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RECTANGULAR NUMBERS AND RATIONAL DISTANCE

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Abstract

We discuss about preliminary properties of rectangular numbers and show how it can be used in rational distances problems. We provide a method to generate set of four points on the parabola, $y = x^2$ so that the distance between any pair of them is rational. We also show that if the following conjecture is true, then there is no point on the plane that has rational distances to all corners of unit square.

Conjecture (*) If we have $m, n \in \mathbb{Q}$ that satisfy $h(m) + \frac{1}{k} = h(n)$, for $k \in \mathbb{Q}$ and $k \neq 1$, then there is no rational number m', n' that satisfy $h(m') + \frac{1}{1 \pm k} = h(n')$.

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1. Introduction

Definition. A rational number, x is called a *rectangular number* if and only if there exists $z \in \mathbb{Q}$ such that $x^2 + 1 = z^2$.

Study of rectangular numbers can be useful in solving rational distance problems, as the equation $x^2 + 1 = z^2$ can represent the distance between two points, where their vertical and horizontal distances are $|x|$ and 1.

Integer distances problem may be reduced and thus the properties of rectangular number can be applied:

If we have $X^2 + Y^2 = Z^2$, where X, Y, Z are non-zero integer, then $(X/Y)^2 + 1 = (Z/Y)^2$ shows that X/Y is actually a rectangular number.

Properties of rectangular numbers were studied in [5]. We will present some preliminary properties of rectangular numbers in Section 2. In Section 3, we will focus on the equality $h(m) + k = h(n)$, where m, n are non-zero rational numbers, k is a fixed rational number and $h(\bar{x})$ is a function defined by $\dfrac{\bar{x}^2 - 1}{2\bar{x}}$ for real number $\bar{x} \neq 0$.

In Section 4, we provide an algorithmic approach to finding non-concylic points on the parabola $y = x^2$ so that the distance between any pair of them is rational.

In [1], it has been proven that there exist infinitely many sets of 4 points on the parabola, such that rational distance between each of them is rational.

In Section 4, we present a method to generate such rational distance sets based on the result from Section 3.

Finally, we will show that how the following unsolved problem, as describe in [2], can be reduced to rectangular numbers.

Problem † Is there a point on the plane whose distances from all the corners of the unit square are rational?

2. Preliminary Properties of Rectangular Numbers

In this section, we shall establish and collect some preliminary properties of rectangular numbers for later use.

Lemma 2.1. *x is a rectangular number if and only if $x = h(m)$, where $h(m) = \frac{m^2-1}{2m}$ for $m \in \mathbb{Q}\backslash\{0\}$.*

Proof. If x is a rectangular number, then there exists $z \in \mathbb{Q}\backslash\{0\}$ such that $x^2+1 = z^2$. So, $(x/z)^2 + (1/z)^2 = 1$ and from the parameterization of circle, we know that $\frac{x}{z} = \frac{m^2-1}{m^2+1}$ and $\frac{1}{z} = \frac{2m}{m^2+1}$; this further implies that $x = \frac{m^2-1}{2m}$ for some $m \in \mathbb{Q}\backslash\{0\}$.

If $x = \frac{m^2-1}{2m}$, then

$$\begin{aligned} x^2+1 &= \left(\frac{m^2-1}{2m}\right)^2 + 1 \\ &= \frac{m^4-2m^2+1}{4m^2} + 1 \\ &= \frac{m^4+2m^2+1}{4m^2} \\ &= \left(\frac{m+1}{2m}\right)^2. \end{aligned}$$

This implies that x is a rectangular number if $x = \frac{m^2-1}{2m}$. □

Lemma 2.2. *If x is a non-zero rectangular number, then $-x$ and $1/x$ are also rectangular numbers.*

Proof. If x is a non-zero rectangular number, then by Lemma 2.1, we have $x = \frac{m^2 - 1}{2m}$ for some $m \in \mathbb{Q}\backslash\{0, 1, -1\}$. Using Lemma 2.1,

$$-x = -\frac{m^2 - 1}{2m} = \frac{(-m)^2 - 1}{2(-m)}$$

and

$$\frac{1}{x} = \frac{2m}{m^2 - 1} = \frac{\left(\frac{m+1}{m-1}\right)^2 - 1}{2\left(\frac{m+1}{m-1}\right)}. \qquad \square$$

Lemma 2.3. *If x is a non-zero rectangular number, then $x + 1$ is not a rectangular number.*

Proof. See [5, Theorem 4.8]. $\square$

Lemma 2.4. *If x is a non-zero rectangular number, then $x + 2$ is not a rectangular number.*

Proof. We proceed with a proof by contradiction as used in [5]. Suppose x is a non-zero rectangular number, we can rewrite $x = \frac{m^2 - 1}{2m}$ for some $m \in \mathbb{Q}\backslash\{0, 1, -1\}$. Let $m = \frac{r}{s}$ for $r, s \in \mathbb{Z}$ and $\gcd(r, s) = 1$. Then $x + 2$ can be written as

$$\frac{r^2 - s^2}{2rs} + 2 = \frac{r^2 + 4rs - s^2}{2rs}.$$

So if $x + 2$ is rectangular, then $(r^2 + 4rs - s^2)^2 + (2rs)^2$ must be a perfect square, say $(r^2 + 4rs + s^2t)^2$, where $t \in \mathbb{Q}$. Rearranging the equation, we derive $r^2s^2(2 - 2t) - 8rs^3(1 + t) + s^4(1 - t^2) = 0$, we can consider this as a quadratic polynomial in terms of r. Since r is an integer, the discriminant of this equation must be a square, say $u^2 = 2(1+t)(7 + 10t - t^2)$.

Let $u = 4y$ and $t = 3 - 2a$, the preceding equation can be simplified to $y^2 = a^3 - 11a + 14$, which is an elliptic curve of conductor 32. Referring to Table 1 of [3], we know that this elliptic curve has rank 0, and its torsion subgroup has order 2. Thus, (2, 0) is the only rational point on the elliptic curve corresponding to $rs = 0$, which leads to a contradiction with $\gcd(r, s) = 1$. □

Remark. There are rectangular numbers that differ from each other by an integer. For example, $\frac{21}{20} = -\frac{99}{20} + 6$. Both $\frac{21}{20}$ and $-\frac{99}{20}$ are rectangular numbers which can be written as $h\left(\frac{5}{2}\right)$ and $h\left(\frac{1}{10}\right)$, respectively, where $h(\bar{x}) = \frac{\bar{x}^2 - 1}{2\bar{x}}$, where $x \in \mathbb{Q}\backslash\{0\}$.

3. Rectangular Numbers of the Form $h(m) + k = h(n)$

Recall that $h(\bar{x})$ is defined by $\frac{\bar{x}^2 - 1}{2\bar{x}}$, where $x \in \mathbb{Q}\backslash\{0\}$. In this section, we will consider the rational solutions of $h(m) + k = h(n)$, where $m, n \in \mathbb{Q}$, $m, n \neq 0$ and k is a fixed rational number. If (x, y) is a rational solution for $h(m) + k = h(n)$, then we mean that $h(y) + k = h(x)$.

Proposition 3.1. *Suppose (q, p) is a rational solution for $h(\bar{y}) + k = h(\bar{x})$, for a fixed $k \in \mathbb{Q}$. Then we will have all the elements in the following set as rational solutions to $h(\bar{y}) + k = h(\bar{x})$:*

$$\{(q, -1/p), (-1/q, p), (-1/q, -1/p), (-p, -q), (-p, 1/q), (1/p, -q), (1/p, 1/q)\}.$$

Proof. Since $(q,\ p)$ is a solution, we have $h(p) + k = h(q)$ and we can rewrite this as:

$$
\begin{aligned}
h(q) &= h(p) + k \\
&= \frac{p^2 - 1}{2p} + k \\
&= \frac{(p^2 - 1)\left(-\frac{1}{p^2}\right)}{2p\left(-\frac{1}{p^2}\right)} + k \\
&= \frac{\frac{1}{p^2} - 1}{2\left(-\frac{1}{p}\right)} + k \\
&= \frac{\left(-\frac{1}{p}\right)^2 - 1}{2\left(-\frac{1}{p}\right)} + k.
\end{aligned}
$$

Thus, $(q, -1/p)$ is also a solution to $h(\bar{y}) + k = h(\bar{x})$. Other solutions can be proved in a similar fashion. □

3.1. Finding other rational points on the curve $h(\bar{y}) + k = h(\bar{x})$

Our objective here is to derive rational points on the curve, $C : h(\bar{y}) + k = h(\bar{x})$, for a fixed $k \in \mathbb{Q}$. Our technique in this section is inspired by the one on elliptic curves. Using the similar technique from page 15 to 22 of [4], we can generate more rational points on an elliptic curve, by one or two known rational points.

Case A. Given two rational points on C, by using the same techniques in elliptic curve, we can generate the third point, P' if we are given two rational points, $P_1 : (n_1,\ m_1),\ P_2 : (n_2,\ m_2)$.

Let $\bar{y} = d\bar{x} + c$ be the straight line connecting P_1 and P_2, where $d, c \in \mathbb{Q}$. If (n, m) is a rational point on this line and the curve C, then $n = dm + c$ and we have

$$\frac{m^2 - 1}{2m} + k = \frac{(dm + c)^2 - 1}{2(dm + c)}. \tag{1}$$

By simplifying the above, we get

$$d(d - 1)m^3 + (2cd - 2kd - c)m^2 + (c^2 - 2kc + d - 1)m + c = 0. \tag{2}$$

The third intersection point, (n_3, m_3) will be:

$$m_3 = \frac{c - 2d(c - k)}{d(d - 1)} - m_1 - m_2;$$

$$n_3 = dm_3 + c;$$

$$d = \frac{n_1 - n_2}{m_1 - m_2};$$

$$c = n_1 - dm_1.$$

Note that if $d = 1$, $d = 0$ or $c = 0$, then equation (2) will be reduced to a quadratic equation. Therefore, either P_1 or P_2 is a multiple point (i.e., the straight line intersects the cubic at two points only). Therefore, we would need another point to generate other points.

Case B. Given one rational point (n_1, m_1) on C, the second point can be found by taking the intersection between C and the tangent line to C at (n_1, m_1):

$$m_3 = \frac{c - 2d(c - k)}{d(d - 1)} - 2m_1;$$

$$n_3 = dm_3 + c;$$

$$d = \frac{m_1^2 + 1}{m_1^2} \cdot \frac{n_1^2}{n_1^2 + 1};$$

$$c = n_1 - dm_1.$$

Since both m_1, n_1 are non-zero, we have d being non-zero. In addition, $d = 1$ when $m_1 = \pm n_1$, and $c = 0$ when $m_1 n_1 = 1$.

If we continue to derive new point P'' by using P' and P_1 or P_2 with the method in Case A, then we will end up with the original points. So we need to define the *third rational point* to be different from P'.

Since (n_3, m_3) is a solution to $h(\bar{y}) + k = h(\bar{x})$, using Proposition 3.1, we can define the *third rational point* to be any of the following:

$$\{(n_3, -1/m_3), (-1/n_3, m_3), (-1/n_3, -1/m_3), (-m_3, -n_3),$$
$$(-m_3, 1/n_3), (1/m_3, -n_3), (1/m_3, 1/n_3)\}$$

provided that the *third rational point* gives non-zero c, d and $d \neq 1$ when we perform calculations as in Case A, in order to produce the new intersection point, P'' on C.

By the techniques above, we can generally generate many rational points by starting with one point (n, m), where $m \neq \pm n$ and $mn \neq 1$ with the following steps:

1. Using Case B to find the second rational point based on (n, m).

2. Using Case A or Case B to find the third rational point.

3. Repeat step 2.

Limitation. One of the downsides of finding rational point on C using this method is that the numerator and denominator will become larger as we derive more points.

It is also possible that we obtain no more new rational points after applying Case A or Case B repeatedly unless the starting point is changed.

We may not be able to generate additional point all the time by using the methodology described in Case A or Case B. If one of the points is a multiple point, then we would not be able to generate new rational point. In this case, we would need another new rational point before we can apply the methodology in Case A and Case B.

4. Application of Section 2 in Solving Rational Distance Problems

Definition. We say that a collection of points are at rational distance if the distance between each pair of points is rational. We will call such collection of points a *rational distance set*.

We recall the following theorem as proven in Theorem 2.4 of [1].

Theorem 4.1. *Suppose* P_1, P_2, P_3 *and* P_4 *are rational points on* $y = x^2$, *where* $P_i = (x_i, x_i^2)$ *for* $i = 1, 2, \ldots, 4$. *The set* $\{P_1, P_2, P_3, P_4\}$ *is a rational distance set if and only if there are rational values* m_{ij} *with* $1 \le i \le j \le 4$ *such that*

$$x_1 = \frac{h(m_{12}) + h(m_{13}) - h(m_{23})}{2},$$

$$x_2 = \frac{h(m_{12}) - h(m_{13}) + h(m_{23})}{2},$$

$$x_3 = \frac{-h(m_{12}) + h(m_{13}) + h(m_{23})}{2},$$

$$x_4 = \frac{-h(m_{12}) - h(m_{13}) + h(m_{23}) + 2h(m_{14})}{2}$$

and

$$h(m_{13}) + h(m_{24}) = h(m_{23}) + h(m_{14}) = h(m_{12}) + h(m_{34}).$$

Consider $h(m_i) + k = h(n_i)$, where $i \in \mathbb{N}$. Rearranging this, we have $h(n_i) - h(m_i) = k$ and from Lemma 2.2, we have $h(n_i) + h(-m_i) = k$.

The techniques within the context of Subsection 3.1 can be used in solving the equation $h(m_{13}) + h(m_{24}) = h(m_{23}) + h(m_{14}) = h(m_{12}) + h(m_{34})$, we can construct a rational distance set of 4 rational points by solving three rational solutions for the equation $h(\bar{y}) + k = h(\bar{x})$.

Let (n_1, m_1), (n_2, m_2), (n_3, m_3) be the solutions to $h(\bar{y}) + k = h(\bar{x})$. Combining these, we have the following:

$$k = h(n_1) + h(-m_1) = h(n_2) + h(-m_2) = h(n_3) + h(-m_3). \qquad (3)$$

These solutions can then be transformed into the rational distance set on parabola $y = x^2$ using Theorem 4.1.

Example. Consider the equality $h(1/2) + 25/12 = h(3)$. Using technique in Case B from Subsection 3.1, we obtain the new intersection point as $(-3/28, -4/21)$, and thus by Proposition 3.1, we can define the *second rational point* as $(-3/28, 21/4)$.

We choose *second rational point* in the way such that when we perform calculations as in Case A, we have non-zero c, d and $d \neq 1$. There are other choices for *second rational point* that meet this criteria but we do not need all of them to arrive at *third rational point*.

Using the two points, we perform calculations as in Case A in Subsection 3.1 and we have $(1711/418, -1121/957)$ as the third intersection point. Since we do not need to find the forth intersection point, we do not need to redefine the *third rational point*.

So, we have these three equations:

1. $h(1/2) + 25/12 = h(3)$,

2. $h(21/4) + 25/12 = h(-3/28)$,

3. $h(-1121/957) + 25/12 = h(1711/418)$.

Rearranging this, we have the following as solutions to the equality $h(\bar{y}) + h(\bar{x}) = 25/12$:

1. $\left(3, -\frac{1}{2}\right)$,

2. $\left(-\frac{3}{28}, -\frac{21}{4}\right)$,

3. $\left(\frac{1711}{418}, \frac{1121}{957}\right)$.

So, $h(3) + h(-1/2) = h(-3/28) + h(-21/4) = h(1711/418) + h(1121/957)$. Let $m_{12} = -1/2$, $m_{13} = -21/4$, $m_{14} = 1711/418$, $m_{23} = 1121/957$, $m_{24} = -3/28$, $m_{34} = 3$ and using Theorem 4.1, we can calculate x_1, x_2, x_3, x_4 as below:

$$x_1 = \frac{h(-1/2) + h(-21/4) - h(-1121/957)}{2} = -116464277/120153264,$$

$$x_2 = \frac{h(-1/2) - h(-21/4) + h(1121/957)}{2} = 206579225/120153264,$$

$$x_3 = \frac{-h(-1/2) + h(-21/4) + h(1121/957)}{2} = -62498291/40051088,$$

$$x_4 = \frac{-h(-1/2) - h(-21/4) + h(1121/957) + 2h(1711/418)}{2}$$

$$= 347699225/120153264.$$

Hence, we get the following points to form a rational distance set on the parabola:

$$P_1 : (-116464277/120153264, (-116464277/120153264)^2),$$

$$P_2 : (206579225/120153264, (206579225/120153267)^2),$$

$$P_3 : (-62498291/40051088, (-62498291/40051088)^2),$$

$$P_4 : (347699225/120153264, (347699225/120153264)^2).$$

Let D_{ij} be the distance between P_i and P_j for $1 \le i < j \le 4$. Then $D_{12} = 5/4$, $D_{13} = 5/4$, $D_{14} = 457/168$, $D_{23} = 1086245/1072797$, $D_{24} = 793/168$, $D_{34} = 5/3$.

We next consider the problem:

Problem † Is there a point all of whose distances from the corners of the unit square are rational?

We will convert this old problem using related rectangular numbers and show that if the following **Conjecture** (*) is true then there is no point on the plane that has rational distances to all corners of unit square.

Conjecture (*) If we have $m, n \in \mathbb{Q}$ that satisfy $h(m) + \frac{1}{k} = h(n)$, for $k \in \mathbb{Q}$ and $k \neq 1$, then there is no rational number m', n' that satisfy $h(m') + \frac{1}{1 \pm k} = h(n')$.

We study **Problem** † in four cases as follows:

Case I. Suppose P_1 is such a point for **Problem** † and P_1 is on the horizontal line l_1 as follows:

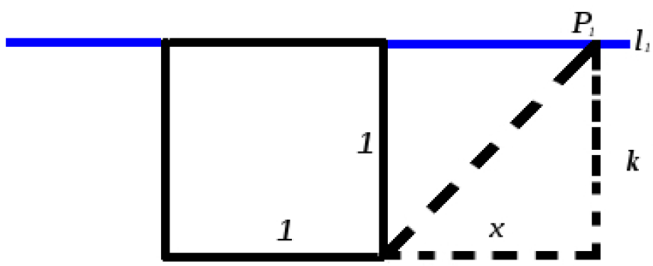

Figure 1. P_1 is at the height of $k = 1$ from the bottom of the unit square.

Then we have $x^2 + 1 = z^2$ and $(x+1)^2 + 1 = u^2$, where $x, z, u \in \mathbb{Q}$. So, we have both $x, x+1$ as rectangular numbers. By Lemma 2.3, this is impossible; thus P_1 does not exist.

Case II. Suppose P_2 is such a point for **Problem** † and P_2 is on the horizontal line l_2 as follows:

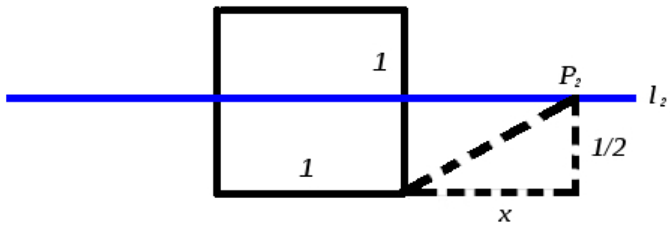

Figure 2. P_2 is at the height of $k = \dfrac{1}{2}$ from the bottom of unit square.

Then we have $x^2 + (1/2)^2 = z^2$ and $(x+1)^2 + (1/2)^2 = u^2$, where x, z, $u \in \mathbb{Q}$. Let $h(n) = 2(x+1)$ and $h(m) = 2x$. Then, we have $h(m) + 2 = h(n)$. By Lemma 2.4, this is impossible; thus P_2 does not exist.

Case III. Suppose P_3 is such a point for **Problem** † and P_3 is on the horizontal line l_3 as follows:

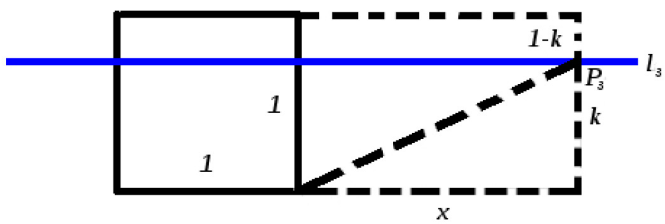

Figure 3. P_3 is at the height of k from the bottom of unit square, where $\dfrac{1}{2} < k < 1$.

If there is a point P_3 on line, l_3, then $x^2 + k^2 = z^2$ and $(x+1)^2 + k^2 = u^2$ for some x, z, $u \in \mathbb{Q}$. Let $h(n) = \dfrac{x+1}{k}$ and $h(m) = \dfrac{x}{k}$. Then we have $h(m) + \dfrac{1}{k} = h(n)$. Also, we have $x^2 + (1-k)^2 = y^2$ and $(x+1)^2 + (1-k)^2 = v^2$ for some y, $v \in \mathbb{Q}$. Let $h(n') = \dfrac{x}{1-k}$ and $h(m') = \dfrac{x+1}{1-k}$, so we have another equality, $h(m') + \dfrac{1}{1-k} = h(n')$.

Case IV. Suppose P_4 is such a point for **Problem** † and P_4 is on the horizontal line l_4 as follows:

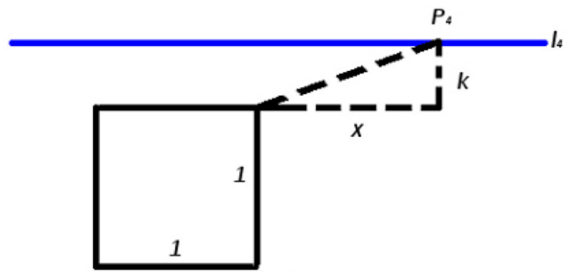

Figure 4. P_4 is at the height of k from the bottom of unit square, where $k > 1$.

If there is a point P_4 on line, l_4, then $x^2 + k^2 = z^2$ and $(x+1)^2 + k^2 = u^2$ for some $x, z, u \in \mathbb{Q}$. Let $h(n) = \frac{x+1}{k}$ and $h(m) = \frac{x}{k}$. Then we have $h(m) + \frac{1}{k} = h(n)$. Also, we have $x^2 + (1+k)^2 = y^2$ and $(x+1)^2 + (1+k)^2 = v^2$ for some $y, v \in \mathbb{Q}$. Let $h(n') = \frac{x}{1+k}$ and $h(m') = \frac{x+1}{1+k}$, so we have another equality, $h(m') + \frac{1}{1+k} = h(n')$.

Situations where $k < 0$ and $k < \frac{1}{2}$ can be reduced to the four cases above due to the symmetry property of unit square. If the point is above the square, then we could rotate it 90° clockwise. Rotate the unit square 180° and 270° clockwise if the point is on the left and below the square, respectively.

If the point is on the edge of square, then it would be Case I with negative x value. Similarly, if the point is inside the square but not on the edge of square, then it would be Case III with negative x value.

Cases I to IV consider all possibilities of the existence of P_1 and the only possible cases for such P_1 to exist are Cases III and IV. So, if **Conjecture** (*) is true then there is no point P such that its distances to all corners of unit square are rational.

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