Far East Journal of Mathematical Sciences (FJMS)

Published Online: July 2014
Available online at http://pphmj.com/journals/fjms.htm
Volume 89, Number 2, 2014, Pages 239-248

SOLVABILITY TO SOME SYSTEMS OF MATRIX EQUATIONS AND MATRIX INEQUALITIES

Zhaohuan Deng

Department of Mathematics
Shanghai University
Shanghai 200444, P. R. China
e-mail: zdeng2007@126.com

Abstract

In this paper, we derive necessary and sufficient conditions for the solvability to the systems of matrix equations and matrix inequalities

$$A_1X = B_1, \quad A_2Y = B_2, \quad A_3Z = B_3, \quad A > BXB^* + CYC^* + DZD^*,$$

$$A_1X = B_1, \quad A_2Y = B_2, \quad A_3Z = B_3, \quad A < BXB^* + CYC^* + DZD^*,$$

$$A_1X = B_1, \quad A_2Y = B_2, \quad A_3Z = B_3, \quad A \geq BXB^* + CYC^* + DZD^*,$$

$$A_1X = B_1, \quad A_2Y = B_2, \quad A_3Z = B_3, \quad A \leq BXB^* + CYC^* + DZD^*.$$

1. Introduction

In recent years, some optimization problems on ranks and inertias of Hermitian matrix expressions have attracted much attention from both theoretical and practical points of view. Chu et al. [2] and Liu and Tian [4]

Received: March 14, 2014; Revised: May 4, 2014; Accepted: May 29, 2014
2010 Mathematics Subject Classification: 15A24, 15A09, 15A03.
Keywords and phrases: matrix equation, Moore-Penrose inverse, rank, inertia, re-nonnegative semidefinite matrix.
Communicated by Qing-Wen Wang

derived the maximal and minimal ranks and inertias of

$$g(X) = A - BXC - (BXC)^*, \tag{1.1}$$

Chu et al. [2] and Liu and Tian [5] provided the maximal and minimal ranks and inertias of

$$h(X, Y) = A - BXB^* - CYC^*. \tag{1.2}$$

Tian [7] presented the maximal and minimal inertias of the Hermitian matrix expression

$$f_k(X_1, ..., X_k) = A - B_1X_1B_1^* - \cdots - B_kX_kB_k^*. \tag{1.3}$$

Chen [1] derived the maximal rank of (1.3).

Our goal of this paper is to give some necessary and sufficient conditions for the solvability to the systems of matrix equations and matrix inequalities

$$A_1X = B_1,\ A_2Y = B_2,\ A_3Z = B_3,\ A > (<, \geq, \leq)BXB^* + CYC^* + DZD^*.$$

In order to do it, we need to derive the maximal rank and extremal inertias of the Hermitian matrix expression of

$$f(X, Y, Z) = A - BXB^* - CYC^* - DZD^*, \tag{1.4}$$

where X, Y and Z are Hermitian solutions of the matrix equations

$$A_1X = B_1, \quad A_2Y = B_2, \quad A_3Z = B_3. \tag{1.5}$$

Throughout this paper, we denote the complex number field by $\mathbb{C}$. The notations $\mathbb{C}^{m\times n}$ and $\mathbb{C}_h^{m\times m}$ stand for the sets of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively. The identity matrix with an appropriate size is denoted by I. For a complex matrix A, the symbols A^* and $r(A)$ stand for the conjugate transpose and the rank of A, respectively. The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the *unique solution* X to the following four matrix equations:

(1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^* = AX$ and (4) $(XA)^* = XA$.

Furthermore, L_A and R_A stand for the two projectors $L_A = I - A^{\dagger}A$ and $R_A = I - AA^{\dagger}$ induced by A, respectively. It is known that $L_A = L_A^*$ and $R_A = R_A^*$. For $A \in \mathbb{C}_h^{m\times m}$, its inertia

$$\mathbb{I}_n(A) = (i_+(A), i_-(A), i_0(A)),$$

is the triple consisting of the numbers of the positive, negative and zero eigenvalues of A, counted with multiplicities, respectively. The two numbers $i_+(A)$ and $i_-(A)$ are usually called the *positive* and *negative signatures*, respectively. It is easy to see that $i_+(A) + i_-(A) = r(A)$.

2. Preliminaries

In this section, we give some lemmas which are used in this paper.

Lemma 2.1 [6]**.** *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, $D \in \mathbb{C}^{m\times p}$, $E \in \mathbb{C}^{q\times n}$, $Q \in \mathbb{C}^{m_1\times k}$ *and* $P \in \mathbb{C}^{l\times n_1}$ *be given. Then*

(1) $r(A) + r(R_A B) = r(B) + r(R_B A) = r[A \quad B]$.

(2) $r(A) + r(CL_A) = r(C) + r(AL_C) = r\begin{bmatrix} A \\ C \end{bmatrix}$.

(3) $r(B) + r(C) + r(R_B A L_C) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$.

Lemma 2.2 [3]**.** *The matrix equation* $AX = B$ *has a Hermitian solution if and only if* $r[A, B] = r(A)$ *and* $AB^* = BA^*$*. In this case, the general Hermitian solution can be expressed as*

$$X = A^{\dagger}B + (A^{\dagger}B)^* - A^{\dagger}BA^{\dagger}A + L_A W L_A, \tag{2.1}$$

where $W = W^*$ *is arbitrary.*

Lemma 2.3 [1]**.** *Let* $A = A^*$, B, C *and* D *be given. Then*

$$\max_{X = X^*, Y = Y^*, Z = Z^*} r[A - BXB^* - CYC^* - DZD^*] = r[A, B, C, D].$$

Lemma 2.4 [7]. *Let* $A = A^*,\ B,\ C$ *and* D *be given, and denote*

$$M = \begin{bmatrix} A & B & C & D \\ B^* & 0 & 0 & 0 \\ C^* & 0 & 0 & 0 \\ D^* & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\max_{X = X^*,\, Y = Y^*,\, Z = Z^*} i_{\pm}[A - BXB^* - CYC^* - DZD^*] = i_{\pm}(M),$$

$$\min_{X = X^*,\, Y = Y^*,\, Z = Z^*} i_{\pm}[A - BXB^* - CYC^* - DZD^*] = r[A,\ B,\ C,\ D] - i_{\mp}(M).$$

3. Maximal Rank of (1.4) Subject to (1.5)

Our goal of this section is to derive the maximal rank of (1.4) subject to (1.5). For convenience of representation, we adopt the following notation:

$$J_1 = \{X = X^* \mid A_1X = B_1\}, \quad J_2 = \{Y = Y^* \mid A_2Y = B_2\},$$

$$J_3 = \{Z = Z^* \mid A_3Z = B_3\}.$$

Theorem 3.1. *Let* $A = A^*,\ B,\ C,\ D,\ A_i,\ B_i\ \ (i = 1,\ 2,\ 3)$ *be given. Assume that the matrix equations in* (1.5) *are consistent, respectively. Then the maximal rank of* (1.4) *subject to* (1.5) *is*

$$\max_{X \in J_1,\, Y \in J_2,\, Z \in J_3} r[f(X,\ Y,\ Z)] = r\begin{bmatrix} A & B & C & D \\ C_1B^* & A_1 & 0 & 0 \\ C_2C^* & 0 & A_2 & 0 \\ C_3D^* & 0 & 0 & A_3 \end{bmatrix}$$

$$- r(A_1) - r(A_2) - r(A_3).$$

Proof. It follows from Lemma 2.2 that the general Hermitian solutions to the matrix equations in (1.5) can be expressed as

$$X = X_0 + L_{A_1} W_1 L_{A_1},\ Y = Y_0 + L_{A_2} W_2 L_{A_2},\ Z = Z_0 + L_{A_3} W_3 L_{A_3}, \tag{3.1}$$

where X_0, Y_0 and Z_0 are special solutions, and W_1, W_2, W_3 are arbitrary Hermitian matrices. Then

$$f(X, Y, Z) = \hat{A} - BL_{A_1} W_1 L_{A_1} B^* - CL_{A_2} W_2 L_{A_2} C^* - DL_{A_3} W_3 L_{A_3} D^*, \tag{3.2}$$

where $\hat{A} = A - BX_0B^* - CY_0C^* - DZ_0D^*$. Applying Lemma 2.3 to (3.2) yields

$$\max_{X \in J_1, Y \in J_2, Z \in J_3} r[f(X, Y, Z)] = r[\hat{A},\ BL_{A_1},\ CL_{A_2},\ DL_{A_3}].$$

Using Lemma 2.1 and $BX_0B^* + CY_0C^* + DZ_0D^* = A$, we have

$$\begin{aligned}
& r[\hat{A},\ BL_{A_1},\ CL_{A_2},\ DL_{A_3}] \\
&= r\begin{bmatrix} \hat{A} & B & C & D \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_3 \end{bmatrix} - r(A_1) - r(A_2) - r(A_3) \\
&= r\begin{bmatrix} A - BX_0B^* - CY_0C^* - DZ_0D^* & B & C & D \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_3 \end{bmatrix} \\
&\quad - r(A_1) - r(A_2) - r(A_3) \\
&= r\begin{bmatrix} A & B & C & D \\ C_1B^* & A_1 & 0 & 0 \\ C_2C^* & 0 & A_2 & 0 \\ C_3D^* & 0 & 0 & A_3 \end{bmatrix} - r(A_1) - r(A_2) - r(A_3).
\end{aligned}$$

□

4. Maximal and Minimal Inertias of (1.4) Subject to (1.5)

We, in this section, consider the maximal and minimal inertias of (1.4) subject to (1.5).

Theorem 4.1. *Let* $A = A^*, B, C, D, A_i, B_i \ (i = 1, 2, 3)$ *be given. Assume that the matrix equations in* (1.5) *are consistent, respectively. Denote*

$$N = \begin{bmatrix} A & B & C & D & \frac{1}{2}BC_1^* & \frac{1}{2}CC_2^* & \frac{1}{2}DC_3^* \\ B^* & 0 & 0 & 0 & A_1^* & 0 & 0 \\ C^* & 0 & 0 & 0 & 0 & A_2^* & 0 \\ D^* & 0 & 0 & 0 & 0 & 0 & A_3^* \\ \frac{1}{2}C_1B^* & A_1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}C_2C^* & 0 & A_2 & 0 & 0 & 0 & 0 \\ \frac{1}{2}C_3D^* & 0 & 0 & A_3 & 0 & 0 & 0 \end{bmatrix}.$$

Then the maximal and minimal inertias of (1.4) *subject to* (1.5) *are*

$$\max_{X \in J_1, Y \in J_2, Z \in J_3} i_{\pm}[f(X, Y, Z)] = i_{\pm}(N) - r(A_1) - r(A_2) - r(A_3),$$

$$\min_{X \in J_1, Y \in J_2, Z \in J_3} i_{\pm}[f(X, Y, Z)] = r\begin{bmatrix} A & B & C & D \\ C_1B^* & A_1 & 0 & 0 \\ C_2C^* & 0 & A_2 & 0 \\ C_3D^* & 0 & 0 & A_3 \end{bmatrix} - i_{\mp}(N).$$

Proof. It follows from Theorem 3.1 that

$$f(X, Y, Z) = \hat{A} - BL_{A_1}W_1L_{A_1}B^* - CL_{A_2}W_2L_{A_2}C^* - DL_{A_3}W_3L_{A_3}D^*, \quad (4.1)$$

where

$$\hat{A} = A - BX_0B^* - CY_0C^* - DZ_0D^*.$$

Applying Lemma 2.4 to (4.1) yields

$$\max_{X\in J_1, Y\in J_2, Z\in J_3} i_{\pm}[f(X, Y, Z)] = i_{\pm}(M),$$

$$\min_{X\in J_1, Y\in J_2, Z\in J_3} i_{\pm}[f(X, Y, Z)] = r[\hat{A}, BL_{A_1}, CL_{A_2}, DL_{A_3}] - i_{\mp}(M),$$

where

$$\hat{A} = A - BX_0B^* - CY_0C^* - DZ_0D^*,$$

$$M = \begin{bmatrix} \hat{A} & BL_{A_1} & CL_{A_2} & DL_{A_3} \\ L_{A_1}B^* & 0 & 0 & 0 \\ L_{A_2}C^* & 0 & 0 & 0 \\ DL_{A_3}D^* & 0 & 0 & 0 \end{bmatrix}.$$

Using $BX_0B^* + CY_0C^* + DZ_0D^* = A,$ we have

$$i_{\mp}(M) = i_{\mp}\begin{bmatrix} \hat{A} & BL_{A_1} & CL_{A_2} & DL_{A_3} \\ L_{A_1}B^* & 0 & 0 & 0 \\ L_{A_2}C^* & 0 & 0 & 0 \\ DL_{A_3}D^* & 0 & 0 & 0 \end{bmatrix}$$

$$= i_{\pm}\begin{bmatrix} \hat{A} & B & C & D & 0 & 0 & 0 \\ B^* & 0 & 0 & 0 & A_1^* & 0 & 0 \\ C^* & 0 & 0 & 0 & 0 & A_2^* & 0 \\ D^* & 0 & 0 & 0 & 0 & 0 & A_3^* \\ 0 & A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_3 & 0 & 0 & 0 \end{bmatrix} - r(A_1) - r(A_2) - r(A_3)$$

$$= i_{\pm}\begin{bmatrix} A - BX_0B^* - CY_0C^* - DZ_0D^* & B & C & D & 0 & 0 & 0 \\ B^* & 0 & 0 & 0 & A_1^* & 0 & 0 \\ C^* & 0 & 0 & 0 & 0 & A_2^* & 0 \\ D^* & 0 & 0 & 0 & 0 & 0 & A_3^* \\ 0 & A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_3 & 0 & 0 & 0 \end{bmatrix}$$

$$- r(A_1) - r(A_2) - r(A_3)$$

$$= i_{\pm}\begin{bmatrix} A & B & C & D & \frac{1}{2}BC_1^* & \frac{1}{2}CC_2^* & \frac{1}{2}DC_3^* \\ B^* & 0 & 0 & 0 & A_1^* & 0 & 0 \\ C^* & 0 & 0 & 0 & 0 & A_2^* & 0 \\ D^* & 0 & 0 & 0 & 0 & 0 & A_3^* \\ \frac{1}{2}C_1B^* & A_1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}C_2C^* & 0 & A_2 & 0 & 0 & 0 & 0 \\ \frac{1}{2}C_1D^* & 0 & 0 & A_3 & 0 & 0 & 0 \end{bmatrix}$$

$$- r(A_1) - r(A_2) - r(A_3). \qquad \square$$

5. Solvability to Some Systems of Matrix Equations and Matrix Inequalities

Theorem 5.1. *Let* $A \in \mathbb{C}_h^{m\times m}$, B, C, D, A_i, B_i $(i = 1, 2, 3)$, N *be as in Theorem* 4.1. *Assume that the matrix equations in* (1.5) *are consistent, respectively. Then*:

(a) *There exist* $X \in J_1$, $Y \in J_2$, $Z \in J_3$ *such that* $A > BXB^* + CYC^* + DZD^*$ *if and only if* $i_+(N) - r(A_1) - r(A_2) - r(A_3) = m.$

(b) *There exist* $X \in J_1$, $Y \in J_2$, $Z \in J_3$ *such that* $A < BXB^* + CYC^* + DZD^*$ *if and only if* $i_-(N) - r(A_1) - r(A_2) - r(A_3) = m.$

(c) *There exist* $X \in J_1, Y \in J_2, Z \in J_3$ *such that* $A \geq BXB^* + CYC^* + DZD^*$ *if and only if*

$$r\begin{bmatrix} A & B & C & D \\ C_1B^* & A_1 & 0 & 0 \\ C_2C^* & 0 & A_2 & 0 \\ C_3D^* & 0 & 0 & A_3 \end{bmatrix} = i_+(N).$$

(d) *There exist* $X \in J_1, Y \in J_2, Z \in J_3$ *such that* $A \leq BXB^* + CYC^* + DZD^*$ *if and only if*

$$r\begin{bmatrix} A & B & C & D \\ C_1B^* & A_1 & 0 & 0 \\ C_2C^* & 0 & A_2 & 0 \\ C_3D^* & 0 & 0 & A_3 \end{bmatrix} = i_-(N).$$

(e) $A > BXB^* + CYC^* + DZD^*$ *holds for any* $X \in J_1, Y \in J_2, Z \in J_3$ *if and only if*

$$r\begin{bmatrix} A & B & C & D \\ C_1B^* & A_1 & 0 & 0 \\ C_2C^* & 0 & A_2 & 0 \\ C_3D^* & 0 & 0 & A_3 \end{bmatrix} = i_-(N) + m.$$

(f) $A < BXB^* + CYC^* + DZD^*$ *holds for any* $X \in J_1, Y \in J_2, Z \in J_3$ *if and only if*

$$r\begin{bmatrix} A & B & C & D \\ C_1B^* & A_1 & 0 & 0 \\ C_2C^* & 0 & A_2 & 0 \\ C_3D^* & 0 & 0 & A_3 \end{bmatrix} = i_+(N) + m.$$

(g) $A \geq BXB^* + CYC^* + DZD^*$ *holds for any* $X \in J_1, Y \in J_2, Z \in J_3$ *if and only if* $i_-(N) = r(A_1) + r(A_2) + r(A_3)$.

(h) $A \leq BXB^* + CYC^* + DZD^*$ *holds for any* $X \in J_1, Y \in J_2, Z \in J_3$ *if and only if* $i_+(N) = r(A_1) + r(A_2) + r(A_3)$.

6. Conclusion

We have derived the maximal rank of the Hermitian matrix expression (1.4) subject to (1.5). We also have presented the maximal and minimal inertias of the Hermitian matrix expression (1.4) subject to (1.5). We have presented necessary and sufficient conditions for the solvability to the systems of matrix equations and matrix inequalities

$$A_1X = B_1, \quad A_2Y = B_2, \quad A_3Z = B_3, \quad A > BXB^* + CYC^* + DZD^*,$$

$$A_1X = B_1, \quad A_2Y = B_2, \quad A_3Z = B_3, \quad A < BXB^* + CYC^* + DZD^*,$$

$$A_1X = B_1, \quad A_2Y = B_2, \quad A_3Z = B_3, \quad A \geq BXB^* + CYC^* + DZD^*,$$

$$A_1X = B_1, \quad A_2Y = B_2, \quad A_3Z = B_3, \quad A \leq BXB^* + CYC^* + DZD^*.$$

References

[1] D. Chen, A decomposition of multiple matrices and its applications, Master's Thesis, 2013.

[2] D. L. Chu, Y. S. Hung and H. J. Woerdeman, Inertia and rank characterizations of some matrix expressions, SIAM J. Matrix Anal. Appl. 31 (2009), 1187-1226.

[3] C. G. Khatri and S. K. Mitra, Hermitian and nonnegative definite solutions of linear matrix equations, SIAM J. Appl. Math. 31 (1976), 579-585.

[4] Y. H. Liu and Y. G. Tian, Max-min problems on the ranks and inertias of the matrix expressions $A - BXC \pm (BXC)^*$, J. Optim. Theory Appl. 148(3) (2011), 593-622.

[5] Y. H. Liu and Y. G. Tian, A simultaneous decomposition of a matrix triplet with applications, Numer. Linear Algebra Appl. 18(1) (2011), 69-85.

[6] G. Marsaglia and G. P. H. Styan, Equalities and inequalities for ranks of matrices, Linear Multilinear Algebra 2 (1974), 269-292.

[7] Y. G. Tian, Equalities and inequalities for inertias of Hermitian matrices with applications, Linear Algebra Appl. 433 (2010), 263-296.