right]$$

$$= \frac{(1+\mu)(2+\mu)}{1 + (2+\mu)\eta^{1+\mu}} \left[\frac{1}{(1+\mu)(2+\mu)} + \frac{\eta^{1+\mu}}{1+\mu} \right]$$

$$= \frac{(1+\mu)(2+\mu)}{1 + (2+\mu)\eta^{1+\mu}} \cdot \frac{1 + (2+\mu)\eta^{1+\mu}}{(1+\mu)(2+\mu)} = 1.$$

This completes the proof.

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