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GEOMETRIC MAPPING MATRICES

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Abstract

The systemic definitions of geometric mapping matrices of point-to-point, point-to-line, point-to-plane, line-to-point, line-to-line, line-to-plane, plane-to-point, plane-to-line and plane-to-plane are presented. In this paper, the projection and mapping relationship between points, lines and planes are developed and defined by matrices similar to 4×4 homogeneous transformation matrices. The matrix is a systemic and useful tool. By the matrix multiplication, we can obtain the mapping results easier and faster than the traditional algebraic operations.

Introduction

In an ancient Chinese mathematics classic called Chuichang suan-shu, or Nine Chapters on the Mathematical Art, numbers arranged in rectangular patterns of rows and columns were used to solve systems of equations. Although this idea appeared in the third century B. C., it was largely ignored until the nineteenth century, until the British mathematician Arthur Cayley and his friend James Joseph Sylvester developed matrix algebra in 1857. The "matrix" was first used in 1857 by these mathematicians to distinguish it from determinants. In fact, the

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term "matrix" was intended to mean "mother of determinants". Matrix has been widely used in several fields of science or engineering. Social sciences, economics, statistics and various other branches of knowledge also make extensive use of matrices to handle their tasks [1-10]. Wu [11] defined a new one-to-one mapping matrix for the cryptography. Wu et al. [12] also presented a modified 4×4 transformation matrices to analyze the kinematic errors.

This paper develops a series of mapping matrices between points, lines and planes. In the traditional algebra, we come across the simultaneous equations frequently and their solutions are to be obtained. For this the utility of matrices is well recognized now. We can use it to accomplish algebraic operations easily especially for the geometric transformation such as coordinate transformation, mapping and coding among many others. By the matrix multiplication, we can obtain the mapping results easier and faster than the traditional algebraic operations.

Geometric Mapping Matrices

As shown in Fig. 1, we first consider the point-to-point mapping. We have the following equation:

$$\frac{x_2 - p}{p - x_1} = \frac{y_2 - q}{q - y_1} = \frac{z_2 - r}{r - z_1} = t, \tag{1}$$

where (x_2, y_2, z_2) is unknown, the other variables are given. Rearrange this equation and apply homogeneous transformation matrix to obtain the following mapping formula:

$$\begin{bmatrix} -t & 0 & 0 & p(t+1) \\ 0 & -t & 0 & q(t+1) \\ 0 & 0 & -t & r(t+1) \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}. \tag{2}$$

Secondly, we consider the point-to-line mapping, as shown in Fig. 2. We write a line in parameter form as

$$L : \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = k \Rightarrow L : \begin{cases} x = x_0 + ak, \\ y = y_0 + bk, \\ z = z_0 + ck, \end{cases} \tag{3}$$

where (x_0, y_0, z_0) is a point which lies on the line and (a, b, c) is the directional vector of this line, these two are given. Our interest is to find the projection point (p, q, r) and the mapping point (x_2, y_2, z_2). As we know, the projection point (p, q, r) lies on the line L, vector $(p - x_1, q - y_1, r - z_1)$ is normal to the vector (a, b, c), so we have the following simultaneous equations:

$$\begin{cases} \dfrac{p - x_0}{a} = \dfrac{q - y_0}{b} = \dfrac{r - z_0}{c} = k \\ a(p - x_1) + b(q - y_1) + c(r - z_1) = 0. \end{cases} \tag{4}$$

Rearrange this equation to get

$$\begin{bmatrix} ak_a & bk_a & ck_a & x_0 - k_a k_0 \\ ak_b & bk_b & ck_b & y_0 - k_b k_0 \\ ak_c & bk_c & ck_c & z_0 - k_c k_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \\ 1 \end{bmatrix}, \tag{5}$$

where

$$k_a = \frac{a}{a^2 + b^2 + c^2}, \quad k_b = \frac{b}{a^2 + b^2 + c^2}, \quad k_c = \frac{c}{a^2 + b^2 + c^2},$$

$$k_0 = ax_0 + by_0 + cz_0. \tag{6}$$

After the projection point (p, q, r) got determined, we can obtain the mapping point (x_2, y_2, z_2) by Eq. (2).

Thirdly, we would like to find the mapping of point-to-plane. Observing Fig. 3, we assume the plane equation

$$E : ax + by + cz = d, \tag{7}$$

where (a, b, c) is the normal vector to the plane. Also, the projection point (p, q, r) lies on the plane and the vector $(p - x_1, q - y_1, r - z_1)$ is parallel to vector (a, b, c). Therefore, we have the following simultaneous equations:

$$\begin{cases} ap + bq + cr = d \\ \dfrac{p - x_1}{a} = \dfrac{q - y_1}{b} = \dfrac{r - z_1}{c} = k. \end{cases} \tag{8}$$

We obtain the projection point (p, q, r) from the above equation:

$$\begin{bmatrix} 1 & 0 & 0 & k_a E_1 \\ 0 & 1 & 0 & k_b E_1 \\ 0 & 0 & 1 & k_c E_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \\ 1 \end{bmatrix}, \tag{9}$$

where

$$k_a = \frac{a}{a^2 + b^2 + c^2}, \quad k_b = \frac{b}{a^2 + b^2 + c^2}, \quad k_c = \frac{c}{a^2 + b^2 + c^2},$$

$$E_1 = d - (ax_1 + by_1 + cz_1). \tag{10}$$

Similarly, the mapping point (x_2, y_2, z_2) can be determined by Eq. (2).

Since all the geometric graphs are constructed by points, we only need to let a line as the points set $(x_0 + ak, y_0 + bk, z_0 + ck)$, $\forall k \in \mathbb{R}$ instead of source point (x_1, y_1, z_1) in Eqs. (2), (5) and (9). The mappings of line-to-point, line-to-line and line-to-plane therefore will be found by these equations. In the same manner, we can let a plane as the points set $\left(\dfrac{d - bk - ch}{a}, k, h\right)$, $\forall k, h \in \mathbb{R}$ to accomplish the mappings of plane-to-point, plane-to-line and plane-to-plane. Two examples are given for illustration.

Example 1. Consider the mapping of line-to-point.

As shown in Fig. 4, assume that the line lies on X-Y plane, passes through $(5, 0)$ and its directional vector is $(-1, 1)$. The point we want to map is $(2, 1)$. Mapping ratio $r = 1/2$. By the processes in the above section, we first represent this line as the points set

$$L : \frac{x_1 - 5}{-1} = \frac{y_1 - 0}{1} = k \Rightarrow L : \begin{cases} x_1 = 5 - k, \\ y_1 = 0 + k. \end{cases} \tag{11}$$

Then using Eq. (2),

$$\begin{bmatrix} -\frac{1}{2} & 0 & 0 & 2\left(\frac{1}{2}+1\right) \\ 0 & -\frac{1}{2} & 0 & 1\left(\frac{1}{2}+1\right) \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5-k \\ 0+k \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(5-k)+2\left(\frac{1}{2}+1\right) \\ -\frac{1}{2}(0+k)+1\left(\frac{1}{2}+1\right) \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ 0 \\ 1 \end{bmatrix}. \quad (12)$$

Eliminating parameter k, we obtain the new line L':

$$L' : x_2 + y_2 = 2 \Rightarrow x + y = 2. \quad (13)$$

Example 2. Consider the mapping of plane-to-point.

If we regard the line L as the plane E in Example 1, Fig. 4, then we write plane E in general form $ax + by + cz = d$, so its points set are $\left(\frac{d - bk - ch}{a}, k, h\right)$, $\forall k, h \in \mathbb{R}$. By Eq. (2), we have

$$\begin{bmatrix} -\frac{1}{2} & 0 & 0 & 2\left(\frac{1}{2}+1\right) \\ 0 & -\frac{1}{2} & 0 & 1\left(\frac{1}{2}+1\right) \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{d-bk-ch}{a} \\ k \\ h \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\left(\frac{d-bk-ch}{a}\right)+2\left(\frac{1}{2}+1\right) \\ -\frac{1}{2}(k)+1\left(\frac{1}{2}+1\right) \\ -\frac{1}{2}(h) \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}. \quad (14)$$

Similarly, eliminating parameters k and h, we obtain the new line E':

$$E' : 2ax_2 + 2by_2 + 2cz_2 = 6a + 3b - d \Rightarrow 2ax + 2by + 2cz = 6a + 3b - d. \quad (15)$$

Since the normal vector to the plane E is $(a, b, c) = (1, 1, 0)$, the plane passes through the point $(5, 0, 0)$, i.e., $1 \times 5 + 1 \times 0 + 0 \times 0 = d$. Therefore, Eq. (15) becomes

$$E' : 2x + 2y = 6 + 3 - 5 \Rightarrow x + y = 2. \quad (16)$$

This result coincides with Eq. (13).

Conclusion

Since 1857, the British mathematician Arthur Cayley and his friend James Joseph Sylvester distinguished matrix from determinant, the matrices got widely applied to several professional fields including mathematics, engineering, social sciences, economics and statistics. Matrices help scientists, engineers and other professional people to arrive at the solution of their problems easily and quickly. With their help, problems become clearer and easily formulated. Their use is well noticed in getting the things easily computerized.

This paper presents some matrix formulas instead of algebraic solutions of simultaneous equations to deal with the projection and mapping problems between points, lines and planes. By the matrix multiplication, the results are obtained easily and faster than the traditional algebraic operation. Every component in matrix is clear and the matrix multiplication is also easy. Since the matrix multiplication is a systematic process, we can use numerically or in symbolic software to establish the mapping formulas rather than solving the simultaneous equations. The new matrix operation makes the matrices more useful. This interesting idea also provides another approach in analysis. We can use these matrices to carry out more assignments. It is hoped that the work presented here will contribute towards the progress in the professional science or engineering or other different fields.

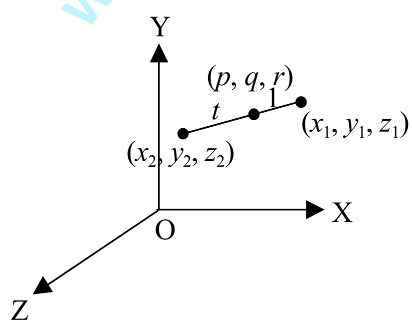

Figure 1. The point-to-point mapping

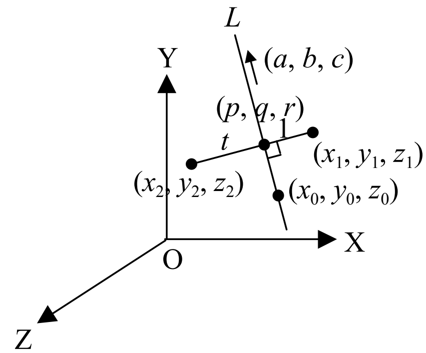

Figure 2. The point-to-line mapping

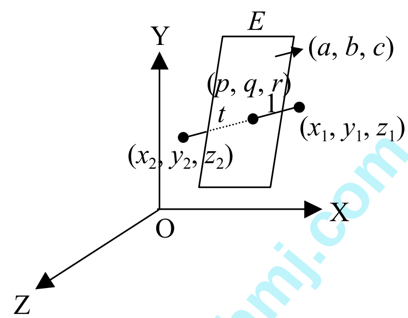

Figure 3. The point-to-plane mapping

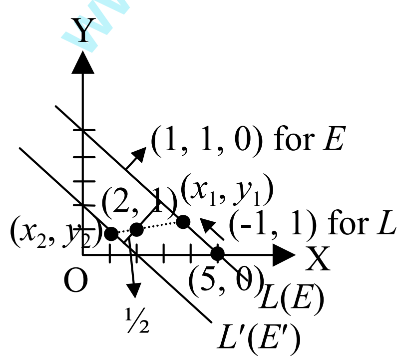

Figure 4. The mapping example of line (plane)-to-point

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