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MEAN AND VARIANCE OF NON-NORMAL FUZZY NUMBERS AND FUZZY QUANTITIES

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Abstract

The concept of mean is extended to non-normal fuzzy numbers and its properties (internality, zero-sum deviation, crispness, invariance, and associativity) are illustrated thoroughly. Analogously, the notion of variance is extended to non-normal fuzzy numbers and its properties (positivity, zero-variance, invariance, change of pole) are reported together with the relationships between two different definitions of variance and different poles. In particular, we demonstrate inequalities between variances of the same fuzzy set with respect to different poles. Finally, the variance is defined for the union of fuzzy sets.

1. Introduction

The concept of fuzzy random variable was introduced by Kwakernaak [8], but its natural characterization, given by the notion of expected value, was introduced only later by Puri and Ralescu [10] and they proved to be a

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fuzzy number. The expectation of a fuzzy number was introduced by Dubois and Prade [3] and it is an interval-valued function. They considered it as a random set and showed that the interval-valued expectation is additive in the sense of an addition of fuzzy numbers. The expected value corresponds to the first moment with respect to the origin and is important in many practical applications. However, some of the most useful information about a real-valued or a fuzzy random variable are provided also by moments greater than the first, especially by that of the second order, which is related to the variance. The first moment is well represented also by a fuzzy number, while the second, more than the first, is better described by crispness rather than fuzziness [9]. In fact, analogously to the case of a real-valued random variable, variance should measure the spread or dispersion of the fuzzy random variable around its expected value and, although it might be interval-valued, generally a crisp value is preferred. The variance or covariance of a fuzzy number and their properties were probably introduced for this reason, some years later, although they usually characterize a random variable in Statistics and are relevant for linear statistical inference with fuzzy random data [2, 5, 7].

The crisp possibilistic mean, variance, and covariance of a fuzzy number were introduced by Carlsson and Fullér [1], considering a continuous possibility distribution, consistently with the extension principle and the current definition of expectation and variance in probability theory.

Few results deal with fuzzy sets that do not verify some properties such as normality and convexity. However, very often there is a necessity to handle a fuzzy set that is obtained by the union of two or more fuzzy numbers. In this case, the resulting fuzzy set is generally non-normal and non-convex, and is termed fuzzy quantity [4]. The variance of a fuzzy quantity, together with its properties, is interesting both theoretically and practically. In the latter case, it may involve the defuzzification step, i.e., the transformation of a fuzzy output into a crisp value, which condenses all the vagueness expressed by a fuzzy quantity. Therefore, it may be useful in many applications of fuzzy systems in control theory, in economics, and in finance.

The present paper analyzes this subject pursuing two aims. The first aim is to illustrate the possibilistic mean, variance, and covariance for a non-normal fuzzy number, summarizing almost all their properties. The second aim is to investigate the possibilistic mean and variance of the union of two fuzzy sets. Therefore, the structure of the paper is as follows: Section 2 reports the usual definitions and basic notations of fuzzy sets and fuzzy numbers. Section 3 describes the possibilistic mean, variance, and covariance of a non-normal fuzzy number, and lists their properties. Section 4 presents the extension of the possibilistic variance to the union of two fuzzy sets. Section 5 concludes with some comments.

2. Basic Definitions and Notations

Let X be a universal set; a fuzzy subset A of X is defined by a function $\mu_A(\cdot) : X \to [0, 1]$, called the *membership function*. Throughout this paper, X is assumed to be the set of real numbers, $\mathbb{R}$, and $\mathcal{F}$, the space of fuzzy sets.

Definition 1. The fuzzy set $A \in \mathcal{F}$ is a fuzzy number iff:

(1) $\forall \gamma \in [0, 1]$, the set $A^{\gamma} = \{x \in \mathbb{R} : \mu_A(x) \geq \gamma\}$, which is called γ-*cut* or γ-*level* of A, is a convex set,

(2) $\mu_A(\cdot)$ is an upper-semicontinuous function,

(3) $\text{supp}(A) = \{x \in \mathbb{R} : \mu_A(x) > 0\}$ is a bounded set in $\mathbb{R}$,

(4) height $A = \max_{x \in X} \mu_A(x) = h > 0$.

By virtue of conditions (1) and (2), each γ-level is a compact and convex subset of $\mathbb{R}$, hence it is a closed interval in $\mathbb{R}$, $A^{\gamma} = [a_L(\gamma), a_R(\gamma)]$. If $h = 1$, then the fuzzy number is normal.

If condition (1) fails, then we say that A is a *fuzzy quantity*. Finally, A is a crisp number with value m if its membership function is given by $\mu_A(x) = 1$ if $x = m$ and $\mu_A(x) = 0$ otherwise.

Definition 2. A fuzzy number A is a so-called left-right fuzzy number, $A = (a_1, a_2, \alpha, \beta)_{LR}$ $a_1 \leq a_2$; $\alpha, \beta > 0$, if the corresponding membership function, for all $x \in \mathbb{R}$, satisfies

$$\mu_A(x) = \begin{cases} L\left(\dfrac{a_1 - x}{\alpha}\right), & a_1 - \alpha < x \leq a_1, \\ h, & a_1 < x \leq a_2, \\ R\left(\dfrac{x - a_2}{\beta}\right), & a_2 < x \leq a_2 + \beta, \\ 0, & \text{elsewhere}, \end{cases} \tag{1}$$

where L and R, called the *left and the right shape functions*, are continuous and decreasing mappings from $[0, 1]$ to $[0, 1]$ such that $L(0) = R(0) = h > 0$ and $L(1) = R(1) = 0$.

For the sake of brevity, this fuzzy number will be denoted by $A = (a_1, a_2, \alpha, \beta)$ and the closure of the support of A will be $[a_1 - \alpha, a_2 + \beta]$.

For a triangular fuzzy number $T = (a, \alpha, \beta)$ with height h, the membership functions are

$$L\left(\frac{a - x}{\alpha}\right) = h\left(1 - \frac{a - x}{\alpha}\right), \tag{2}$$

$$R\left(\frac{x - a}{\beta}\right) = h\left(1 - \frac{x - a}{\beta}\right). \tag{3}$$

In the following, we will represent a general L-R fuzzy number A by means of its γ-levels, i.e., $A = A^{\gamma} = [a_L(\gamma), a_R(\gamma)]$, $\forall \gamma \in [0, h]$.

3. Mean, Variance, and Covariance for Non-normal Fuzzy Numbers

The concept of mean or expected value corresponds to the notion of the center of gravity, which is an imaginary point in a body of matter or in a distribution of any type, where for convenience in calculation or representative synthesis, the total weight of the body may be considered to be concentrated. Therefore, it is also termed center of mass. In the case of uniform density, the concept of mean or expected value corresponds to the notion of a barycenter, which is the point at the center of any shape and it is also termed a centroid, center of area or center of volume. Such notions involve an idea of crispness. However, within the fuzzy set theory, in the framework devised by Puri and Ralescu [10], the expected value is a fuzzy number and in the context of Dubois and Prade [3], it is an interval-valued function.

Let A be a normal fuzzy number of the LR type with strictly monotonic shape functions. Then the γ-level sets of A are $[A]^{\gamma} = [a_1 - \alpha L^{-1}(\gamma), a_2 + \beta R^{-1}(\gamma)], \ \forall \gamma \in [0, 1],$ and the lower and upper mean probability values (see Dubois and Prade [3, pp. 292-293]) are

$$E_*(A) = a_1 - \alpha \int_{-\infty}^{a_1} L(x) dx \quad \text{and} \quad E^*(A) = a_2 + \beta \int_{a_2}^{+\infty} R(x) dx.$$

Therefore, the interval-valued expectation is $E(A) = [E_*(A), E^*(A)].$

The possibilistic mean, introduced by Carlsson and Fullér [1], equally generates an interval-valued function, with the lower and upper bounds indicated below:

$$M_*(A) = \frac{\int_0^1 \gamma[a_1 - \alpha L^{-1}(\gamma)] d\gamma}{\int_0^1 \gamma d\gamma} = 2\int_0^1 \gamma[a_1 - \alpha L^{-1}(\gamma)] d\gamma,$$

$$M^*(A) = \frac{\int_0^1 \gamma[a_2 - \beta R^{-1}(\gamma)] d\gamma}{\int_0^1 \gamma d\gamma} = 2\int_0^1 \gamma[a_2 - \beta R^{-1}(\gamma)] d\gamma$$

and the interval-valued possibilistic mean is $M(A) = [M_*(A), M^*(A)].$ The interval-valued possibilistic mean is a proper subset of the corresponding interval-valued probabilistic mean: $M(A) \subset E(A),$ as Carlsson and Fullér [1] proved in their paper.

The crisp possibilistic mean value [1] of A is given by

$$\overline{M}(A) = \frac{M_*(A) + M^*(A)}{2} \tag{4}$$

and, obviously, an analogous crisp value could be given for the crisp expectation [3].

3.1. Mean for non-normal fuzzy numbers

Let A be an L-R fuzzy number with strictly monotonic and continuous shape functions $L(x)$ and $R(x)$, whose γ-level sets are $[A]^\gamma = [a_L(\gamma), a_R(\gamma)]$, $\forall \gamma \in [0, h]$. The definition of the crisp possibilistic mean value [1, 6] is extended now to the case where A is a non-normal fuzzy number and some of its properties, which might be useful subsequently, are stated afterwards.

Definition 3. The crisp possibilistic mean value of A with height h is the arithmetic mean of the lower possibility-weighted average of the left-hand endpoint of the γ-cut and of the upper possibility-weighted average of the right-hand endpoint of the γ-cut, with the final expression given by

$$\overline{M}(A) = \frac{1}{h^2}\int_0^h \gamma[a_L(\gamma) + a_R(\gamma)]d\gamma. \tag{5}$$

In fact,

$$\overline{M}(A) = \frac{1}{2}[\overline{M}_*(A) + \overline{M}^*(A)] = \frac{1}{2}\left[\frac{\int_0^h \gamma a_L(\gamma)d\gamma}{\int_0^h \gamma d\gamma} + \frac{\int_0^h \gamma a_R(\gamma)d\gamma}{\int_0^h \gamma d\gamma}\right]$$

$$= \frac{1}{h^2}\int_0^h \gamma[a_L(\gamma) + a_R(\gamma)]d\gamma.$$

The factor $1/h^2$ originates from the weight $\left(\int_0^h \gamma d\gamma\right)^{-1}$ applied to the two types of means.

The properties of the crisp possibilistic mean value of A can be summarized as follows:

1. Internality

$\overline{M}(A)$ lies between the upper and lower bounds of the support. In fact, from the inequalities: $a_L(0) \leq a_L(\gamma) \leq a_R(\gamma) \leq a_R(0), \quad \forall\gamma,$ we can immediately infer that $\overline{M}(A) \in [a_L(0), a_R(0)]$.

2. Zero-sum deviation

The expected value of the negative deviations from the mean equals the expected value of the positive deviations. Let D be the deviation from $\overline{M}(A)$ with $D^+ = a_R(\gamma) - \overline{M}(A)$ and $D^- = a_L(\gamma) - \overline{M}(A)$ or vice versa. Then

$$\overline{M}(D) = \frac{1}{2}\left(\frac{2}{h^2}\int_0^h \gamma[a_R(\gamma) - \overline{M}(A)]d\gamma + \frac{2}{h^2}\int_0^h \gamma[a_L(\gamma) - \overline{M}(A)]d\gamma\right)$$

$$= \frac{1}{2}\frac{2}{h^2}\left(\int_0^h \gamma[a_R(\gamma) - \overline{M}(A) + a_L(\gamma) - \overline{M}(A)]d\gamma\right)$$

$$= \frac{1}{h^2}\int_0^h \gamma[a_R(\gamma) + a_L(\gamma)]d\gamma - \frac{1}{h^2}\int_0^h 2\gamma\overline{M}(A)d\gamma$$

$$= \overline{M}(A) - \frac{1}{h^2}2\overline{M}(A)\int_0^h \gamma d\gamma = \overline{M}(A) - \overline{M}(A) = 0.$$

3. Crispness

If A is a crisp real number $\overline{a}$, then $a_L(\gamma) = a_R(\gamma) = \overline{a}, \ \forall\gamma \in [0, h]$ and

$$\overline{M}(A) = \frac{1}{h^2}\int_0^h \gamma(\overline{a} + \overline{a})d\gamma = \overline{a}.$$

4. Invariance

For each $\lambda \in [0, h]$ and $\theta \in \mathbb{R}$, let us consider the linear transformation $C = \lambda A + \theta$. Hence, $C^\gamma = [\lambda a_L(\gamma) + \theta, \lambda a_R(\gamma) + \theta]$. Then the possibilistic mean (or expected) value of C is given by

$$\overline{M}(C) = \overline{M}(\lambda A + \theta) = \frac{1}{h^2}\int_0^h \gamma[\lambda a_L(\gamma) + \theta + \lambda a_R(\gamma) + \theta]d\gamma$$

$$= \frac{1}{h^2}\lambda\int_0^h \gamma[a_L(\gamma) + a_R(\gamma)]d\gamma + \frac{1}{h^2}2\theta\int_0^h \gamma d\gamma = \lambda\overline{M}(A) + \theta.$$

In a similar way, we might prove that the possibilistic mean of a linear combination of fuzzy numbers is equal to the linear combinations of the possibilistic means of fuzzy numbers, that is,

$$\overline{M}(\lambda A + \theta B) = \lambda\overline{M}(A) + \theta\overline{M}(B).$$

5. Associativity

This property implies that the total mean of a set A is equal to the weighted mean of the means of disjoint subsets, partitioning A. More formally, let A be an L-R fuzzy number with strictly monotonic and continuous shape functions and let $A_1, \ldots, A_k$ be subsets of A such that $\text{supp}(A_i) \cap \text{supp}(A_j) = \varnothing$ if $i \neq j$, $\mu_A(x) = \sum_{i=1}^{k} \mu_{A_i}(x)$, and $\max[\mu_{A_i}(x)] = h_i$. For the sake of simplicity, the partition is carried out in such a way that each A_i is only left or right. Then there exist two elements, A_{i_L} and A_{i_R}, of the partition such that $\max[\mu_{A_{i_L}}(x)] = \max[\mu_{A_{i_R}}(x)] = h$ and $(i_R - i_L) \geq 1$. Consequently, for $1 \leq i \leq i_L$, $a_i(\gamma) = a_L(\gamma)$ if $h_{i-1} \leq \gamma \leq h_i$ and $a_i(\gamma) = 0$ elsewhere. For $i_R \leq i \leq k$, $a_i(\gamma) = a_R(\gamma)$ if $h_{i-1} \leq \gamma \leq h_i$ and $a_i(\gamma) = 0$ elsewhere. Therefore, $\sum_{i=1}^{i_L} a_i(\gamma) = a_L(\gamma)$ and $\sum_{i=i_R}^{k} a_i(\gamma) = a_R(\gamma)$. Let $w_i = (h_i^2 - h_{i-1}^2)/(2h^2)$ be the weight of A_i. Then the possibilistic mean of A is given by the weighted mean of the possibilistic means of the subsets A_i:

$$\overline{M}(A) = \sum_{i=1}^{k} w_i\overline{M}(A_i) = \sum_{i=1}^{k} w_i \frac{\int_{h_{i-1}}^{h_i} \gamma a_i(\gamma)d\gamma}{\int_{h_{i-1}}^{h_i} \gamma d\gamma}$$

$$= \sum_{i=1}^{k} w_i \frac{2}{h_i^2 - h_{i-1}^2} \int_{h_{i-1}}^{h_i} \gamma a_i(\gamma) d\gamma$$

$$= \sum_{i=1}^{k} \frac{h_i^2 - h_{i-1}^2}{2h^2} \frac{2}{h_i^2 - h_{i-1}^2} \int_{h_{i-1}}^{h_i} \gamma a_i(\gamma) d\gamma$$

$$= \sum_{i=1}^{k} \frac{1}{h^2} \int_{h_{i-1}}^{h_i} \gamma a_i(\gamma) d\gamma = \frac{1}{h^2} \left[\int_0^h \gamma a_L(\gamma) d\gamma + \int_0^h \gamma a_R(\gamma) d\gamma \right]$$

$$= \frac{1}{h^2} \int_0^h \gamma [a_L(\gamma) + a_R(\gamma)] d\gamma.$$

The previous list reports the main properties of the crisp possibilistic mean of a fuzzy number that are similar to those of the arithmetic mean for a set of numbers or to those of the expected value for a real-valued random variable. Moreover, it should be noted that when $h = 1$, the results can be easily deduced from those explicitly reported in Fullér and others [1, 6]. Now, let us calculate the crisp possibilistic mean for non-normal trapezoidal and triangular fuzzy numbers.

Example 3.1. Let $A = (a_1, a_2, \alpha, \beta)$ be a non-normal trapezoidal fuzzy number. Then $a_L(\gamma) = a_1 - (1 - \gamma/h)\alpha$ and $a_R(\gamma) = a_2 + (1 - \gamma/h)\beta$, while the possibilistic mean (or expected) value of A (without detailed algebraic passages) is given by

$$\overline{M}(A) = \frac{1}{h^2} \int_0^h \gamma [a_1 - (1 - \gamma/h)\alpha + a_2 + (1 - \gamma/h)\beta] d\gamma$$

$$= \frac{(a_1 + a_2)}{2} + \frac{(\beta - \alpha)}{6}$$

and it could be read as equal to the midpoint of the minor basis plus a fraction of the skewness, i.e., an asymmetric measure of the trapezium, but it does not depend on the height, h, of the non-normal fuzzy number.

If $a_2 = a_1$, then A is a non-normal triangular fuzzy number and $\overline{M}(A)$

$= a_1 + (\beta - \alpha)/6.$ Therefore, analogously to the case of a trapezoidal fuzzy number, it could be read as equal to the abscissa of the vertex plus a fraction of the skewness, i.e., a measure of the triangle's asymmetry.

These results show that triangular and trapezoidal non-normal fuzzy numbers with different heights, h, have the same interval-valued possibilistic mean as normal ones, as could be argued intuitively from Figure 1, where the trapezoidal fuzzy number, $A(a_1, a_2, \alpha, \beta)$, is reported with respect to different heights $(h_1, h_2, h_3, \ldots)$, implying different slopes for the membership functions. However, any non-normal fuzzy number of the LR type with strictly monotonic functions, $A(a_1, a_2, a_L(\gamma), a_R(\gamma))$, should provide the same outcome, if $\max[\mu_A(x)] = h$, $\forall h \in [0, 1]$, whereas at its extreme values: $a_L(0) = a_1 - \alpha$ and $a_L(h) = a_1$, as well $a_R(0) = a_2 - \beta$ and $a_R(h) = a_2$.

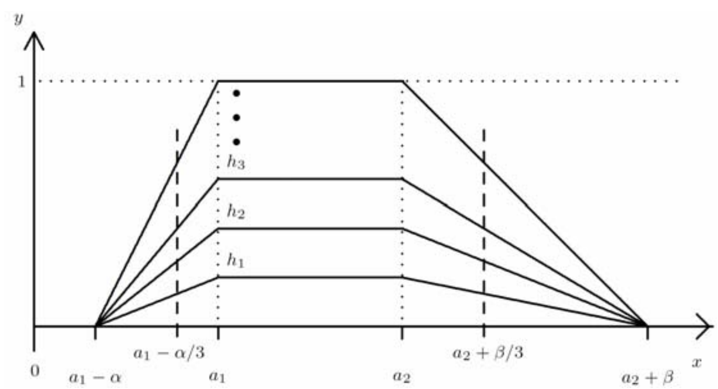

Figure 1. Trapezoidal fuzzy numbers with different heights, h, setup on $A(a_1, a_2, \alpha, \beta)$.

In general, the variance may depend on the height, h, as shown in the following example:

Example 3.2. Let $A = (a_1, a_2, \alpha, \beta)$ be a non-normal trapezoidal fuzzy number with $a_L(\gamma) = a_1 - \alpha + \alpha \log(1 + h\gamma)/\log(1 + h^2)$ and $a_R(\gamma) = a_2 + (1 - \gamma/h)\beta$. The possibilistic mean (or expected) value of A (again without detailed algebraic passages) is

$$\overline{M}(A) = \frac{1}{h^2}\int_0^h \gamma\left[a_1 - \alpha + \alpha\frac{\log(1+h\gamma)}{\log(1+h^2)} + a_2 + \left(1 - \frac{\gamma}{h}\right)\beta\right]d\gamma$$

$$= \frac{(a_1 + a_2)}{2} + \frac{\beta}{6} - \frac{\alpha(h^2 - 2)}{4h^2\log(1+h^2)} - \frac{\alpha}{2h^4}.$$

On the contrary, if $a_L(\gamma)$ and/or $a_R(\gamma)$ depend only on γ/h, then a simple change in the variable leads to independence of the variance on h. For example, let $A = (a_1, a_2, \alpha, \beta)$ be a non-normal trapezoidal fuzzy number with $a_L(\gamma) = a_1 - \alpha + \alpha\log[1 + (e-1)\gamma/h]$ and $a_R(\gamma) = a_2 + (1 - \gamma/h)\beta$, where e is the base of the natural logarithms. It can be shown that the possibilistic mean (or expected) value of A is

$$\overline{M}(A) = (a_1 + a_2)/2 + \beta/6 - \alpha(e^2 - 4e + 5)/[4(e-1)^2].$$

3.2. Variance and covariance for non-normal fuzzy numbers

Let A be a non-normal fuzzy number, $A^\gamma = [a_L(\gamma), a_R(\gamma)]$ be its γ-cuts, and $\max_{x\in\mathbb{R}}[\mu(x)]$ be equal to h, where $h \leq 1$. Let $E(\gamma) = \frac{1}{2}(a_L(\gamma) + a_R(\gamma))$ be the arithmetic mean of the γ-levels. Following Carlsson and Fullér [1], the definition of the variance of A can be stated as follows:

Definition 4. The crisp possibilistic variance of A is the lower possibility-weighted average of the squared distance between $E(\gamma)$ and the left-hand endpoint of the γ-cut plus the upper possibility-weighted average of the squared distance between $E(\gamma)$ and the right-hand endpoint of the γ-cut, i.e.,

$$V(A) = \frac{1}{h^2}\left(\int_0^h \operatorname{Pos}[A \leq a_L(\gamma)][E(\gamma) - a_L(\gamma)]^2 d\gamma + \int_0^h \operatorname{Pos}[A \geq a_R(\gamma)][E(\gamma) - a_R(\gamma)]^2 d\gamma\right). \tag{6}$$

Therefore, the crisp possibilistic variance of A is the arithmetic mean of the lower and upper possibilistic variances: $V_*(A)$ and $V^*(A)$, respectively. The factor $1/h^2$ is derived from the weights, as in the definition of the possibilistic mean. Since $\mathrm{Pos}[A \le a_L(\gamma)] = \mathrm{Pos}[A \ge a_R(\gamma)] = \gamma$, elementary calculus leads to the following formula:

$$V(A) = \frac{1}{2h^2}\int_0^h [a_R(\gamma) - a_L(\gamma)]^2 \gamma d\gamma. \tag{7}$$

Example 3.3. Let $A = (a_1, a_2, \alpha, \beta)$ be a trapezoidal fuzzy number with central values a_1 and a_2, spreads α and β and height h. Then

$$V(A) = \left(\frac{a_2 - a_1}{2} + \frac{\alpha + \beta}{6}\right)^2 + \frac{(\alpha + \beta)^2}{72}.$$

Example 3.4. If $a_2 = a_1$, then $A = (a_1, \alpha, \beta)$ is a triangular fuzzy number with central value a_1, spreads α and β and height h. Its variance is

$$V(A) = \frac{(\alpha + \beta)^2}{24}$$

and it follows immediately from the preceding result. Similarly to the crisp possibilistic mean, the crisp possibilistic variance does not depend on height, *h*, if the fuzzy number is a trapezoidal or triangular fuzzy number. Therefore, in these cases, as could be argued from Figure 1, the interval-valued possibilistic variance does not depend on *h*, because when *h* changes, the spread measured by variance remains unchanged.

The variance may be defined also as the second moment about a given value, termed *pole* and generally assumed to be equal to the expected value, but any other value might be chosen.

Definition 5. Let $x_0 \in \mathbb{R}$ be a fixed number and let A be a non-normal fuzzy number as defined above. Then, analogously to the previous definition, the variance of A, with respect to the *pole* x_0, is given by

$$V_{x_0}(A) = \frac{1}{h^2}\left(\int_0^h \mathrm{Pos}[A \le a_L(\gamma)][x_0 - a_L(\gamma)]^2 d\gamma \right.$$

$$\left. + \int_0^h \mathrm{Pos}[A \ge a_R(\gamma)][x_0 - a_R(\gamma)]^2 d\gamma\right). \tag{8}$$

The following definition of the covariance between two fuzzy numbers is useful for the development of this expression and to obtain an interesting result as reported below [6].

Definition 6. Let A and B be two non-normal fuzzy numbers of the LR type with strictly monotonic and continuous shape functions and with the same height, h. The crisp possibilistic covariance of A and B is the lower possibility-weighted average of the distance between $E(\gamma)$ and the left-hand endpoint of the γ-cut of A, multiplied by the corresponding distance of B plus the upper possibility-weighted average of the distance between the right-hand endpoint of the γ-cut and $E(\gamma)$ of A, multiplied by the corresponding distance of B, i.e.,

$$C(A, B) = \frac{1}{h^2}\left(\int_0^h \gamma[E_A(\gamma) - a_L(\gamma)][E_B(\gamma) - b_L(\gamma)]d\gamma \right.$$

$$\left. + \int_0^h \gamma[a_R(\gamma) - E_A(\gamma)][b_R(\gamma) - E_B(\gamma)]d\gamma\right). \tag{9}$$

Therefore, the crisp possibilistic covariance of A and B is the arithmetic mean of the lower and upper possibilistic covariances: $C_*(A, B)$ and $C^*(A, B)$, respectively. The factor $1/h^2$ is derived from the weights, as in the variance. Simple passages lead to

$$C(A, B) = \frac{1}{2h^2}\int_0^h \gamma[a_R(\gamma) - a_L(\gamma)][b_R(\gamma) - b_L(\gamma)]d\gamma. \tag{10}$$

Note that if $A \equiv B$, i.e., A coincides with B, then $C(A, A) = V(A)$.

Example 3.5. Let $A = (a_1, a_2, \alpha, \beta)$ and $B = (b_1, b_2, \varepsilon, \delta)$ be two trapezoidal fuzzy numbers. It can be shown that the covariance between A

and B is given by

$$C(A,\ B) = \frac{1}{2h^2}\int_0^h \gamma[a_2 + (1-\gamma/h)\beta - a_1 + (1-\gamma/h)\alpha]$$

$$\cdot [b_2 + (1-\gamma/h)\delta - b_1 + (1-\gamma)\varepsilon]d\gamma$$

$$= \left(\frac{a_2 - a_1}{2} + \frac{\alpha+\beta}{6}\right)\left(\frac{b_2 - b_1}{2} + \frac{\varepsilon+\delta}{6}\right) + \frac{1}{72}(\alpha+\beta)(\varepsilon+\delta).$$

Example 3.6. If $a_2 = a_1$ and $b_2 = b_1$, then A and B are triangular fuzzy numbers and their covariance becomes

$$C(A,\ B) = \frac{1}{24}(\alpha+\beta)(\varepsilon+\delta).$$

Also, the covariance of two trapezoidal or triangular fuzzy numbers does not depend on h, but in the case of functions of other shapes, it might depend on h.

Example 3.7. Let A be a non-normal fuzzy number of the LR type with strictly decreasing and continuous shape functions, and with the centre $c = (a_1 + a_2)/2$. Let $a_L(h) = a_1$ and $a_R(h) = a_2$ be the internal values corresponding to the maximum heights of the memberships functions. Let us consider two symmetrical fuzzy numbers, A' and A'' such that $E(A') = E(A'') = c$ and $A'^{\gamma} = [a_L(\gamma),\ \phi(\gamma)]$ and $A''^{\gamma} = [\psi(\gamma),\ a_L(\gamma)]$, where ϕ and $\psi : [0,\ h] \to \mathbb{R}$ are defined as follows: $\phi(\gamma) = a_1 + a_2 - a_L(\gamma)$ and $\psi(\gamma) = a_1 + a_2 - a_R(\gamma)$. Then A' and A'' are two symmetrical non-normal fuzzy numbers of the LR type, derived from A, and their covariance will be

$$C(A',\ A'') = \frac{1}{h^2}\left(\int_0^h \gamma[E_{A'}(\gamma) - a_L(\gamma)][E_{A''}(\gamma) - a_R(\gamma)]d\gamma\right.$$

$$\left. + \int_0^h \gamma[a_L(\gamma) - E_{A'}(\gamma)][a_R(\gamma) - E_{A''}(\gamma)]d\gamma\right)$$

$$= \frac{2}{h^2}\int_0^h \gamma[c - a_L(\gamma)][a_R(\gamma) - c]d\gamma.$$

Example 3.8. Let $A = (a_1, a_2, \alpha, \beta)$ be a trapezoidal fuzzy number and $c = (a_1 + a_2)/2$. Let $A' = (a_1, a_2, \alpha)$ and $A'' = (a_1, a_2, \beta)$ be two symmetrical trapezoidal fuzzy numbers setup on A, as indicated in the previous Example 3.7. The covariance between A' and A'' is given by

$$C(A', A'') = \frac{2}{h^2}\int_0^h \gamma[c - a_1 + (1-\gamma/h)\alpha][a_2 + (1-\gamma/h)\beta - c]d\gamma$$

$$= \frac{(a_2 - a_1)^2}{4} + \frac{(a_2 - a_1)(\alpha + \beta)}{6} + \frac{\alpha\beta}{6}.$$

Example 3.9. If $a_2 = a_1$, then A is a triangular fuzzy number. Let $A' = (a_1, \alpha)$ and $A'' = (a_1, \beta)$ be two symmetrical triangular fuzzy numbers setup on A. The covariance between A' and A'' is given by

$$C(A', A'') = \frac{\alpha\beta}{6}.$$

The variance $V(A)$ satisfies the following properties:

1. Positivity: $V(A) \geq 0$.

2. Zero-variance: $V(A) = 0$ if and only if A is a crisp (constant) value, implying $a_L(\gamma) = a_R(\gamma) = \bar{a}$.

3. Invariance for linear transformation: Let $\lambda \in [0, 1]$ and $\theta \in \mathbb{R}$ be real numbers, let A be a fuzzy number, and let $B = \lambda A + \theta$. Then

$$V(B) = V(\lambda A + \theta) = \lambda^2 V(A).$$

The following theorem could be seen also as a property of variance.

Theorem 1. *Let $x_0 \in \mathbb{R}$ be a constant number and let A be a non-normal fuzzy number of the LR type with strictly decreasing and continuous shape functions, and with centre c. Then*

$$V_{x_0}(A) = 2V(A) + [x_0 - \overline{M}(A)]^2 - [c - \overline{M}(A)]^2 - C(A', A''), \tag{11}$$

where A' *and* A'' *are two symmetrical non-normal fuzzy numbers of the LR type*, *constructed upon A*, *as indicated in Example* 3.7.

Proof.

$$V_{x_0}(A) = \frac{1}{h^2}\left[\int_0^h \gamma[x_0 - a_L(\gamma)]^2 d\gamma + \int_0^h \gamma[x_0 - a_R(\gamma)]^2 d\gamma\right]$$

Simple passages lead to

$$= \frac{1}{h^2}\int_0^h \gamma\{[a_R(\gamma) - a_L(\gamma)]^2 + 2x_0^2 - 2x_0[a_L(\gamma) + a_R(\gamma)]$$

$$+ 2a_L(\gamma)a_R(\gamma)\}d\gamma$$

$$= 2V(A) + \frac{1}{h^2}x_0^2[\gamma^2|_0^h] - 2x_0\overline{M}(A)$$

$$- \frac{1}{h^2}2\int_0^h \gamma\{c^2 - c[a_L(\gamma) + a_R(\gamma)] + [c - a_L(\gamma)][a_R(\gamma) - c]\}d\gamma$$

$$= 2V(A) + x_0^2 - 2x_0\overline{M}(A) - \frac{1}{h^2}c^2[\gamma^2|_0^h] + 2c\overline{M}(A) - C(A', A'')$$

$$= 2V(A) + x_0^2 - 2x_0\overline{M}(A) - c^2 + 2c\overline{M}(A) \pm [\overline{M}(A)]^2 - C(A', A'')$$

$$= 2V(A) + x_0^2 - 2x_0\overline{M}(A) + [\overline{M}(A)]^2 - c^2 + 2c\overline{M}(A)$$

$$- [\overline{M}(A)]^2 - C(A', A'')$$

$$= 2V(A) + [x_0 - \overline{M}(A)]^2 - [c - \overline{M}(A)]^2 - C(A', A'').$$

Theorem 2. *Let* $x_0 \in \mathbb{R}$ *be a constant number and let A be a non-normal fuzzy number of the LR type with strictly monotonic and continuous shape functions*, *and* $E(\gamma) = [a_L(\gamma) + a_R(\gamma)]/2.$ *Then the following equality also holds*:

$$V_{x_0}(A) = V(A) + \frac{2}{h^2}\int_0^h \gamma[x_0 - E(\gamma)]^2 d\gamma. \tag{12}$$

Proof.

$$V_{x_0}(A) = \int_0^h \gamma[x_0 - a_L(\gamma)]^2 d\gamma + \int_0^h \gamma[x_0 - a_R(\gamma)]^2 d\gamma$$

$$= \int_0^h \gamma\{[x_0 - a_L(\gamma) \pm E(\gamma)]^2 + [x_0 - a_R(\gamma) \pm E(\gamma)]^2\} d\gamma$$

$$= \int_0^h \gamma\{[x_0 - E(\gamma)]^2 + [-a_L(\gamma) + E(\gamma)]^2$$

$$+ 2[x_0 - E(\gamma)][-a_L(\gamma) + E(\gamma)] + [x_0 - E(\gamma)]^2$$

$$+ [-a_R(\gamma) + E(\gamma)]^2 + 2(x_0 - E(\gamma))[-a_R(\gamma) + E(\gamma)]\} d\gamma$$

$$= \int_0^h \gamma\{2[x_0 - E(\gamma)]^2 + 2[x_0 - E(\gamma)][E(\gamma) - a_L(\gamma) + E(\gamma) - a_R(\gamma)]$$

$$+ [E(\gamma) - a_L(\gamma)]^2 + [E(\gamma) - a_R(\gamma)]^2\} d\gamma$$

$$= \int_0^h \gamma\{[E(\gamma) - a_L(\gamma)]^2 + [E(\gamma) - a_R(\gamma)]^2 + 2[x_0 - E(\gamma)]^2$$

$$+ 2[x_0 - E(\gamma)][2E(\gamma) - a_L(\gamma) - a_R(\gamma)]\} d\gamma$$

$$= \int_0^h \gamma\{[E(\gamma) - a_L(\gamma)]^2 + [E(\gamma) - a_R(\gamma)]^2 + 2[x_0 - E(\gamma)]^2\} d\gamma$$

$$= \int_0^h \gamma\{[E(\gamma) - a_L(\gamma)]^2 + [E(\gamma) - a_R(\gamma)]^2\} d\gamma + \int_0^h \gamma 2[x_0 - E(\gamma)]^2 d\gamma$$

$$= V(A) + 2\int_0^h \gamma[x_0 - E(\gamma)]^2 d\gamma.$$

Note that the 9th line disappears in 10th line because $[2E(\gamma) - a_L(\gamma) - a_R(\gamma)] = 0.$ The expressions given by the two theorems are obviously equivalent. The proof is carried out through the expressions reported below:

$$V_{x_0}(A) = V(A) + \frac{2}{h^2}\int_0^h \gamma[x_0 - E(\gamma)]^2 d\gamma$$

$$= V(A) + \frac{2}{h^2}\int_0^h \gamma\left[x_0 - \frac{a_L(\gamma) + a_R(\gamma)}{2}\right]^2 d\gamma$$

$$= V(A) + \frac{2}{h^2}\int_0^h \gamma\left[\frac{2x_0 - [a_L(\gamma) + a_R(\gamma)]}{2}\right]^2 d\gamma$$

$$= V(A) + \frac{2}{h^2}\int_0^h \gamma\,\frac{1}{4}\{4x_0^2 - 4x_0[a_L(\gamma) + a_R(\gamma)]$$

$$+ [a_L(\gamma) + a_R(\gamma)]^2\}d\gamma$$

$$= V(A) + \frac{1}{2h^2}\int_0^h \gamma 4x_0^2 d\gamma - \frac{1}{2h^2}\int_0^h \gamma 4x_0[a_L(\gamma) + a_R(\gamma)]d\gamma$$

$$+ \frac{1}{2h^2}\int_0^h \gamma[a_L(\gamma) + a_R(\gamma)]^2 d\gamma$$

$$= V(A) + \frac{2}{h^2}x_0^2\left[\frac{\gamma^2}{2}\Bigg|_0^h\right] - \frac{2}{h^2}x_0\int_0^h \gamma[a_L(\gamma) + a_R(\gamma)]d\gamma$$

$$+ \frac{1}{2h^2}\int_0^h \gamma[a_L(\gamma) + a_R(\gamma) \pm a_L(\gamma)]^2 d\gamma$$

$$= V(A) + x_0^2 - 2x_0\overline{M}(A) + \frac{1}{2h^2}\int_0^h \gamma\{[a_R(\gamma) - a_L(\gamma)]^2$$

$$+ 4a_L(\gamma)[a_R(\gamma) - a_L(\gamma)] + 4a_L(\gamma)^2\}d\gamma$$

$$= V(A) + x_0^2 - 2x_0\overline{M}(A) + \frac{1}{2h^2}\int_0^h \gamma[a_R(\gamma) - a_L(\gamma)]^2 d\gamma$$

$$+ \frac{1}{2h^2}\int_0^h \gamma[4a_L(\gamma)a_R(\gamma) - 4a_L(\gamma)^2 + 4a_L(\gamma)^2]d\gamma$$

$$= V(A) + x_0^2 - 2x_0\overline{M}(A) + V(A) + \frac{2}{h^2}\int_0^h \gamma[a_L(\gamma)a_R(\gamma)]d\gamma$$

$$= 2V(A) + x_0^2 - 2x_0\overline{M}(A) + \frac{2}{h^2}\int_0^h \gamma[a_L(\gamma)a_R(\gamma)]d\gamma.$$

Example 3.10. Let $A = (a_1, a_2, \alpha, \beta)$ be a trapezoidal non-normal fuzzy number and $c = (a_1 + a_2)/2$. Let $A' = (a_1, a_2, \alpha)$ and $A'' = (a_1, a_2, \beta)$ be two symmetrical trapezoidal fuzzy numbers setup on A. The crisp possibilistic variance of A, with respect to $x_0 \in \mathbb{R}$, is given by

$$V_{x_0}(A) = \frac{(a_2 - a_1)^2}{4} + \frac{(\alpha + \beta)^2}{12} + \left(x_0 - \frac{a_1 + a_2}{2}\right)^2$$

$$- (x_0 - a_2)\frac{\beta - \alpha}{3} - \frac{\alpha\beta}{6}.$$

The expression for a triangular fuzzy number is immediately obtained, assuming $a_1 = a_2$.

Proof.

$$V_{x_0}(A) = 2V(A) + [x_0 - \overline{M}(A)]^2 - [c - \overline{M}(A)]^2 - C(A', A'')$$

$$= 2\left\{\left[\frac{a_2 - a_1}{2} + \frac{\alpha + \beta}{6}\right]^2 + \frac{(\alpha + \beta)^2}{72}\right\}$$

$$+ \left(x_0 - \frac{(a_1 + a_2)}{2} - \frac{(\beta - \alpha)}{6}\right)^2$$

$$- \left[\frac{a_1 + a_2}{2} - \frac{(a_1 + a_2)}{2} - \frac{(\beta - \alpha)}{6}\right]^2$$

$$- \left[\frac{(a_2 - a_1)^2}{4} + \frac{(a_2 - a_1)(\alpha + \beta)}{6} + \frac{\alpha\beta}{6}\right]$$

$$= 2\left(\frac{a_2 - a_1}{2}\right)^2 + 2\left(\frac{\alpha + \beta}{6}\right)^2 + 2(a_2 - a_1)\frac{\alpha + \beta}{6} + 2\,\frac{(\alpha + \beta)^2}{72}$$

$$+\left(x_0-\frac{a_1+a_2}{2}\right)^2+\left(\frac{\beta-\alpha}{6}\right)^2-2\left(x_0-\frac{a_1+a_2}{2}\right)\left(\frac{\beta-\alpha}{6}\right)$$

$$-\left[-\frac{\beta-\alpha}{6}\right]^2-\frac{(a_2-a_1)^2}{4}-\frac{(a_2-a_1)(\alpha+\beta)}{6}-\frac{\alpha\beta}{6}$$

$$=\frac{(a_2-a_1)^2}{2}-\frac{(a_2-a_1)^2}{4}+\frac{(\alpha+\beta)^2}{18}+\frac{(\alpha+\beta)^2}{36}$$

$$+\left(x_0-\frac{a_1+a_2}{2}\right)^2+(a_2-a_1)\frac{\alpha+\beta}{3}-(a_2-a_1)\frac{\alpha+\beta}{6}$$

$$+\left(\frac{\beta-\alpha}{6}\right)^2-\left(\frac{\beta-\alpha}{6}\right)^2-\left(x_0-\frac{a_1+a_2}{2}\right)\frac{\beta-\alpha}{3}-\frac{\alpha\beta}{6}$$

$$=\frac{(a_2-a_1)^2}{4}+\frac{(\alpha+\beta)^2}{12}+\left(x_0-\frac{a_1+a_2}{2}\right)^2$$

$$+\left[(a_2-a_1)-\frac{a_2-a_1}{2}-\left(x_0-\frac{a_1+a_2}{2}\right)\right]\frac{\beta-\alpha}{3}-\frac{\alpha\beta}{6}$$

$$=\frac{(a_2-a_1)^2}{4}+\frac{(\alpha+\beta)^2}{12}+\left(x_0-\frac{a_1+a_2}{2}\right)^2$$

$$-(x_0-a_2)\frac{\beta-\alpha}{3}-\frac{\alpha\beta}{6}.$$

In order to introduce one more property of variance, it is useful to define the variance in a form analogous to the one used for a real-valued random variable, already reported by Carlsson and Fullér [1] and adapted here for a non-normal fuzzy number.

Definition 7. Let A be a non-normal fuzzy number of the LR type with strictly decreasing and continuous shape functions. The crisp possibilistic variance of A is the lower possibility-weighted average of the squared distance between $\overline{M}(A)$ and the left-hand endpoint of the γ-cut plus the upper possibility-weighted average of the squared distance between $\overline{M}(A)$

and the right-hand endpoint of the γ-cut, i.e., the pole is the constant real value, $\overline{M}(A) \in \mathbb{R}$, and the variance becomes

$$V_{\overline{M}}(A) = \frac{1}{h^2}\int_0^h \gamma\{[\overline{M}(A) - a_L(\gamma)]^2 + [\overline{M}(A) - a_R(\gamma)]^2\}d\gamma. \tag{13}$$

It can be proved that

$$V_{\overline{M}}(A) = \frac{1}{h^2}\int_0^h \gamma[a_L^2(\gamma) + a_R^2(\gamma)]d\gamma - \overline{M}^2(A). \tag{14}$$

Proof.

$$V'(A) = \frac{1}{2}[V'_*(A) + V'^*(A)]$$

$$= \frac{1}{2}\left(\frac{1}{\int_0^h \gamma d\gamma}\int_0^h \mathrm{Pos}[A \le a_L(\gamma)][\overline{M}(A) - a_L(\gamma)]^2 d\gamma\right.$$

$$\left.+ \frac{1}{\int_0^h \gamma d\gamma}\int_0^h \mathrm{Pos}[A \ge a_R(\gamma)][\overline{M}(A) - a_R(\gamma)]^2 d\gamma\right)$$

$$= \frac{1}{2}\left(\frac{2}{h^2}\int_0^h \gamma[\overline{M}(A) - a_L(\gamma)]^2 d\gamma + \frac{2}{h^2}\int_0^h \gamma[\overline{M}(A) - a_R(\gamma)]^2 d\gamma\right)$$

$$= \frac{1}{h^2}\int_0^h \gamma\{[\overline{M}(A) - a_L(\gamma)]^2 + [\overline{M}(A) - a_R(\gamma)]^2\}d\gamma$$

$$= \frac{1}{h^2}\int_0^h \gamma[\overline{M}^2(A) + a_L^2(\gamma) - 2\overline{M}(A)a_L(\gamma)$$

$$+ \overline{M}^2(A) + a_R^2(\gamma) - 2\overline{M}(A)a_R(\gamma)]d\gamma$$

$$= \frac{1}{h^2}\int_0^h \gamma[a_L^2(\gamma) + a_R^2(\gamma)]d\gamma + \frac{1}{h^2}\int_0^h \gamma 2\overline{M}^2(A)d\gamma$$

$$-\frac{1}{h^2}\int_0^h \gamma 2\overline{M}(A)[a_L(\gamma)+a_R(\gamma)]d\gamma$$

$$=\frac{1}{h^2}\int_0^h \gamma[a_L^2(\gamma)+a_R^2(\gamma)]d\gamma+\overline{M}^2(A)-2\overline{M}^2(A)$$

$$=\frac{1}{h^2}\int_0^h \gamma[a_L^2(\gamma)+a_R^2(\gamma)]d\gamma-\overline{M}^2(A).$$

There is a property of $V_{\overline{M}}(A)$ that could also be ascribed to the property of the crisp possibilistic mean, specifically to its weighted squared deviation, whose average or expected value is minimum when the pole of deviation is assumed to be the crisp possibilistic mean.

Theorem 3. *Let $x_0 \in \mathbb{R}$ be a constant number and let A be a non-normal fuzzy number of the LR type with strictly decreasing and continuous shape functions. Then the variance with respect to the pole x_0, $V_{x_0}(A)$, always exceeds the variance with respect to the pole $\overline{M}(A)$, $V_{\overline{M}}(A)$, i.e.,*

$$V_{x_0}(A) \geq V_{\overline{M}}(A). \tag{15}$$

Proof.

$$V_{\overline{M}}(A)=\frac{1}{h^2}\int_0^h \gamma\{[\overline{M}(A)\pm x_0-a_L(\gamma)]^2+[\overline{M}(A)\pm x_0-a_R(\gamma)]^2\}d\gamma$$

$$=\frac{1}{h^2}\int_0^h \gamma\{[\overline{M}(A)-x_0]^2+[x_0-a_L(\gamma)]^2+2[\overline{M}(A)-x_0][x_0-a_L(\gamma)]$$

$$+[\overline{M}(A)-x_0]^2+[x_0-a_R(\gamma)]^2$$

$$+2[\overline{M}(A)-x_0][x_0-a_R(\gamma)]\}d\gamma$$

$$=\frac{1}{h^2}\int_0^h \gamma\{[x_0-a_L(\gamma)]^2+[x_0-a_R(\gamma)]^2\}d\gamma$$

$$+\frac{1}{h^2}\int_0^h \gamma 2[\overline{M}(A)-x_0]^2 d\gamma$$

$$+\frac{1}{h^2}\int_0^h \gamma 2[\overline{M}(A)-x_0][x_0-a_L(\gamma)+x_0-a_R(\gamma)]d\gamma$$

$$= V_{x_0}(A)+[\overline{M}(A)-x_0]^2+[\overline{M}(A)-x_0]\frac{1}{h^2}\int_0^h 2\gamma 2x_0 d\gamma$$

$$-[\overline{M}(A)-x_0]\frac{1}{h^2}\int_0^h 2\gamma[a_L(\gamma)+a_R(\gamma)]d\gamma$$

$$= V_{x_0}(A)+[\overline{M}(A)-x_0]^2+2x_0[\overline{M}(A)-x_0]-2\overline{M}(A)[\overline{M}(A)-x_0]$$

$$= V_{x_0}(A)+[\overline{M}(A)-x_0]^2+2[\overline{M}(A)-x_0][x_0-\overline{M}(A)]$$

$$= V_{x_0}(A)-[\overline{M}(A)-x_0]^2 \le V_{x_0}(A).$$

The relationships between the variances $V_{x_0}(A)$, $V_{\overline{M}}(A)$ and $V(A)$ are stated in the following theorem:

Theorem 4. *Let A be a non-normal fuzzy number of the LR type with strictly decreasing and continuous shape functions. Then the variance is minimum when the pole is the mean* $E(\gamma)=[a_L(\gamma)+a_R(\gamma)]/2,$ *i.e., for any constant value* $x_0 \in \mathbb{R},$ *the following inequalities hold*:

$$V_{x_0}(A) \ge V_{\overline{M}}(A) \ge V(A). \tag{16}$$

Therefore, the variance, $V_{\overline{M}}(A),$ is formally similar to the corresponding notion of variance defined in statistics, but, unlike the latter, $V_{\overline{M}}(A)$ does not attain the minimum value, i.e., the pole equal to $\overline{M}(A)$ is not the pole providing the minimum value for the variance of *A*. On the contrary, the definitions of $V(A),$ given in equations (6) or (7), which assume the mean of the γ-levels, $E(\gamma),$ as pole, have the minimum value, although the pole is not a fixed point in the support of *A*. The proof is analogous to the previous:

$$V_{\overline{M}}(A) = \frac{1}{h^2}\int_0^h \gamma\{[\overline{M}(A) \pm E(\gamma) - a_L(\gamma)]^2 + [\overline{M}(A) \pm E(\gamma) - a_R(\gamma)]^2\}d\gamma$$

$$= \frac{1}{h^2}\int_0^h \gamma\{[\overline{M}(A) - E(\gamma)]^2 + [E(\gamma) - a_L(\gamma)]^2$$

$$+ 2[\overline{M}(A) - E(\gamma)][E(\gamma) - a_L(\gamma)] + [\overline{M}(A) - E(\gamma)]^2$$

$$+ [E(\gamma) - a_R(\gamma)]^2 + 2[\overline{M}(A) - E(\gamma)][E(\gamma) - a_R(\gamma)]\}d\gamma$$

$$= \frac{1}{h^2}\int_0^h \gamma\{[E(\gamma) - a_L(\gamma)]^2 + [E(\gamma) - a_R(\gamma)]^2\}d\gamma$$

$$+ \frac{1}{h^2}\int_0^h \gamma\{2[\overline{M}(A) - E(\gamma)]^2$$

$$+ 2[\overline{M}(A) - E(\gamma)][E(\gamma) - a_L(\gamma) + E(\gamma) - a_R(\gamma)]\}d\gamma$$

$$= V(A) + \frac{1}{h^2}\int_0^h \gamma\{2[\overline{M}(A) - E(\gamma)]^2$$

$$+ 2[\overline{M}(A) - E(\gamma)][2E(\gamma) - a_L(\gamma) - a_R(\gamma)]\}d\gamma$$

$$= V(A) + \frac{1}{h^2}\int_0^h \gamma\{2[\overline{M}(A) - E(\gamma)]^2\}d\gamma \geq V(A).$$

Let $A = (a_1, a_2, \alpha, \beta)$ be a trapezoidal non-normal fuzzy number with a centre $c = (a_1 + a_2)/2$, left-width $\alpha > 0$ and right-width $\beta > 0$ or let $A = (a_1, \alpha, \beta)$ be a triangular non-normal fuzzy number with a "centre" $c \equiv a_1$, left-width $\alpha > 0$ and right-width $\beta > 0$. Then, the results reported above, some of which previously illustrated in the literature [1, 6] are summarized in Table 1.

Table 1. Mean, variances, and covariances of a non-normal trapezoidal and a triangular fuzzy number with height h

Moment	Trapezoidal fuzzy number	Triangular fuzzy number
$\overline{M}(A)$	$\frac{(a_1 + a_2)}{2} + \frac{(\beta - \alpha)}{6}$	$a_1 + \frac{\beta - \alpha}{6}$
$V(A)$	$\left(\frac{a_2 - a_1}{2} + \frac{\alpha + \beta}{6}\right)^2 + \frac{(\alpha + \beta)^2}{72}$	$\frac{(\alpha + \beta)^2}{24}$
$C(A, B)$	$\left(\frac{a_2 - a_1}{2} + \frac{\alpha + \beta}{6}\right)\left(\frac{b_2 - b_1}{2} + \frac{\varepsilon + \delta}{6}\right)$ $+ \frac{1}{72}(\alpha + \beta)(\varepsilon + \delta)$	$\frac{1}{24}(\alpha + \beta)(\varepsilon + \delta)$
$C(A', A'')$	$\frac{(a_2 - a_1)^2}{4} + \frac{(a_2 - a_1)(\alpha + \beta)}{6} + \frac{\alpha\beta}{6}$	$\frac{\alpha\beta}{6}$
$V_{x_0}(A)$	$\frac{(a_2 - a_1)^2}{4} + \frac{(\alpha + \beta)^2}{12} + \left(x_0 - \frac{a_1 + a_2}{2}\right)^2$ $- (x_0 - a_2)\frac{\beta - \alpha}{3} - \frac{\alpha\beta}{6}$	$\frac{(\alpha + \beta)^2}{12} + (x_0 - a_1)^2$ $- (x_0 - a_1)\frac{\beta - \alpha}{3} - \frac{\alpha\beta}{6}$

4. Variance of Fuzzy Quantities

The previous definition is extended to the case of fuzzy quantities, namely to the union of two or more fuzzy numbers, which is not, in general, a fuzzy number.

Let $A = (a_1, a_2, \alpha, \beta)_{LR}$ and $B = (b_1, b_2, \varepsilon, \delta)_{LR}$, where $a_1 < b_1$, are normal fuzzy numbers with continuous monotonic shape functions. Let $A^\gamma = [a_L(\gamma), a_R(\gamma)]$ and $B^\gamma = [b_L(\gamma), b_R(\gamma)]$ be their γ-cuts.

The union and the intersection of A and B could be defined by means of several t-norms and t-conorms, well-known in literature. We choose here the most usual ones, that is, we define

$$D = A \cap B = \{(x, \mu_D(x)) \text{ s.t. } \mu_D(x) = \min[\mu_A(x), \mu_B(x)]\}, \tag{17}$$

$$C = A \bigcup B = \{(x, \mu_C(x)) \text{ s.t. } \mu_C(x) = \max[\mu_A(x), \mu_B(x)]\}. \tag{18}$$

D is always a fuzzy number, with height being h_D.

Three cases may occur:

1. $\operatorname{supp} A \bigcap \operatorname{supp} B \neq \varnothing$, that is, $D \neq \varnothing$ and $A \bigcup B$ has convex γ-levels. Therefore, it is a fuzzy number.

2. $\operatorname{supp} A \bigcap \operatorname{supp} B \neq \varnothing$, that is, $D \neq \varnothing$, but $A \bigcup B$ does not have convex γ-levels. Therefore, it is a fuzzy quantity, but not a fuzzy number.

3. $\operatorname{supp} A \bigcap \operatorname{supp} B \neq \varnothing$, that is, $D \neq \varnothing$.

From our point of view, the last two are the most relevant cases.

If $E(\gamma)$ is the mean value, $E(\gamma) = \frac{1}{4}(a_L + a_R + b_L + b_R)(\gamma)$, using the above mentioned notations, the variance of the union of A and B could be stated as follows.

Definition 8. The variance of $A \bigcup B$ is given by

$$V(A \bigcup B) = \sum_{i=1}^{4} \int_{h_i}^{1} \gamma[\phi_i(\gamma) - E(\gamma)]^2 d\gamma, \tag{19}$$

where

$$\phi_i(\gamma) = \begin{cases} \min[a_L(\gamma), b_L(\gamma)], & i = 1, \\ a_R(\gamma), & i = 2, \\ b_L(\gamma), & i = 3, \\ \max[a_R(\gamma), b_R(\gamma)], & i = 4 \end{cases} \tag{20}$$

and $h_i = 0$ if $i = 1, 4$; $h_i = h_D$ if $i = 2, 3$.

Just to consider one case, suppose that $\min[a_L(\gamma), b_L(\gamma)] = a_L(\gamma)$ and $\max[a_R(\gamma), b_R(\gamma)] = b_R(\gamma)$, $\forall \gamma \in [0, 1]$. We can write

$$V(A \cup B) = \int_0^1 \gamma[a_L(\gamma) - E(\gamma)]^2 d\gamma + \int_{h_D}^1 \gamma[a_R(\gamma) - E(\gamma)]^2 d\gamma$$

$$+ \int_{h_D}^1 \gamma[b_L(\gamma) - E(\gamma)]^2 d\gamma + \int_0^1 \gamma[b_R(\gamma) - E(\gamma)]^2 d\gamma \quad (21)$$

$$= \int_0^1 \gamma\{[a_L(\gamma) - E(\gamma)]^2 + [a_R(\gamma) - E(\gamma)]^2\} d\gamma$$

$$+ \int_0^1 \gamma\{[b_L(\gamma) - E(\gamma)]^2 + [b_R(\gamma) - E(\gamma)]^2\} d\gamma$$

$$- \int_0^{h_D} \gamma\{[b_L(\gamma) - E(\gamma)]^2 + [a_R(\gamma) - E(\gamma)]^2\} d\gamma \quad (22)$$

$$= V_E(A) + V_E(B) - h_D^2 V_E(A \cap B), \quad (23)$$

where $V_E(\cdot)$ denotes the variance with respect to the pole specified by the subscript index, $E(\gamma)$.

Let us observe that if $A \cap B = \varnothing$, then the relationship still holds. Let us now compute $V(A \cup B)$ through the variances of the sets A, B, and $A \cap B$, each calculated with respect to its own mean value. Let $E_A(\gamma)$ be the arithmetic mean of the γ-levels of A and, similarly, let $E_B(\gamma)$ and $E_D(\gamma)$ be the arithmetic mean of the γ-levels of B and D, respectively.

Property 4.1. The variance of the union of two sets is expressed also by the following equality:

$$V(A \cup B) = V(A) + V(B) - h_D^2 V(A \cap B) + \int_0^1 \gamma[E_B(\gamma) - E_A(\gamma)]^2 d\gamma$$

$$- 2\int_0^{h_D} \gamma[E_D(\gamma) - E(\gamma)]^2 d\gamma. \quad (24)$$

Note that the absence of a subscript index in the symbol of variance means that the variance is calculated with respect to the arithmetic mean of the γ-levels of its argument, i.e., $V(A) = V_{E_A}(A)$, $V(A \cap B) = V_{E_{A\cap B}}(A \cap B)$, and so on.

To prove equation (24), the following equalities may be considered:

$$[a_L(\gamma) - E(\gamma)]^2 + [a_R(\gamma) - E(\gamma)]^2$$

$$= [a_L(\gamma) - E_A(\gamma)]^2 + [a_R(\gamma) - E_A(\gamma)]^2 + 2[E(\gamma) - E_A(\gamma)]^2,$$

$$[b_L(\gamma) - E(\gamma)]^2 + [b_R(\gamma) - E(\gamma)]^2$$

$$= [b_L(\gamma) - E_B(\gamma)]^2 + [b_R(\gamma) - E_B(\gamma)]^2 + 2[E(\gamma) - E_B(\gamma)]^2,$$

$$[b_L(\gamma) - E(\gamma)]^2 + [a_R(\gamma) - E(\gamma)]^2$$

$$= [b_L(\gamma) - E_D(\gamma)]^2 + [a_R(\gamma) - E_D(\gamma)]^2 + 2[E(\gamma) - E_D(\gamma)]^2$$

leading to

$$V(A \cup B) = V(A) + V(B) - h_D^2 V(A \cap B) + 2\int_0^1 \gamma[E_A(\gamma) - E(\gamma)]^2 d\gamma$$
$$+ 2\int_0^1 \gamma[E_B(\gamma) - E(\gamma)]^2 d\gamma - 2\int_0^{h_D} \gamma[E_D(\gamma) - E(\gamma)]^2 d\gamma. \quad (25)$$

Since

$$[E_A(\gamma) - E(\gamma)]^2 + [E_B(\gamma) - E(\gamma)]^2$$

$$= \left[\frac{E_A(\gamma) - E_B(\gamma)}{2}\right]^2 + \left[\frac{E_B(\gamma) - E_A(\gamma)}{2}\right]^2$$

$$= \frac{1}{2}[E_B(\gamma) - E_A(\gamma)]^2,$$

equation (24) is immediately obtained.

Three special cases are described in the following:

1. If $A \equiv B$, i.e., A coincides with B, then $V(A) = V(B)$, $h_D = 1$, $A \cap B = A$, $E_B(\gamma) = E_A(\gamma)$, $E_D(\gamma) = E_A(\gamma) = E(\gamma)$, and

$$V(A \cup B) = V(A) + V(A) - V(A) + \int_0^1 \gamma \cdot 0 d\gamma - 2\int_0^{h_D} \gamma \cdot 0 d\gamma = V(A).$$

In other terms, if $A \equiv B$, then the variance of $A \cup B$ is equal to the variance of A.

2. If $A \cap B = \emptyset$, i.e., A and B are disjoint sets, then $h_D = 0$, $V(A \cap B) = V(\emptyset) = 0$ and

$$V(A \cup B) = V(A) + V(B) + \int_0^1 \gamma[E_B(\gamma) - E_A(\gamma)]^2 d\gamma.$$

In other words, if A and B are disjoint sets, then the variance of $A \cup B$ is equal to the sum of the variances of A and B plus an additive term, which could be intended as the "variance" of the difference between the means of their γ-levels.

3. If $A \supset B$, i.e., B is a subset of A, then $h_D = 1$, $A \cap B = B$, $E_D(\gamma) = E_B(\gamma)$, and

$$V(A \cup B) = V(A) + V(B) - V(B) + \int_0^1 \gamma[E_B(\gamma) - E_A(\gamma)]^2 d\gamma$$

$$- 2\int_0^1 \gamma\left[E_B(\gamma) - \left(\frac{E_A(\gamma) + E_B(\gamma)}{2}\right)\right]^2 d\gamma$$

$$= V(A) + \frac{1}{2}\int_0^1 \gamma[E_B(\gamma) - E_A(\gamma)]^2 d\gamma.$$

This result is not intuitive because one would expect to obtain only $V(A)$: the additive term might be intended as a measure of the asymmetry of subset B with respect to A. More specifically, it is somehow a measure of the diversity of the two shapes. If $L_B(x) = L_A(x - \tau)$ with $\tau \in \mathbb{R}^+$ and $R_B(x) = R_A(x + \tau)$, then $E_B(\gamma) = E_A(\gamma)$, and $V(A \cup B) = V(A)$.

Note also that the crisp possibilistic mean of $A \cup B$ presents similar difficulties because it is equal to the mean of the crisp possibilistic means of *A* and *B*, which will not coincide with the crisp possibilistic mean of *A*.

4.1. Numerical example of the variance of the union of two fuzzy sets

Let $A = A(1, 2, 1, 2)$ and $B = B(4, 5, 1, 1/5)$ be two trapezoidal normal fuzzy numbers, as illustrated in Figure 2. Therefore, $a_L(\gamma) = \gamma$ and $a_R(\gamma) = 4 - 2\gamma$ for *A*, $b_L(\gamma) = 3 + \gamma$ and $b_R(\gamma) = 11/2 - \gamma/2$ for *B*. Then $E(\gamma) = (1/4)[\gamma + 4 - 2\gamma + 3 + \gamma + 11/2 - \gamma/2] = (25/8) - (\gamma/8)$. Other terms useful in the formulae are the means of the membership functions of *A* and *B*: $E_A(\gamma) = (1/2)[\gamma + 4 - 2\gamma] = 2 - \gamma/2$ and $E_B(\gamma) = (1/2)[3 + \gamma + 11/2 - \gamma/2] = 17/4 + \gamma/4$. Let *D* be the fuzzy number resulting from the intersection of *A* and *B*. Then $D = D(3 + 1/3, 1/3, 2/3)$ with $h_D = 1/3$ and the mean of its membership functions will be: $E_D(\gamma) = (1/2)[3 + \gamma + 4 - 2\gamma] = 7/2 - \gamma/2$.

$$
\begin{aligned}
V(A \cup B) &= \int_0^1 \gamma\left(\gamma - \frac{25}{8} + \frac{\gamma}{8}\right)^2 d\gamma + \int_{h_D}^1 \gamma\left(4 - 2\gamma - \frac{25}{8} + \frac{\gamma}{8}\right)^2 d\gamma \\
&\quad + \int_{h_D}^1 \gamma\left(3 + \gamma - \frac{25}{8} + \frac{\gamma}{8}\right)^2 d\gamma + \int_0^1 \gamma\left(\frac{11}{2} - \frac{\gamma}{2} - \frac{25}{8} + \frac{\gamma}{8}\right)^2 d\gamma \\
&= \int_0^1 \gamma\left[\left(\gamma - \frac{25}{8} + \frac{\gamma}{8}\right)^2 + \left(4 - 2\gamma - \frac{25}{8} + \frac{\gamma}{8}\right)^2\right] d\gamma \\
&\quad + \int_0^1 \gamma\left[\left(3 + \gamma - \frac{25}{8} + \frac{\gamma}{8}\right)^2 + \left(\frac{11}{2} - \frac{\gamma}{2} - \frac{25}{8} + \frac{\gamma}{8}\right)^2\right] d\gamma \\
&\quad - \int_0^{h_D} \gamma\left[\left(3 + \gamma - \frac{25}{8} + \frac{\gamma}{8}\right)^2 + \left(4 - 2\gamma - \frac{25}{8} + \frac{\gamma}{8}\right)^2\right] d\gamma \\
&= V_E(A) + V_E(B) - h_D^2 V_E(A \cap B) \\
&= \frac{387}{128} + \frac{319}{128} - \frac{49}{3456} = \frac{19013}{3456}
\end{aligned}
$$

or, following equation (24)

$$V(A \cup B) = V(A) + V(B) - h_D^2 V(A \cap B) + \int_0^1 \gamma\left(\frac{17}{4} + \frac{\gamma}{4} - 2 + \frac{\gamma}{2}\right)^2 d\gamma$$

$$- 2\int_0^{1/3} \gamma\left(\frac{7}{2} - \frac{\gamma}{2} - \frac{25}{8} + \frac{\gamma}{8}\right)^2 d\gamma$$

$$= \frac{9}{8} + \frac{19}{32} - \frac{1}{216} + \frac{729}{192} - 2\frac{11}{2304} = \frac{19013}{3456}.$$

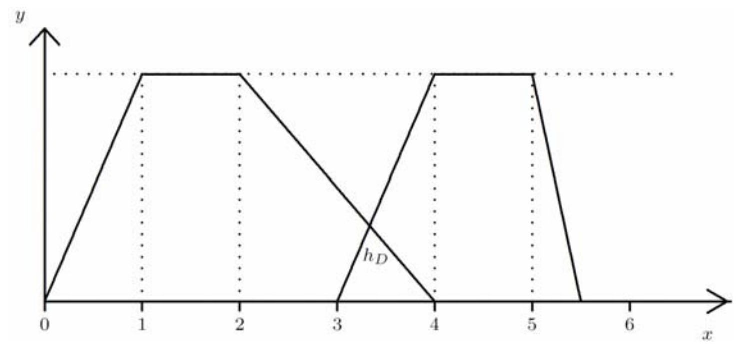

Figure 2. Two trapezoidal fuzzy numbers: $A(1, 2, 1, 2)$ and $B(4, 5, 1, 1/2)$.

5. Conclusions

The definitions of mean and variance have been extended to non-normal fuzzy numbers of the LR type and it has been shown that for the most common shapes (linear and power functions), mean and variance do not depend on the non-normality, i.e., the crisp possibilistic mean and variance do not depend on the height, *h*, of a fuzzy number with linear or power shape functions. Their properties have been illustrated thoroughly, more than in any other previous publications. Moreover, variance has been extended to the union of fuzzy sets, leading up to an appealing expression involving the variances of the two original fuzzy numbers minus the variance of their intersection multiplied by the square of the height of their intersection, plus two additional terms contributing to balance the simplification of the relationship.

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