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EXTENDED CALCULI ON Z_3-GRADED QUANTUM SUPERPLANE

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Abstract

In this work, we explicitly set up extended calculi on the Z_3-graded quantum superplane using approach of [2].

1. Introduction

Noncommutative geometry has started to play an important role in different fields of mathematics and mathematical physics over the past decade. The basic structure giving a direction to the noncommutative geometry is a differential calculus on an associative algebra. The noncommutative differential geometry of quantum groups was introduced in [17]. In this approach, the differential calculus on the group is deduced from the properties of the group and it involves functions on the group, differentials, differential forms and derivatives. The other approach, initiated

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in [16], followed Manin's emphasis [11] on the quantum spaces as the primary objects. Differential forms are defined in terms of noncommuting coordinates, and the differential and algebraic properties of quantum groups acting on these spaces are obtained from the properties of the spaces.

In [12], Manin extended the notation of quantum space to that of quantum superspace, called also *quantum superplane*, of which the defining quadratic relations remain invariant under linear transformations. These endomorphisms constitute the quantum supergroup. From a wary mathematical point of view, the quantum superplane appears in this approach as a comodule over the corresponding quantum supergroup. The quantum (super)space has been then visualized by many as a paradigm for the general program of quantum deformed physics. There have been many attempts to generalize Z_2-graded constructions to the Z_3-graded case lately [1, 3, 4, 7-10]. Chung [4] studied the Z_3-graded quantum space that generalizes the Z_2-graded space called a *superspace*, using the methods of Wess and Zumino [16]. Çelik [3] studied the noncommutative geometry of the Z_3-graded superplane. Let us shortly investigate a general Z_3-graded algebraic structure.

The cyclic group Z_3 can be represented in the complex plane by means of the cubic roots of unity $j = e^{\frac{2\pi i}{3}}$ $(i^2 = -1)$,

$$j^3 = 1 \quad \text{and} \quad j^2 + j + 1 = 0 \quad \text{or} \quad (j+1)^2 = j.$$

One can define the Z_3-graded commutator $[A, B]$ as

$$[A, B]_{Z_3} = AB - j^{ab}BA,$$

where $grad(A) = a$ and $grad(B) = b.$ If A and B are j-commutative, then we have

$$AB = j^{ab}BA.$$

2. Review of Calculus on the Z_3-graded Quantum Superplane

Elementary properties of differential geometry of the Z_3-graded quantum superplane are described in [3]. We state briefly the properties, we will need in this work.

2.1. The algebra of functions on the Z_3-graded quantum superplane

It is well-known that the Z_2-graded quantum plane or quantum superplane is defined as an associative algebra whose even coordinate x and odd coordinate θ satisfy the relations

$$x\theta = q\theta x, \quad \theta^2 = 0,$$

where q is a non-zero complex deformation parameter. One of the possible ways to generalize the quantum superplane is to increase the power of nilpotency of its odd generator. This fact gives the motivation for the following definition.

Definition 2.1. Let $O(\mathbb{C}_q^{1|1})$ be the algebra with the generators x and θ satisfying the relations

$$x\theta = q\theta x, \quad \theta^3 = 0, \tag{2.1}$$

where the coordinate x with respect to the Z_3-grading is of grade 0 and the coordinate θ with respect to the Z_3-grading is of grade 1. We call $O(\mathbb{C}_q^{1|1})$ the *algebra of functions* on the Z_3-graded quantum superplane $\mathbb{C}_q^{1|1}$.

Definition 2.2. Let $\Lambda(\mathbb{C}_q^{1|1})$ be the algebra with the generators φ and y satisfying the relations

$$\varphi y = qjy\varphi, \quad \varphi^3 = 0, \tag{2.2}$$

where the coordinate φ with respect to the Z_3-grading is of grade 1 and the coordinate *y* with respect to the Z_3-grading is of grade 2. We call $\Lambda(\mathbb{C}_q^{1|1})$ the *quantum exterior algebra* of the Z_3-graded quantum superplane $\mathbb{C}_q^{1|1}$.

Obviously, in the classical case $q = 1$, the algebra $O(\mathbb{C}_1^{1|1})$ is the Z_3-graded polynomial algebra in two commuting indeterminates and the algebra $\Lambda(\mathbb{C}_j^{1|1})$ is the exterior algebra of $\mathbb{C}^{1|1}$.

2.2. The Hopf algebra $\mathcal{A}$

Let $\mathcal{A}$ be the algebra $O(\mathbb{C}_q^{1|1})$. If we extend the algebra $\mathcal{A}$ by adding the inverse of *x* which obeys

$$xx^{-1} = 1 = x^{-1}x,$$

then we know that the algebra $\mathcal{A}$ is a Z_3-graded Hopf algebra [3]:

Theorem 2.3. *The algebra $\mathcal{A}$ is a graded Hopf algebra with the following co-structures: the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is defined by*

$$\Delta(x) = x \otimes x, \quad \Delta(\theta) = \theta \otimes x + x \otimes \theta. \tag{2.3}$$

The counit $\varepsilon : \mathcal{A} \to \mathcal{C}$ is given by

$$\varepsilon(x) = 1, \quad \varepsilon(\theta) = 0. \tag{2.4}$$

The coinverse $S : \mathcal{A} \to \mathcal{A}$ is defined by

$$S(x) = x^{-1}, \quad S(\theta) = -x^{-1}\theta x^{-1}. \tag{2.5}$$

Note that

$$\Delta(1) = 1 \otimes 1, \quad \Delta(x^{-1}) = x^{-1} \otimes x^{-1}.$$

Here, the multiplication in $\mathcal{A} \otimes \mathcal{A}$ is defined with the rule

$$(A \otimes B)(C \otimes D) = j^{grad(B)\,grad(C)} AC \otimes BD. \tag{2.6}$$

2.3. Review of a differential calculus on Z_3-graded quantum superplane

The quantum superplane underlies a noncommutative differential calculus on a smooth manifold with exterior differential d satisfying $\mathsf{d}^2 = 0$. So the above mentioned generalization of the superplane raise a natural question of possible generalization of differential calculus to one with exterior differential d satisfying $\mathsf{d}^3 = 0$. From an algebraic point of view, a sufficient algebraic structure underlying a differential calculus is the notion of the Z_3-graded differential algebra. Therefore, we can generalize a differential calculus with the help of an appropriate generalization of Z_3-graded differential algebra.

Notice that linear operator d applied to x produces a 1-form whose Z_3-grade is 1 by definition. Similarly, application of d to θ produces a 1-form whose Z_3-grade is 2. We will denote the obtained quantities by $\mathsf{d}x$, and $\mathsf{d}\theta$, respectively. When the linear operator d is applied to $\mathsf{d}x$ (or twice by iteration to x), it will produce a new entity we will call a 1-*form of grade* 2, denoted by d^2x and applied to $\mathsf{d}\theta$ produces a 1-form of grade 0, modulo 3, denoted by $\mathsf{d}^2\theta$. Finally, we require that $\mathsf{d}^3 = 0$.

A differential calculus on an arbitrary algebra $\mathcal{X}$ is an $\mathcal{X}$-bimodule Γ with a $\mathbb{C}$-linear exterior differential operator $\mathsf{d} : \mathcal{X} \to \Gamma$ such that

(i) d satisfies the Leibniz rule $\mathsf{d}(f \cdot g) = (\mathsf{d}f) \cdot g + f \cdot \mathsf{d}g$ for any $f, g \in \mathcal{X}$,

(ii) Γ is the linear span of elements of the form $a \cdot \mathsf{d}b \cdot c$ with $a, b, c \in \mathcal{X}$.

Natural generalization of a usual calculus leads to the following definition: Let $\mathcal{X}$ be a quantum space for a Hopf algebra $\mathcal{H}$ and $\tau : \Gamma \to \Gamma$ is the linear map of grade zero which gives

$$\tau(a) = j^{grad(a)}a, \quad \forall a \in \mathcal{X}. \tag{2.7}$$

Definition 2.4. We consider a map $\phi_L : \Gamma \to \Gamma \otimes \mathcal{A}$ such that

$$\phi_L \circ \mathsf{d} = (\tau \otimes \mathsf{d}) \circ \Delta. \tag{2.8}$$

Then we have

$$\phi_L(\mathsf{d}x) = x \otimes \mathsf{d}x, \quad \phi_L(\mathsf{d}\theta) = j\theta \otimes \mathsf{d}x + x \otimes \mathsf{d}\theta. \tag{2.9}$$

We now define a map Δ_L as follows:

$$\Delta_L(a_1 \cdot \mathsf{d}b_1 + \mathsf{d}b_2 \cdot a_2) = \Delta(a_1)\phi_L(\mathsf{d}b_1) + \phi_L(\mathsf{d}b_2)\Delta(a_2), \tag{2.10}$$

for all $a_i \in \mathcal{X}$ and $\mathsf{d}a_i \in \Gamma$.

Definition 2.5. A differential calculus over the algebra Γ on the quantum space $\mathcal{X}$ with left coaction $\varphi : \mathcal{X} \to \mathcal{H} \otimes \mathcal{X}$ is called *left covariant* with respect to $\mathcal{H}$ if there exists a left coaction $\Delta_L : \Gamma \to \mathcal{H} \otimes \Gamma$ of $\mathcal{H}$ on Γ satisfying equation (2.10) and such that $\Delta_L(\mathsf{d}a) = (\tau \otimes \mathsf{d})\varphi(a)$ for all $a \in \mathcal{X}$.

A noncommutative differential calculus on the Z_3-graded quantum superplane was given in [3].

Theorem 2.6. *The commutation relations for the differential calculus* Γ *on the* Z_3*-graded quantum superplane take the following explicit form*:

(1) *the commutation relations with the coordinates of the first order differentials*

$$\begin{aligned} x \cdot \mathsf{d}x &= j^2 \mathsf{d}x \cdot x, \\ x \cdot \mathsf{d}\theta &= q\mathsf{d}\theta \cdot x + (j^2 - 1)\mathsf{d}x \cdot \theta, \\ \theta \cdot \mathsf{d}x &= q^{-1} j\mathsf{d}x \cdot \theta, \\ \theta \cdot \mathsf{d}\theta &= j\mathsf{d}\theta \cdot \theta, \end{aligned} \tag{2.11}$$

(2) *the commutation relations between the first order differentials have the form*

$$\mathsf{d}x \wedge \mathsf{d}\theta = qj\mathsf{d}\theta \wedge \mathsf{d}x, \quad \mathsf{d}x \wedge \mathsf{d}x \wedge \mathsf{d}x := (\mathsf{d}x)^3 = 0, \tag{2.12}$$

(3) *the commutation relations with the coordinates of the second order differentials have the form*

$$x \cdot \mathrm{d}^2 x = j^2 \mathrm{d}^2 x \cdot x,$$

$$x \cdot \mathrm{d}^2\theta = q\mathrm{d}^2\theta \cdot x + (j^2 - 1)\mathrm{d}^2 x \cdot \theta,$$

$$\theta \cdot \mathrm{d}^2 x = q^{-1}\mathrm{d}^2 x \cdot \theta,$$

$$\theta \cdot \mathrm{d}^2\theta = \mathrm{d}^2\theta \cdot \theta, \tag{2.13}$$

(4) *the commutation relations between the first order differentials and the second order differentials have the form*

$$\mathrm{d}x \wedge \mathrm{d}^2 x = j\mathrm{d}^2 x \wedge \mathrm{d}x,$$

$$\mathrm{d}x \wedge \mathrm{d}^2\theta = q\mathrm{d}^2\theta \wedge \mathrm{d}x + (j - j^2)\mathrm{d}^2 x \wedge \mathrm{d}\theta,$$

$$\mathrm{d}\theta \wedge \mathrm{d}^2 x = q^{-1} j^2 \mathrm{d}^2 x \wedge \mathrm{d}\theta,$$

$$\mathrm{d}\theta \wedge \mathrm{d}^2\theta = \mathrm{d}^2\theta \wedge \mathrm{d}\theta, \tag{2.14}$$

(5) *the commutation relations between the second order differentials have the form*

$$\mathrm{d}^2 x \wedge \mathrm{d}^2\theta = qj^2\mathrm{d}^2\theta \wedge \mathrm{d}^2 x, \tag{2.15}$$

(6) *the relations of the coordinates with their partial derivatives*:

$$\partial_x x = 1 + j^2 x\partial_x + (j^2 - 1)\theta\partial_\theta,$$

$$\partial_x \theta = q^{-1} j^2 \theta\partial_x,$$

$$\partial_\theta x = qx\partial_\theta,$$

$$\partial_\theta\theta = 1 + j^2\theta\partial_\theta, \tag{2.16}$$

(7) *the relations of partial derivatives*:

$$\partial_x\partial_\theta = qj\partial_\theta\partial_x, \quad \partial_\theta^3 = 0. \tag{2.17}$$

Definition 2.7. If f is a differentiable function of x and θ, then the first order differential of f is defined as

$$\mathrm{d}f = (\mathrm{d}x\partial_x + \mathrm{d}\theta\partial_\theta)f.$$

Now, we need some useful relations which will be necessary to construct Z_3 extended calculi.

Proposition 2.8. *The relations between partial derivatives and first order differentials are*

$$\partial_x \mathrm{d}x = j\mathrm{d}x\partial_x,$$

$$\partial_x \mathrm{d}\theta = q^{-1}\mathrm{d}\theta\partial_x,$$

$$\partial_\theta \mathrm{d}x = qj^2\mathrm{d}x\partial_\theta,$$

$$\partial_\theta \mathrm{d}\theta = j^2\mathrm{d}\theta\partial_\theta + (j^2 - j)\mathrm{d}x\partial_x. \tag{2.18}$$

Proof. For completing the proof, we will assume the following form:

$$\partial_x \mathrm{d}x = F_1\mathrm{d}x\partial_x + F_2\mathrm{d}\theta\partial_\theta,$$

$$\partial_x \mathrm{d}\theta = F_3\mathrm{d}\theta\partial_x + F_4\mathrm{d}x\partial_\theta,$$

$$\partial_\theta \mathrm{d}x = F_5\mathrm{d}x\partial_\theta + F_6\mathrm{d}\theta\partial_x,$$

$$\partial_\theta \mathrm{d}\theta = F_7\mathrm{d}\theta\partial_\theta + F_8\mathrm{d}x\partial_x.$$

Applying ∂_x and ∂_θ to the relations (2.11), we obtain

$$F_1 = j, \quad F_2 = 0, \quad F_3 = q^{-1}, \quad F_4 = 0,$$

$$F_5 = qj^2, \quad F_6 = 0, \quad F_7 = j^2, \quad F_8 = (j^2 - j).$$

Proposition 2.9. *The relations between partial derivatives and second order differentials are*

$$\partial_x \mathrm{d}^2x = j\mathrm{d}^2x\partial_x,$$

$$\partial_x \mathrm{d}^2\theta = q^{-1}\mathrm{d}^2\theta\partial_x,$$

$$\partial_\theta \mathrm{d}^2 x = q \mathrm{d}^2 x \partial_\theta,$$

$$\partial_\theta \mathrm{d}^2 \theta = \mathrm{d}^2 \theta \partial_\theta + (1 - j^2) \mathrm{d}^2 x \partial_x. \tag{2.19}$$

Proof. In order to complete the proof, we will assume the following form:

$$\partial_x \mathrm{d}^2 x = E_1 \mathrm{d}^2 x \partial_x + E_2 \mathrm{d}^2 \theta \partial_\theta,$$

$$\partial_x \mathrm{d}^2 \theta = E_3 \mathrm{d}^2 \theta \partial_x + E_4 \mathrm{d}^2 x \partial_\theta,$$

$$\partial_\theta \mathrm{d}^2 x = E_5 \mathrm{d}^2 x \partial_\theta + E_6 \mathrm{d}^2 \theta \partial_x,$$

$$\partial_\theta \mathrm{d}^2 \theta = E_7 \mathrm{d}^2 \theta \partial_\theta + E_8 \mathrm{d}^2 x \partial_x.$$

Applying ∂_x and ∂_θ to the relations (2.13), (2.14) and (2.15), we obtain

$$E_1 = j, \quad E_2 = 0, \quad E_3 = q^{-1}, \quad E_4 = 0,$$

$$E_5 = q, \quad E_6 = 0, \quad E_7 = 1, \quad E_8 = (1 - j^2).$$

Proposition 2.10. *The relations between exterior derivative and first order differentials are*

$$\mathrm{d}(\mathrm{d}x) = j(\mathrm{d}x)\mathrm{d}, \quad \mathrm{d}(\mathrm{d}\theta) = j^2(\mathrm{d}\theta)\mathrm{d} \tag{2.20}$$

and the relations between exterior derivative and second order differentials are

$$\mathrm{d}(\mathrm{d}^2 x) = j^2(\mathrm{d}^2 x)\mathrm{d}, \quad \mathrm{d}(\mathrm{d}^2\theta) = (\mathrm{d}^2\theta)\mathrm{d} \tag{2.21}$$

and the relations between exterior derivative and partial derivatives are

$$\partial_x \mathrm{d} = j \mathrm{d} \partial_x, \quad \partial_\theta \mathrm{d} = j^2 \mathrm{d} \partial_\theta. \tag{2.22}$$

Proof. If we apply the exterior differential d to $\mathrm{d}x$ and use the relations (2.18), then we get

$$\begin{aligned}
\mathrm{d}(\mathrm{d}x) &= (\mathrm{d}x\partial_x + \mathrm{d}\theta\partial_\theta)\mathrm{d}x \\
&= \mathrm{d}x\partial_x\mathrm{d}x + \mathrm{d}\theta\partial_\theta\mathrm{d}x \\
&= j\mathrm{d}x\mathrm{d}x\partial_x + qj^2\mathrm{d}\theta\mathrm{d}x\partial_\theta \\
&= j\mathrm{d}x\mathrm{d}x\partial_x + j\mathrm{d}x\mathrm{d}\theta\partial_\theta \\
&= j\mathrm{d}x(\mathrm{d}x\partial_x + \mathrm{d}y\partial_y) \\
&= j(\mathrm{d}x)\mathrm{d}.
\end{aligned}$$

Accordingly, if we apply it to the second order differentials and partial derivatives (by using (2.19) and (2.18)), then other relations can be found in the similar manner.

3. Extended Calculi on the Z_3-graded Quantum Superplane

There is a relationship of the exterior derivative with the Lie derivative and to describe this relation, we introduce a new operator: the inner derivation. Hence the differential calculi on Z_3-graded quantum superplane can be extended into a large calculi. We call this new calculus the *Cartan calculi*. The connection of the inner derivation, denoted by $\boldsymbol{i}_a$, and the Lie derivative, denoted by $\mathcal{L}_a$, is given by the Cartan formula:

$$\mathcal{L}_a = \boldsymbol{i}_a \circ \mathrm{d} + \mathrm{d} \circ \boldsymbol{i}_a.$$

This and other formulae are explained in [5, 13-15].

The exterior derivative and the Lie derivative are set to cover the idea of a derivative in different ways. These differences can be hasped together by introducing the idea of an antiderivation which is called an *inner derivation*.

3.1. Inner derivations

Let us begin with some information about the inner derivations. Generally, for a smooth vector field X on a manifold the inner derivation, denoted by $\boldsymbol{i}_X$, is a linear operator which maps k-forms to $(k-1)$-forms. If we define the inner derivation $\boldsymbol{i}_X$ on the set of all differential forms on a manifold, then we know that $\boldsymbol{i}_X$ is an antiderivation:

$$\boldsymbol{i}_X(\alpha \wedge \beta) = (\boldsymbol{i}_X\alpha) \wedge \beta + j^k \alpha \wedge (\boldsymbol{i}_X\beta),$$

where α and β are both differential forms. The inner derivation $\boldsymbol{i}_X$ acts on 0- and 1-forms as follows:

$$\boldsymbol{i}_X(f) = 0, \quad \boldsymbol{i}_X(\mathrm{d}f) = X(f).$$

In order to obtain the commutation rules of the coordinates with inner derivations and the other relations, we will use the approach of [2].

Proposition 3.1. *The commutation relations of the inner derivations with the partial derivatives are*

$$\begin{aligned}
\boldsymbol{i}_x\partial_x &= j\partial_x\boldsymbol{i}_x,\\
\boldsymbol{i}_x\partial_\theta &= qj^2\partial_\theta\boldsymbol{i}_x + (j^2 - j)\partial_x\boldsymbol{i}_\theta,\\
\boldsymbol{i}_\theta\partial_x &= q^{-1}\partial_x\boldsymbol{i}_\theta,\\
\boldsymbol{i}_\theta\partial_\theta &= j^2\partial_\theta\boldsymbol{i}_\theta.
\end{aligned} \tag{3.1}$$

Proof. If we assume that the commutation relations of the inner derivations with the partial derivatives ∂_x and ∂_θ are in the following form:

$$\begin{aligned}
\boldsymbol{i}_x\partial_x &= B_1\partial_x\boldsymbol{i}_x + B_2\partial_\theta\boldsymbol{i}_\theta,\\
\boldsymbol{i}_x\partial_\theta &= B_3\partial_\theta\boldsymbol{i}_x + B_4\partial_x\boldsymbol{i}_\theta,\\
\boldsymbol{i}_\theta\partial_x &= B_5\partial_x\boldsymbol{i}_\theta + B_6\partial_\theta\boldsymbol{i}_x,\\
\boldsymbol{i}_\theta\partial_\theta &= B_7\partial_\theta\boldsymbol{i}_\theta + B_8\partial_x\boldsymbol{i}_x,
\end{aligned}$$

then the proof reduces to find the coefficients B_k $(1 \le k \le 8)$. To find them, if we apply $\boldsymbol{i}_x$ and $\boldsymbol{i}_\theta$ to the relations (2.11), then we obtain

$$B_1 = j, \quad B_2 = 0, \quad B_3 = qj^2, \quad B_4 = j^2 - j,$$

$$B_5 = q^{-1}, \quad B_6 = 0, \quad B_7 = j^2, \quad B_8 = 0.$$

We now wish to find the commutation relations between the coordinates x, θ and the inner derivations associated with them.

Proposition 3.2. (a) *The commutation relations of the inner derivations with* x *and* θ *are*

$$\begin{aligned} \boldsymbol{i}_x x &= j^2 x \boldsymbol{i}_x + (j^2 - 1)\theta \boldsymbol{i}_\theta, \\ \boldsymbol{i}_x \theta &= q^{-1} j \theta \boldsymbol{i}_x, \\ \boldsymbol{i}_\theta x &= q x \boldsymbol{i}_\theta, \\ \boldsymbol{i}_\theta \theta &= j \theta \boldsymbol{i}_\theta, \end{aligned} \tag{3.2}$$

(b) *the commutation relations between the first order differentials and the inner derivations are*

$$\begin{aligned} \boldsymbol{i}_x \wedge \mathrm{d}x &= 1 + j^2 \mathrm{d}x \wedge \boldsymbol{i}_x + (j^2 - 1)\mathrm{d}\theta \wedge \boldsymbol{i}_\theta, \\ \boldsymbol{i}_x \wedge \mathrm{d}\theta &= q^{-1} j \mathrm{d}\theta \wedge \boldsymbol{i}_x, \\ \boldsymbol{i}_\theta \wedge \mathrm{d}x &= q j \mathrm{d}x \wedge \boldsymbol{i}_\theta, \\ \boldsymbol{i}_\theta \wedge \mathrm{d}\theta &= 1 + j^2 \mathrm{d}\theta \wedge \boldsymbol{i}_\theta \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} \boldsymbol{i}_x \wedge \mathrm{d}x &= 1 + j \mathrm{d}x \wedge \boldsymbol{i}_x + (j - j^2)\mathrm{d}\theta \wedge \boldsymbol{i}_\theta, \\ \boldsymbol{i}_x \wedge \mathrm{d}\theta &= q^{-1} \mathrm{d}\theta \wedge \boldsymbol{i}_x, \\ \boldsymbol{i}_\theta \wedge \mathrm{d}x &= q \mathrm{d}x \wedge \boldsymbol{i}_\theta, \\ \boldsymbol{i}_\theta \wedge \mathrm{d}\theta &= 1 + j \mathrm{d}\theta \wedge \boldsymbol{i}_\theta. \end{aligned} \tag{3.4}$$

Proof. In order to obtain the commutation rules of the coordinates with inner derivations, we shall assume that they are of the following form:

$$\begin{aligned} \boldsymbol{i}_x x &= A_1 x \boldsymbol{i}_x + A_2 \theta \boldsymbol{i}_\theta, \\ \boldsymbol{i}_x \theta &= A_3 \theta \boldsymbol{i}_x + A_4 x \boldsymbol{i}_\theta, \end{aligned}$$

$$\boldsymbol{i}_\theta x = A_5 x \boldsymbol{i}_\theta + A_6 \theta \boldsymbol{i}_x,$$

$$\boldsymbol{i}_\theta \theta = A_7 \theta \boldsymbol{i}_\theta + A_8 x \boldsymbol{i}_x.$$

The coefficients A_k $(1 \le k \le 8)$ will be determined in terms of the deformation parameters q and j. But the use of the relations (2.1) does not give rise any solution in terms of the parameters q and j. However, we have

$$A_4(A_1 - qA_5) = 0, \quad A_2(A_7 - qA_3) = 0, \quad A_2 A_8 - qA_4 A_6 = 0,$$

$$A_8(A_5 - qA_1) = 0, \quad A_6(A_3 - qA_7) = 0, \quad A_4 A_6 - q^2 A_8 A_2 = 0.$$

To find the coefficients, we need the commutation relations of the inner derivations with the differentials of x and θ. Since

$$\boldsymbol{i}_{X_i}(\mathrm{d}X_j) = \delta_{ij}, \quad (X_1 = x,\ X_2 = \theta),$$

we can assume that the relations between the differentials and the inner derivations are of the following form:

$$\boldsymbol{i}_x \wedge \mathrm{d}x = 1 + a_1 \mathrm{d}x \wedge \boldsymbol{i}_x + a_2 \mathrm{d}\theta \wedge \boldsymbol{i}_\theta,$$

$$\boldsymbol{i}_x \wedge \mathrm{d}\theta = a_3 \mathrm{d}\theta \wedge \boldsymbol{i}_x + a_4 \mathrm{d}x \wedge \boldsymbol{i}_\theta,$$

$$\boldsymbol{i}_\theta \wedge \mathrm{d}x = a_5 \mathrm{d}x \wedge \boldsymbol{i}_\theta + a_6 \mathrm{d}\theta \wedge \boldsymbol{i}_x,$$

$$\boldsymbol{i}_\theta \wedge \mathrm{d}\theta = 1 + a_7 \mathrm{d}\theta \wedge \boldsymbol{i}_\theta + a_8 \mathrm{d}x \wedge \boldsymbol{i}_x.$$

Applying $\boldsymbol{i}_x$ and $\boldsymbol{i}_\theta$ to the relations (2.11), one gets

$$A_1 = j^2, \quad A_2 = j^2 - 1, \quad A_3 = q^{-1} j, \quad A_4 = 0,$$

$$A_5 = q, \quad A_6 = 0, \quad A_7 = j, \quad A_8 = 0$$

and

$$a_2(qA_1 - j^2 A_5) = 0, \quad jA_2 a_6 - j^2 a_2 A_6 = 0, \quad a_2 A_8 = 0,$$

$$A_4(qa_7 - ja_3) = 0, \quad j^2 A_4 a_8 - ja_4 A_8 = 0, \quad a_6 A_4 = 0, \text{ etc.}$$

Using the relations (3.2), (2.12) and (3.1), we obtain

$$a_4 = a_6 = 0, \quad a_1^2 + a_1 + 1 = 0.$$

Thus, we can take $a_1 = j$ or $a_1 = j^2$. If we demand that the commutation rules of the inner derivations with d are of the form

$$\boldsymbol{i}_x \circ \mathsf{d} + C_1 \mathsf{d} \circ \boldsymbol{i}_x = \partial_x, \quad \boldsymbol{i}_\theta \circ \mathsf{d} + C_2 \mathsf{d} \circ \boldsymbol{i}_\theta = \partial_\theta,$$

then one has

$$a_1 = j^2, \quad C_1 = -1, \quad a_2 = j^2 - 1, \quad a_3 = q^{-1} j, \quad a_4 = 0, \quad a_5 = qj,$$

$$C_2 = -j, \quad a_6 = 0, \quad a_7 = j^2, \quad a_8 = 0$$

or

$$a_1 = j, \quad C_1 = -j^2, \quad a_2 = j - j^2, \quad a_3 = q^{-1}, \quad a_4 = 0, \quad a_5 = q,$$

$$C_2 = -1, \quad a_6 = 0, \quad a_7 = j, \quad a_8 = 0.$$

Proposition 3.3. *The commutation relations between the second order differentials and the inner derivations are*

$$\boldsymbol{i}_x \wedge \mathsf{d}^2 x = j^2 \mathsf{d}^2 x \wedge \boldsymbol{i}_x,$$

$$\boldsymbol{i}_x \wedge \mathsf{d}^2 \theta = q^{-1} \mathsf{d}^2 \theta \wedge \boldsymbol{i}_x,$$

$$\boldsymbol{i}_\theta \wedge \mathsf{d}^2 x = qj \mathsf{d}^2 x \wedge \boldsymbol{i}_\theta,$$

$$\boldsymbol{i}_\theta \wedge \mathsf{d}^2 \theta = \mathsf{d}^2 \theta \wedge \boldsymbol{i}_\theta + (1 - j^2) \mathsf{d}^2 x \wedge \boldsymbol{i}_x. \tag{3.5}$$

Proof. In order to find the relations between the second order differentials and the inner derivations, we will assume the following form:

$$\boldsymbol{i}_x \wedge \mathsf{d}^2 x = C_3 + k_1 \mathsf{d}^2 x \wedge \boldsymbol{i}_x + k_2 \mathsf{d}^2 \theta \wedge \boldsymbol{i}_\theta,$$

$$\boldsymbol{i}_x \wedge \mathsf{d}^2 \theta = k_3 \mathsf{d}^2 \theta \wedge \boldsymbol{i}_x + k_4 \mathsf{d}^2 x \wedge \boldsymbol{i}_\theta,$$

$$\boldsymbol{i}_\theta \wedge \mathsf{d}^2 x = k_5 \mathsf{d}^2 x \wedge \boldsymbol{i}_\theta + k_6 \mathsf{d}^2 \theta \wedge \boldsymbol{i}_x,$$

$$\boldsymbol{i}_\theta \wedge \mathrm{d}^2\theta = C_4 + k_7 \mathrm{d}^2\theta \wedge \boldsymbol{i}_\theta + k_8 \mathrm{d}^2 x \wedge \boldsymbol{i}_x.$$

Using the relations (2.13)-(2.15) with $\boldsymbol{i}_x$ and $\boldsymbol{i}_\theta$, we obtain

$$C_3 = 0, \quad k_1 = j^2, \quad k_3 = q^{-1}, \quad k_5 = qj, \quad k_7 = 1,$$

$$C_4 = 0, \quad k_2 = k_4 = k_6 = 0, \quad k_8 = 1 - j^2.$$

3.2. Lie derivatives

We know, from the classical differential geometry that the Lie derivative $\mathcal{L}$ can be defined as a linear map from the exterior algebra into itself which takes k-forms to k-forms. For a 0-form, that is, an ordinary function f, the Lie derivative is just the contraction of the exterior derivative with the vector field X:

$$\mathcal{L}_X f = \boldsymbol{i}_X \mathrm{d} f.$$

For a general differential form, the Lie derivative is likewise a contraction, taking into account the variation in X:

$$\mathcal{L}_X \alpha = \boldsymbol{i}_X \mathrm{d}\alpha + \mathrm{d}(\boldsymbol{i}_X \alpha).$$

The Lie derivative has an important property when acting on differential forms. If α and β are two differential forms on M, then

$$\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + j^k \alpha \wedge (\mathcal{L}_X \beta),$$

where α is a k-form.

In this section, we find the commutation rules of the Lie derivatives with the elements of the algebra $\mathcal{A}$, their differentials, etc., using the approach of [2].

Proposition 3.4. *In the* Z_3*-graded space, the formulae of the Lie derivative are given by*

$$\mathcal{L}_x = \boldsymbol{i}_x \circ \mathrm{d} - j\mathrm{d} \circ \boldsymbol{i}_x,$$

$$\mathcal{L}_\theta = \boldsymbol{i}_\theta \circ \mathsf{d} - j^2\mathsf{d} \circ \boldsymbol{i}_\theta. \tag{3.6}$$

Proof. Let

$$\mathcal{L}_x = \boldsymbol{i}_x \circ \mathsf{d} + C\mathsf{d} \circ \boldsymbol{i}_x, \quad \mathcal{L}_\theta = \boldsymbol{i}_\theta \circ \mathsf{d} + D\mathsf{d} \circ \boldsymbol{i}_\theta.$$

If we apply d to these formulae, using the relations (2.20), we get

$$C = -j, \quad D = -j^2.$$

Proposition 3.5. *The commutation relations of the Lie derivatives with the elements of* $\mathcal{A}$ *are*

$$\mathcal{L}_x x = 1 + j^2 x\mathcal{L}_x + (j^2 - 1)\theta\mathcal{L}_\theta + j^2(j-1)^2\mathsf{d}\theta \wedge \boldsymbol{i}_\theta + (j^2 - 1)\mathsf{d}x \wedge \boldsymbol{i}_x,$$

$$\mathcal{L}_x\theta = q^{-1}j^2\theta\mathcal{L}_x + q^{-1}(j - j^2)\mathsf{d}\theta \wedge \boldsymbol{i}_x,$$

$$\mathcal{L}_\theta x = qx\mathcal{L}_\theta + q(j - j^2)\mathsf{d}x \wedge \boldsymbol{i}_\theta,$$

$$\mathcal{L}_\theta\theta = 1 + j^2\theta\mathcal{L}_\theta + (j^2 - 1)\mathsf{d}\theta \wedge \boldsymbol{i}_\theta \tag{3.7}$$

or

$$\mathcal{L}_x x = 1 + j^2 x\mathcal{L}_x + (j^2 - 1)\theta\mathcal{L}_\theta - j(j-1)^2\mathsf{d}\theta \wedge \boldsymbol{i}_\theta + (j - 1)\mathsf{d}x \wedge \boldsymbol{i}_x,$$

$$\mathcal{L}_x\theta = q^{-1}j^2\theta\mathcal{L}_x + q^{-1}(1 - j^2)\mathsf{d}\theta \wedge \boldsymbol{i}_x,$$

$$\mathcal{L}_\theta x = qx\mathcal{L}_\theta + q(1 - j^2)\mathsf{d}x \wedge \boldsymbol{i}_\theta,$$

$$\mathcal{L}_\theta\theta = 1 + j^2\theta\mathcal{L}_\theta + (j^2 - 1)\mathsf{d}\theta \wedge \boldsymbol{i}_\theta. \tag{3.8}$$

Proof. Using the relations (3.3) and (3.4), we get

$$\begin{aligned}
\mathcal{L}_x x &= (\boldsymbol{i}_x \circ \mathsf{d} - j\mathsf{d} \circ \boldsymbol{i}_x)x \\
&= 1 + j^2 x(\boldsymbol{i}_x \circ \mathsf{d} - j\mathsf{d} \circ \boldsymbol{i}_x) + (j^2 - 1)\theta(\boldsymbol{i}_\theta \circ \mathsf{d} - j^2\mathsf{d} \circ \boldsymbol{i}_\theta) \\
&\quad + (j^2 - 1)\mathsf{d}x \wedge \boldsymbol{i}_x + \cdots \\
&= 1 + j^2 x\mathcal{L}_x + (j^2 - 1)\theta\mathcal{L}_\theta + (j^2 - 1)[(1 - j)\mathsf{d}\theta \wedge \boldsymbol{i}_\theta + \mathsf{d}x \wedge \boldsymbol{i}_x].
\end{aligned}$$

Other relations can be similarly obtained.

Proposition 3.6. *The relations of the Lie derivatives with the first order differentials are*

$$\mathcal{L}_x \mathrm{d}x = \mathrm{d}x\mathcal{L}_x + (1-j)\mathrm{d}\theta\mathcal{L}_\theta,$$

$$\mathcal{L}_x \mathrm{d}\theta = q^{-1}\mathrm{d}\theta\mathcal{L}_x,$$

$$\mathcal{L}_\theta \mathrm{d}x = qj^2\mathrm{d}x\mathcal{L}_\theta,$$

$$\mathcal{L}_\theta \mathrm{d}\theta = j\mathrm{d}\theta\mathcal{L}_\theta \tag{3.9}$$

or

$$\mathcal{L}_x \mathrm{d}x = j^2\mathrm{d}x\mathcal{L}_x + (j^2-1)\mathrm{d}\theta\mathcal{L}_\theta,$$

$$\mathcal{L}_x \mathrm{d}\theta = q^{-1}j^2\mathrm{d}\theta\mathcal{L}_x,$$

$$\mathcal{L}_\theta \mathrm{d}x = qj\mathrm{d}x\mathcal{L}_\theta,$$

$$\mathcal{L}_\theta \mathrm{d}\theta = \mathrm{d}\theta\mathcal{L}_\theta. \tag{3.10}$$

Proof. Using the relations (2.20) and (3.3) or (3.4), we get

$$\begin{aligned}
\mathcal{L}_x \mathrm{d}x &= (\boldsymbol{i}_x \circ \mathrm{d} - j\mathrm{d} \circ \boldsymbol{i}_x) \wedge \mathrm{d}x \\
&= (\boldsymbol{i}_x \circ \mathrm{d}) \wedge (\mathrm{d}x) - j(\mathrm{d} \circ \boldsymbol{i}_x) \wedge \mathrm{d}x \\
&= j\boldsymbol{i}_x \wedge \mathrm{d}x \wedge \mathrm{d} - j\mathrm{d} - \mathrm{d} \wedge (\mathrm{d}x)\boldsymbol{i}_x - j(j^2-1)(\mathrm{d} \wedge \mathrm{d}\theta) \wedge \boldsymbol{i}_\theta \\
&= j\mathrm{d} + \mathrm{d}x \wedge (\boldsymbol{i}_x \circ \mathrm{d}) + j(j^2-1)\mathrm{d}\theta \wedge (\boldsymbol{i}_x \circ \mathrm{d}) - j\mathrm{d} - j\mathrm{d}x \wedge \mathrm{d} \circ \boldsymbol{i}_x \\
&\quad - (j^2-1)(\mathrm{d} \wedge \mathrm{d}\theta) \wedge \boldsymbol{i}_\theta \\
&= \mathrm{d}x \wedge (\boldsymbol{i}_x \circ \mathrm{d} - j\mathrm{d} \circ \boldsymbol{i}_x) + (1-j)\mathrm{d}\theta \wedge (\boldsymbol{i}_\theta \circ \mathrm{d} - j^2\mathrm{d} \circ \boldsymbol{i}_\theta) \\
&= \mathrm{d}x\mathcal{L}_x + (1-j)\mathrm{d}\theta\mathcal{L}_\theta.
\end{aligned}$$

The other relations can be similarly obtained.

Proposition 3.7. *The relations of the Lie derivatives with the second order differentials are*

$$\mathcal{L}_x \mathrm{d}^2 x = j\mathrm{d}^2 x \mathcal{L}_x,$$

$$\mathcal{L}_x \mathrm{d}^2 \theta = q^{-1}\mathrm{d}^2 \theta \mathcal{L}_x,$$

$$\mathcal{L}_\theta \mathrm{d}^2 x = q\mathrm{d}^2 x \mathcal{L}_\theta,$$

$$\mathcal{L}_\theta \mathrm{d}^2 \theta = \mathrm{d}^2 \theta \mathcal{L}_\theta + (1 - j^2)\mathrm{d}^2 x \mathcal{L}_x. \tag{3.11}$$

Proof. Using the relations (3.7), (2.21) and (2.22), the relations of the Lie derivatives with second order differentials also can be similarly obtained. Other commutation relations can be similarly obtained.

To complete the description of the above scheme, below we get the remaining commutation relations as follows:

Proposition 3.8. *The Lie derivatives and partial derivatives are*

$$\mathcal{L}_x \partial_x = \partial_x \mathcal{L}_x,$$

$$\mathcal{L}_x \partial_\theta = q\partial_\theta \mathcal{L}_x + (1 - j^2)\partial_x \mathcal{L}_\theta,$$

$$\mathcal{L}_\theta \partial_x = q^{-1} j^2 \partial_x \mathcal{L}_\theta,$$

$$\mathcal{L}_\theta \partial_\theta = \partial_\theta \mathcal{L}_\theta. \tag{3.12}$$

Proof. Using the relations (2.22) and (3.6), the relations of the Lie derivatives with partial derivatives also can be similarly obtained.

Proposition 3.9. *The relations of the inner derivations are*

$$\boldsymbol{i}_x \wedge \boldsymbol{i}_\theta = qj^2 \boldsymbol{i}_\theta \wedge \boldsymbol{i}_x, \quad \boldsymbol{i}_x^3 = 0. \tag{3.13}$$

Proposition 3.10. *The commutation relations between the Lie derivatives and the inner derivations are*

$$\boldsymbol{i}_x \mathcal{L}_x = \mathcal{L}_x \boldsymbol{i}_x,$$

$$\boldsymbol{i}_x \mathcal{L}_\theta = q\mathcal{L}_\theta \boldsymbol{i}_x,$$

$$\boldsymbol{i}_\theta \mathcal{L}_x = q^{-1}\mathcal{L}_x \boldsymbol{i}_\theta,$$

$$\boldsymbol{i}_\theta \mathcal{L}_\theta = j\mathcal{L}_\theta \boldsymbol{i}_\theta. \tag{3.14}$$

Proposition 3.11. *The commutation relations of the Lie derivatives are*

$$\mathcal{L}_x \mathcal{L}_\theta = q\mathcal{L}_\theta \mathcal{L}_x + (j^2 - j)\mathrm{d}\boldsymbol{i}_x \mathcal{L}_\theta. \tag{3.15}$$

Note. The Lie derivatives can be written as follows:

$$\mathcal{L}_x = \partial_x + (1 - j)\mathrm{d}\boldsymbol{i}_x, \quad \mathcal{L}_\theta = \partial_\theta + (j - j^2)\mathrm{d}\boldsymbol{i}_\theta. \tag{3.16}$$

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