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NOTE ON NON-NEWTONIAN FLOW DUE TO A VARIABLE SHEAR STRESS

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Abstract

The flow of an incompressible second-grade fluid along an infinite permeable wall is discussed. The velocity field due to a variable shear stress on the porous boundary is examined. The non-linear partial differential equation resulting from the momentum equation is solved analytically. Effects of suction and material parameter of second-grade fluid on the flow phenomena are analyzed. The solution of the problem indicates that for small time a strong non-Newtonian effect occurs in the velocity field and for large time the velocity field gives the results for Newtonian case.

1. Introduction

It is now generally recognized that in industrial applications non-Newtonian fluids are more appropriate than Newtonian fluids. Numerous models have been suggested for non-Newtonian fluids with their constitutive equations varying greatly in complexity. Already the

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class of flows for which an exact solution is possible for Navier-Stokes equations that govern the flow of Newtonian fluids is rather restricted. This class is further narrowed down for non-Newtonian fluids on account of the non-linear relationship between the stress and the rate of strain at any point of the flow. One particular class of fluids for which one can reasonably hope to derive exact solutions is the class of viscoelastic fluids, which was first introduced by Rivlin and Ericksen [7]. The constitutive equation for one of the simpler models in this class is

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where $\mathbf{T}$ is the stress tensor, p is the scalar pressure and μ, α_1 and α_2 are measurable material constants. They denote, respectively, the viscosity, elasticity and cross-viscosity. These material constants can be determined from viscometric flows for any real fluid. $\mathbf{A}_1$ and $\mathbf{A}_2$ are Rivlin-Ericksen tensors and they denote, respectively, the rate of strain and acceleration. $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined by

$$\begin{aligned} \mathbf{A}_n &= \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(\nabla\mathbf{V}) + (\nabla\mathbf{V})^T\mathbf{A}_{n-1}, \; n = 1, 2, \\ \mathbf{A}_0 &= 1. \end{aligned} \tag{2}$$

Here $\frac{d}{dt}$ is the material time derivative, and $\mathbf{V}$ is the velocity at a point.

Second order fluids are dilute polymeric solutions (e.g., polyisobutylene, methyl-methacrylate in n butyl acetate, polyethylene oxide in water, etc.). The equation is frame invariant and applicable for low shear rates. A detailed account on the characteristics of second-grade fluids is well documented by Dunn and Rajagopal [2]. Theoretical investigations by Dunn and Fosdick [1] and Fosdick and Rajagopal [3] have indicated that for an exact model, satisfying the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy be a minimum in equilibrium, the following conditions must hold:

$$\mu \geq 0, \;\; \alpha_1 \geq 0, \;\; \alpha_1 + \alpha_2 = 0. \tag{3}$$

The analysis of the flow of the second-grade fluids, in particular, and the viscoelastic fluids, in general, is more challenging mathematically and computationally, because of a peculiarity in the equations governing the fluid motion; namely, the order of the differential equation(*s*) characterizing the flow of these fluids is more than the number of the available boundary conditions. This issue is discussed in reference [6]. Later a detailed discussion is presented by Rajagopal [5] and relevant references are given in that study. In our special type of flow considered here, however, there is no need for additional boundary conditions.

In this study, the flow on a porous plate is discussed. The fluid considered is second-grade. The porous character of the plate and viscoelastic nature of the fluid increase the order the differential equation. The flow is due to a variable shear stress of magnitude of $cU_0(t)$. Finally, the effects of the material parameter of second-grade fluid are examined.

2. Mathematical Formulation

Suppose that a second-grade fluid, at rest, occupies the space above an infinitely extended porous plate in the (x, z)-plane. At time $t = 0^+$ the plate is under the action of variable shear stress with $U_0(t)$. By the influence of the shear stress, the fluid above the plate is gradually moved. The velocity field will be of the form

$$\mathbf{V} = [u(y, t), V(t), 0], \tag{4}$$

where u is the x-component of the velocity $\mathbf{V}$, $V(t) < 0$ is the suction velocity and $V(t) > 0$ is the blowing velocity. The governing equations are given by

$$div\mathbf{V} = 0, \tag{5}$$

$$\rho \frac{d\mathbf{V}}{dt} = div\mathbf{T}, \tag{6}$$

where ρ is the density of the fluid. By virtue of (4), the continuity equation is identically satisfied and the x-momentum equation reduces to

$$\rho\left[\frac{\partial u}{\partial t} + V(t)\frac{\partial u}{\partial y}\right] = \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\left[\frac{\partial^3 u}{\partial y^2 \partial t} + V(t)\frac{\partial^3 u}{\partial y^3}\right], \; y \geq 0, \; t > 0. \tag{7}$$

The corresponding boundary condition is

$$\left[\mu \frac{\partial u}{\partial y} + \alpha_1 \left\{\frac{\partial^2 u}{\partial y \partial t} + V(t) \frac{\partial^2 u}{\partial y^2}\right\}\right]_{y=0} = cU_0(t),\ t > 0. \tag{8}$$

Moreover, the natural condition

$$u(y, t) \to 0, \text{ as } y \to \infty, \tag{9}$$

also has to be satisfied.

3. Solution of the Problem

In order to obtain the solution for (7) we introduce the similarity parameter

$$\eta = \frac{y - \int V(t)dt}{2\sqrt{\nu t}} \tag{10}$$

and expand the velocity field in a series with respect to the second-grade parameter as [8]

$$u(\eta) = \left[f_0(\eta) + \left(\frac{\beta}{\nu t}\right) f_1(\eta) + \left(\frac{\beta}{\nu t}\right)^2 f_2(\eta) + \cdots\right], \tag{11}$$

where ν is the kinematic viscosity and $\beta = \frac{\alpha_1}{\rho}$.

Substituting (11) into (7) and boundary conditions (8) and (9) and then equating the equal powers, we obtain the following systems:

System of order zero

$$f_0'' + 2\eta f_0' = 0, \tag{12}$$

$$[f_0']_{\eta=\lambda(t)} = \bar{c}U_0(t), \tag{13}$$

$$f_0(\eta) \to 0 \text{ as } \eta \to \infty. \tag{14}$$

System of order one

$$f_1'' + 2\eta f_1' + 4f_1 = f_0'' + \frac{1}{2}\eta f_0''', \tag{15}$$

$$\left[f_1' + \frac{1}{2}(f_0' + \eta f_0'')\right]_{\eta=\lambda(t)} = 0, \tag{16}$$

$$f_1(\eta) \to 0 \text{ as } \eta \to \infty, \tag{17}$$

where prime denotes the differentiation with respect to η and

$$\lambda(t) = -\frac{\int V(t)dt}{2\sqrt{\nu t}}, \quad \bar{c} = \frac{2}{\rho}\sqrt{\frac{t}{\nu}}c. \tag{18}$$

Zeroth-order solution

The solution of (12) subject to boundary conditions (13) and (14) is given by

$$f_0 = A[1 - \text{erf}(\eta)]. \tag{19}$$

In order to satisfy the condition (13), we require

$$A = \bar{c}U_0(t)\left[-\frac{\sqrt{\pi}}{2}\exp\left\{\frac{-\int V(t)dt}{2\sqrt{\nu t}}\right\}^2\right]. \tag{20}$$

It is obvious from (20) that A is a function of time only. However, A must be a constant, and this is possible only if

$$\bar{c}U_0(t) = (\text{const.})\left[-\frac{2}{\sqrt{\pi}}\exp\left\{-\left(\frac{-\int V(t)dt}{2\sqrt{\nu t}}\right)^2\right\}\right]. \tag{21}$$

We observe that (21) is a relationship between $U_0(t)$ and $V(t)$ which ensures the constancy of A. So it follows that A is constant.

We further examine that (21) significantly narrows the class of possible solution of the system of order zero and even in this case the solution is two parametric

$$f_0 = f_0[y, t, V(t), U_0(t)]. \tag{22}$$

It is necessary to specify the function $U_0(t)$ in the form of (21) and at the same time $U_0(t)$ defines the value of constant A, which significantly affects the value of $f_0(\eta)$. Thus, we conclude that (19) and (21) give the solution for the system of order zero.

First-order solution

Inserting the zeroth order solution into (15)-(17) we obtain

$$f_1'' + 2\eta f_1' + 4f_1 = \frac{2A}{\sqrt{\pi}}\eta e^{-\eta^2}(3 - 2\eta^2), \tag{23}$$

$$[f_1']_{\eta=\lambda(t)} = \frac{2A}{\sqrt{\pi}} e^{-\lambda^2}(1 - 2\lambda^2), \tag{24}$$

$$f_1(\eta) \to 0 \text{ as } \eta \to \infty. \tag{25}$$

The solution of (23) subject to the conditions (24) and (25) is given by

$$f_1(\eta) = \frac{A}{\sqrt{\pi}}\left(\frac{1 - 5\lambda^2 + 2\lambda^4}{1 - 2\lambda^2}\right)\eta e^{-\eta^2} + \frac{A}{\sqrt{\pi}}\eta^3 e^{-\eta^2}. \tag{26}$$

The complete solution up to order $\left(\frac{\beta}{\nu t}\right)$ is of the following form:

$$u(\eta) = A\left[1 - \int_0^\eta e^{-\xi^2} d\xi\right] + \left(\frac{\beta}{\nu t}\right)\left[\frac{A}{\sqrt{\pi}}\left(\frac{1 - 5\lambda^2 + 2\lambda^4}{1 - 2\lambda^2}\right)\eta e^{-\eta^2} + \frac{A}{\sqrt{\pi}}\eta^3 e^{-\eta^2}\right]. \tag{27}$$

The graphs are shown in Figures 1, 2 and 3 in which velocity varies with respect to the wall, for various values of time, second-grade parameter, and suction/blowing parameter, i.e., $t = 1, \beta = 0, \cdots t = 1, \beta = 0.2, \cdots t = 1, \beta = 0.5$, and $V = -0.1, 0, 0.1$.

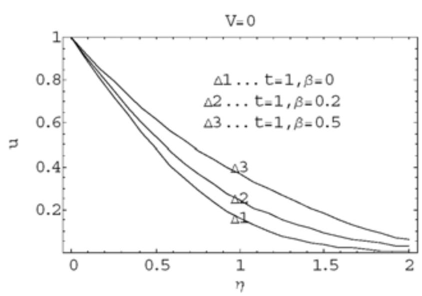

Figure 1

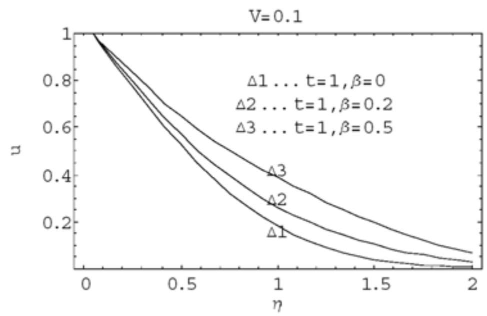

Figure 2

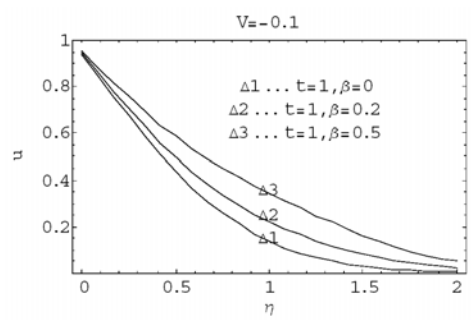

Figure 3

4. Special Cases

Case 1. If $V(t) = 0, \beta = 0,$ and the condition at the plate is of impulsive motion, then we obtain the familiar first Stokes' problem of a wall suddenly set into motion. The solution is given by

$$f(\eta) = U[1 - \operatorname{erf}(\eta^*)], \tag{28}$$

where

$$\eta^* = \frac{y}{2\sqrt{\nu t}}.$$

Case 2. If $V(t) = 0, \beta \neq 0$ and the condition at the plate is of impulsive motion, then we readily recover the result of Teipel [8], and is given by

$$f = U\left[f_0(\eta) + \left(\frac{\beta}{\nu t}\right) f_1(\eta) + \left(\frac{\beta}{\nu t}\right)^2 f_2(\eta) + \cdots\right],$$

$$f_0(\eta) = \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\xi^2} d\xi\right),$$

$$f_1(\eta) = \left(a_1 \eta + \frac{1}{\sqrt{\pi}} \eta^3\right) e^{-\eta^2},$$

$$f_2(\eta) = \left[b_1 \eta + \frac{5}{2}\left(\frac{1}{\sqrt{\pi}} - a_1 - \frac{4}{15} b_1\right)\eta^3 - \frac{1}{4}\left(\frac{11}{\sqrt{\pi}} - 4a_1\right)\eta^5 + \frac{1}{2\sqrt{\pi}}\eta^7\right] e^{-\eta^2}, \tag{29}$$

where

$$a_1 = -\frac{1}{2\sqrt{\pi}}, \quad b_1 = -\frac{3}{2}\frac{1}{\sqrt{\pi}}.$$

Case 3. For $V(t) = \frac{K}{2}\sqrt{\frac{\nu}{t}}$, $\beta = 0$, and sudden motion of the plate, we obtain the result of Jahnke et al. [4] and is given as

$$F(\eta^*) = U\left[1 - \frac{\operatorname{erf}\left(\eta^* - \frac{K}{2}\right)}{1 + \operatorname{erf}\left(\frac{K}{2}\right)}\right]. \tag{30}$$

The graph is shown for $K = -2, 0, 1, 2, 4, 6$ in Figure 4. It is observed that for $K > 23$, $F(\eta^*)$ is exactly one and for $K < -11$ it is no more real.

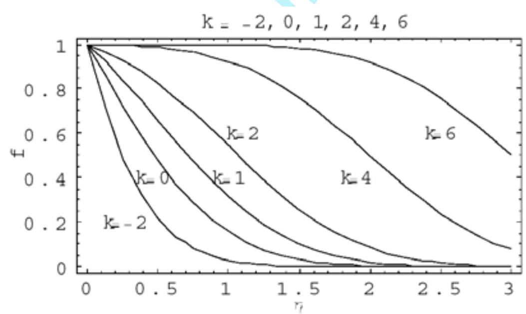

Figure 4

5. Conclusions

The main features of the solution (27) are:

1. The boundary layer thickness decreases with increase of suction

velocity and increase with large values of blowing velocity. It is also noted that the blowing velocity and the material parameter of the second-grade fluid has the similar effect on boundary layer thickness.

2. It is observed that for short time $(t \leq 5)$ a strong non-Newtonian effect is present in the velocity field and velocity behaves as a Newtonian case for large time $(t > 5)$.

3. Introduction of the similarity parameter η leads to an exact solution of the governing non-linear partial differential equation.

4. It is to remark that the viscous flow due to a variable shear stress has not been reported yet in the literature and can be obtained as a special case of the presented analysis by taking $\alpha_1 = 0$.

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