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ON ISOMORPHIC PROPERTIES IN GSA III

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Abstract

In this paper, we are concerned with some concepts of R-groups, where R is a near-ring, and their related algebraic structures such as group semiautomata: the first and second isomorphism theorems of group semiautomata have already been studied by us in earlier papers.

1. Introduction

The purpose of this paper is to study the algebraic theory of group semiautomata related with the papers [1-4].

A *group semiautomaton* (in short, *GSA*) is a quadruple $(Q, +, X, \delta)$, where $(Q, +)$, as the set of states, is a group (not necessarily abelian), X is the set of inputs and $\delta : Q \times X \to Q$ is the state transition function.

Let R be a (left) near-ring and G be an additive group. Then G is called an *R-group* if there exists a mapping $\mu : G \times R \to G$ defined by $\mu(x, a) = xa$ which satisfies: (i) $x(a + b) = xa + xb$, (ii) $x(ab) = (xa)b$ and (iii)

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$x1 = x$ (if R has a unity 1) for all $x \in G$ and $a, b \in R$. We denote it by G_R.

For an R-group G, a subgroup T of G such that $TR \subset T$ is called an *R-subgroup* of G, and an *R-ideal* of G is a normal subgroup N of G such that $(N + x)a - xa \subset N$ for all $x \in G$, $a \in R$. This is denoted by $N \triangleleft G$.

For the remainder basic concepts and results on near-rings, we refer to [6].

2. The Third Isomorphic Properties in GSA

From now on, the input set X is fixed for every GSA. So a GSA on the set Q of states will be denoted by $(Q, +, \delta)$, or sometimes, briefly, denoted by Q if there arises no confusion. Moreover, $\delta(q, x)$ is written simply as qx.

Let $(Q, +, \delta)$ and $(Q', +, \delta')$ be two GSAs with the same input set X and $f : Q \to Q'$ a group homomorphism. We call that f is a *GSA homomorphism* from $(Q, +, \delta)$ into $(Q', +, \delta')$ if f satisfies $(qx)f = (qf)x$ for all $q \in Q$, $x \in X$.

Let $f : (Q, +, \delta) \to (Q', +, \delta')$ be a GSA homomorphism. Then (1) f is called a *GSA monomorphism* if f is injective, (2) f is called a *GSA epimorphism* if f is surjective and (3) f is called a *GSA isomorphism* if f is bijective. In this case, we denote that $Q \cong Q'$.

A normal subgroup K of $(Q, +)$ is called an *ideal* of a GSA $(Q, +, \delta)$ if $(K + q)x - qx \subseteq K$ for all $q \in Q$, $x \in X$. This will be denoted by $K \triangleleft (Q, +, \delta)$.

Clearly, 0 and Q are trivial ideals of $(Q, +, \delta)$.

Let $(Q, +, \delta)$ be a GSA with the input set X and K be an ideal of a GSA $(Q, +, \delta)$. Then the set $Q/K = \{q + K \mid q \in Q\}$ is a quotient group under addition as group theory. Define $\overline{\delta} : Q/K \times X \to Q/K$ by $(q + K, x) \mapsto qx + K$. Then $\overline{\delta}$ becomes a state transition function. This GSA $(Q/K, +, \overline{\delta})$ is called a *quotient GSA*.

A subgroup S of $(Q, +)$ is called a *subgroup semiautomaton* or simply *SGSA* of a GSA $(Q, +, \delta)$ if $(S, +, \delta)$ is a GSA. This will be denoted by $S < (Q, +, \delta)$.

We do remark that there is no direct connection between ideals and SGSA as we see in the following examples.

Example 2.1. Let us consider the integer group modulo 4 $(Q, +) = (Z_4, +) = \{0, 1, 2, 3\}$ as the set of states and $X = \{x, y\}$ the set of inputs.

Define the state transition function δ by the following rule: $(a, x) \mapsto 0$, $\forall a \in Q$, and $(a, y) \mapsto 2$, $\forall a \in Q$.

It is easy to see that $S = \{0, 2\}$ forms an SGSA of $(Q, +, \delta)$ with $SX = \{0, 2\}$. Also, S is an ideal of $(Q, +, \delta)$, for instance, $(1 + 0)x - 1x = 1x - 1x = 0 - 0 = 0$, $(1 + 0)y - 1y = 1y - 1y = 2 - 2 = 0$, $(3 + 2)x - 3x = 1x - 3x = 0 - 0 = 0$ and $(3 + 2)y - 3y = 1y - 3y = 2 - 2 = 0$. They are all elements of S.

Example 2.2. Let us consider the Klein 4-group $(Q, +) = (K_4, +) = \{0, a, b, c\}$ as the set of states and $X = \{x\}$ the set of inputs.

Define the state transition function δ by the following graph:

$$a \mapsto b \mapsto c \mapsto 0 \mapsto 0$$

using x-action. It is easy to see that $S = \{0, c\}$ forms an SGSA of $(Q, +, \delta)$ with $SX = \{0\}$. However, S is not an ideal of $(Q, +, \delta)$ for $(b + c)x - bx = ax - bx = b - c = a$. This is not an element of S.

Let $f : (Q, +, \delta) \to (Q', +, \delta')$ be a GSA homomorphism. Then (1) $0f^{-1} = \{a \in Q \mid af = 0\}$ is called the *Kernel* of *f*, which is denoted by *Kerf* and (2) *Qf* is called the *image* of *f*, which is denoted by *Imf*. Obviously, *Kerf* and *Imf* are GSA.

First, we introduce some basic properties of GSA.

Theorem 2.3 [2]. *Let* $(Q, +, \delta)$, $(Q', +, \delta')$ *be two GSAs and* $f : (Q, +, \delta) \to (Q', +, \delta')$ *be a GSA epimorphism. Then the set* $Kerf = \{q \in Q \mid qf = 0\}$ *is an ideal of* $(Q, +, \delta)$.

Conversely, if $K \lhd (Q, +, \delta)$, *then the canonical group homomorphism* $\pi : Q \to Q/K$ *by* $q\pi = q + K$ *is a GSA epimorphism from* $(Q, +, \delta)$ *onto* $(Q/K, +, \mu)$ *for all* $q \in Q$ *and* $Ker\pi = K$, *where* $\mu : Q/K \times X \to Q/K$ *via* $(q + K, x) \mapsto qx + K$.

Proposition 2.4. *Let* $(Q, +, \delta)$ *be a GSA. Then an ideal K of* $(Q, +, \delta)$ *is an SGSA if and only if* $\{0\}X \subseteq K$.

Proof. Suppose that an ideal *K* of $(Q, +, \delta)$ is an SGSA. Then, for all $k \in K$ and $x \in X$, $kx - 0x = (0 + k)x - 0x$, which is contained in *K*, so $0k \in K$. Hence $\{0\}X \subseteq K$. (Since K is an SGSA, $kx \in K$, $-0x \in K - kx \subseteq K$.)

Conversely, suppose that $\{0\}X \subseteq K$. Let $k \in K$ and $x \in X$. To show that $kx \in K$, consider $kx - 0x = (0 + k)x - 0x \in K$, because that $K \lhd (Q, +, \delta)$. This equation implies $kx \in K + 0x \subseteq K$. $\square$

Obviously, we get the following important but straightforward statement.

Theorem 2.5. *Let* $(Q, +, \delta)$, $(Q', +, \delta')$ *be two GSAs and* $f : (Q, +, \delta) \to (Q', +, \delta')$ *be a GSA homomorphism. Then the set* $Kerf = \{0\}$ *if and only if f is a GSA monomorphism*.

Theorem 2.6 (Fundamental Theorem) [2]**.** *Let* $(Q, +, \delta), (Q', +, \delta')$ *be two GSAs and* $f : (Q, +, \delta) \to (Q', +, \delta')$ *be a GSA homomorphism. Then*

(1) *Kerf is an ideal of* $(Q, +, \delta)$.

(2) *There exists a unique GSA monomorphism* $g : Q/Kerf \to Q'$ *such that* $f = \pi g$, *where* $\pi : Q \to Q/Kerf$ *is a canonical GSA epimorphism. In particular,* $Q/Kerf \cong Imf$.

For further discussion, in a GSA $(Q, +, \delta)$, let S, T be the subsets of Q. We define their sum as the set

$$S + T = \{s + t \mid s \in S, t \in T\}.$$

It is not hard to see that the sum of two SGSAs is not an SGSA in general, from Example 2.2. For example, $S = \{0, c\}$ and $T = \{0, b\}$ are two SGSAs of $(K_4, +, \delta)$, but $S + T$ is not an SGSA of $(K_4, +, \delta)$.

However, we have the following substructures:

Theorem 2.7 [4]**.** *Let* $(Q, +, \delta)$ *be a GSA. Then*

(1) *If S is an SGSA of* $(Q, +, \delta)$ *and I is an ideal of* $(Q, +, \delta)$, *then* $S + I$ *is an SGSA of* $(Q, +, \delta)$.

(2) *If S and I are ideals of* $(Q, +, \delta)$, *then* $S + I$ *is an ideal of* $(Q, +, \delta)$.

Theorem 2.8 (Second Isomorphism Theorem) [4]**.** *Let* $(Q, +, \delta)$ *be a GSA, S be an SGSA of* $(Q, +, \delta)$ *and I be an ideal of* $(Q, +, \delta)$. *Then*

(1) $S + I = I + S$, $S + I$ *is an SGSA of* $(Q, +, \delta)$, $I \lhd (S + I, +, \delta)$ *and* $I \cap S \lhd (S, +, \delta)$.

(2) $S/(I \cap S) \cong (I + S)/I$ *as quotient GSA.*

Now, we establish our main result.

Theorem 2.9 (Third Isomorphism Theorem). *Let* $(Q, +, \delta)$ *be a GSA*, $I, K \triangleleft (Q, +, \delta)$ *and* $K < (Q, +, \delta)$ *with* $I \subseteq K$. *Then*

(1) $K/I \triangleleft Q/I$, *as GSA*.

(2) $(Q/I)/(K/I) \cong Q/K$, *as GSA-isomorphism*.

Proof. (1) Suppose that $K \triangleleft (Q, +, \delta)$ and $K < (Q, +, \delta)$ with $I \subseteq K$. Then obviously, $(K/I, +) \triangleleft (Q/I, +)$ in group theory. Next, let $q + I \in Q/I$, $a + I \in K/I$, and let $x \in X$. Then we have

$$\begin{aligned} & (q + I + a + I)x - (q + I)x \\ &= (q + a + I)x - (q + I)x = (q + a)x + I - qx + I \\ &= (q + a)x - qx + I + I = (q + a)x - qx + I \in (K/I), \end{aligned}$$

since $(q + a)x - qx \in K$. Hence $K/I \triangleleft Q/I$.

(2) Define $\phi : Q/I \to Q/K$ by $(q + I)\phi = q + K$ for all $q \in Q$. Then ϕ is well defined and GSA-epimorphism. Because of $q + I = p + I \Leftrightarrow -p + q \in I$, since $I \subseteq K$, $q + K = p + K$. Clearly, ϕ is a group homomorphism, to show that ϕ is a GSA homomorphism, we get

$$\{(q + I)x\}\phi = (qx + I)\phi = qx + K = (q + K)x = (q + I)\phi x$$

for all $q \in Q$, $x \in X$. Finally, we see that $Ker\phi = K/I$. Therefore, by the Fundamental Theorem, we have that $(Q/I)/(K/I) \cong Q/K$. □

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