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LINEAR STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY MULTIFRACTIONAL BROWNIAN MOTION

K. Narita

Department of Information Systems Creation
Faculty of Engineering
Kanagawa University
Rokkakubashi 3-27-1, Kanagawa-ku
Yokohama 221-8686, Japan
e-mail: narita@kanagawa-u.ac.jp

Abstract

Theory of fractional Brownian motion (fBm) is exclusively developed by many researchers and used for modeling long-range dependence when studying processes in computer networks, in financial markets as well as in hydromechanics, climatology, and hydrography.

The fBm $B_H(t)$, where $t \geq 0$, is a Gaussian process with stationary increments and has the so-called Hurst parameter $H \in (0, 1)$ which characterizes self-similarity of distributions and roughness of paths. However, the stationarity of increments of fBm restricts substantially its applicability for modeling processes with long memory. In particular, it does not allow us to model processes whose regularity of paths and "memory depth" change in time. A generalization of the fBm is the multifractional Brownian motion (mBm), denoted by

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$B_h(t)$, where the constant Hurst parameter H in $B_H(t)$ is substituted by a time-dependent Hölder continuous Hurst function $h(t)$ taking its values in $(0, 1)$. As such, mBms are useful as stochastic models for phenomena that exhibit non-stationarity, for example, risky asset in financial market, traffic in modern telecommunication networks or signal processing.

0. Introduction

In this paper, we shall obtain explicit expression for solutions of linear stochastic differential equations (SDEs) driven by mBm with time-dependent coefficients. Here SDEs are described in a space $(\mathcal{S})^*$ of generalized random processes (the Hida space of stochastic distributions) and the stochastic integrals with respect to mBms are defined in $(\mathcal{S})^*$ as the multifractional Wick-Ito sense. More precisely, we introduce the *S*-transform of a mean square integrable random variable Φ; for Φ fixed, this is the functional $S[\Phi](\eta)$ operating on deterministic functions η and is fully characterized as the expectation on Φ multiplied by the factor, like the exponential martingale induced by η. Then the Wick product $\Phi \Diamond \Psi$ of Φ and Ψ is characterized by the defining equation such that $S[\Phi \Diamond \Psi](\eta) = S[\Phi](\eta) \cdot S[\Psi](\eta)$. Further, the *S*-transform and the diamond $\Diamond$ denoting the Wick product can be extended to $(\mathcal{S})^*$. This leads us to the white noise approach such that the multifractional Wick-Ito integral can be defined as the $(\mathcal{S})^*$-valued integral of $F(t) \Diamond \frac{d}{dt} B_h(t)$, if it exists; $\int_a^b F(t) d^{\Diamond} B_h(t) := \int_a^b F(t) \Diamond \frac{d}{dt} B_h(t) dt$, where $\frac{d}{dt} B_h(t)$ denotes the multifractional white noise. We notice that the multifractional Wick-Ito integral is the extension of the fractional Wick-Ito integral with respect to fBm with the constant Hurst parameter $H \neq 1/2$ and the classical Ito integral with respect to the standard Brownian motion that is the fBm with $H = 1/2$.

Our investigation is based on the method of the *S*-transform which is much simpler and does not make use of the complicated constructions from the white noise calculus. Moreover, the injective property of the *S*-transform enables us to solve the above-mentioned SDEs.

Thus, for fundamental results, referring to Bender [3, 4], Corlay et al. [15], Lebovits and Lévy-Véhel [27] and Lebovits et al. [28], we shall proceed to discuss important notions such as fBm (Section 1), mBm (Section 2), *S*-transform approach (Section 3), white noise setting (Section 4), white noise operators M_H (Section 5) and stochastic integral with respect to mBm (Section 6). In the sequel, we shall obtain explicit expression for solutions of the linear SDEs driven by mBm (Theorem 7.1, Section 7). Moreover, we shall derive Ito formula for geometric mBm from the simple one for mBm $B_h(t)$ (Theorem 8.3, Section 8). As an application, we shall obtain a multifractional version of the Black-Scholes equation, that is, the pricing partial differential equation (PDE) related to European call option, where a risky asset process is modeled by geometric mBm (Theorem 9.1, Section 9).

Our theorem on expression for solutions of SDEs corresponds to an extension of the result in Lebovits and Lévy-Véhel [27] where the solutions of SDEs are limited to geometric mixed multifractional Brownian motion and mixed multifractional Ornstein-Uhlenbeck process. Ito formula for geometric mBm corresponds to an extension of that in Lebovits and Lévy-Véhel [27] where the simple case for mBm is investigated. In addition, the multifractional version of the pricing PDE corresponds to an extension of the result in Necula [33] where the market is considered under the fBm environment with the constant Hurst parameter *H* in $(1/2, 1)$.

1. Fractional Brownian Motion

Fractional Brownian motion was introduced in 1940 by Kolmogorov [24] as a way to generate Gaussian "spirals" in a Hilbert space, and then popularized in 1968 by Mandelbrot and Van Ness [31] by its relevance to model natural phenomena; hydrology, finance, signals and images processing, and telecommunications.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Then the fractional Brownian motion is defined as a Brownian motion with a constant parameter H which is called *Hurst parameter*:

Definition 1.1. A real-valued random process $B_H = (B_H(t))_{t\in\mathbb{R}}$ is called (*two-sided*, *normalized*) *fractional Brownian motion* (*fBm*) with Hurst parameter $H \in (0, 1)$ provided that

(i) $B_H(t)$ is a Gaussian process,

(ii) $B_H(0) = 0$ a.s.,

(iii) $E[B_H(t)] = 0,$

(iv) $E[B_H(t)B_H(s)] = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}],\ t,\ s \in \mathbb{R},$

where $E[\cdot]$ denotes the mathematical expectation.

Especially the case $H = 1/2$ leads to the *standard Brownian motion* (*sBm*). In this sense, fBm appears as a generalization of sBm. It is well known that fBm B_H is a semimartingale if and only if $H = 1/2$, i.e., in the case of a classical sBm. Hence, Ito's stochastic integration theory for semimartingales cannot be applied, if $H \neq 1/2$.

Remark 1.2. FBm B_H is not a stationary process, but has *stationary increments*. In fact, from Definition 1.1(iv), we can deduce the following expression for the variance of the increment of the process in an interval:

$$E[(B_H(t) - B_H(s))^2] = |t-s|^{2H}.$$

Moreover, since B_H is a Gaussian process, we have that for all $n \geq 1$,

$$E[|B_H(t) - B_H(s)|^n] = \frac{2^{n/2}}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right)|t-s|^{nH}.$$

Remark 1.3. By the Kolmogorov criterion, a process $X = (X(t))_{t\in\mathbb{R}}$ admits a continuous modification if there exist constants $\alpha \geq 1$, $\beta > 0$, and

$k > 0$ such that

$$E[|X(t) - X(s)|^{\alpha}] \leq k|t - s|^{1+\beta}$$

for all $s, t \in \mathbb{R}$. Noticing the *n*th moment of the fBm in Remark 1.2, by the Kolmogorov continuity criterion, we deduce that fBm has a version with continuous trajectories.

FBm is the only centered Gaussian process with stationary increments. The Hurst parameter *H* governs different properties of the fBm, for instance, the self-similarity of the process, the correlation of the increments and the roughness of the path. We summarize these according to Bertrand et al. [8] as follows:

1. Self-similarity. For all $c > 0$,

$$\{B_H(ct);\, t \in \mathbb{R}\} \overset{\text{law}}{\sim} \{c^H B_H(t);\, t \in \mathbb{R}\},$$

where $\overset{\text{law}}{\sim}$ means the equivalence in the sense of probability law. Thus, *H* means the self-similar index.

2. Correlation of the increments. Stationary increments means that for all $h, s, t \in \mathbb{R}$,

$$\begin{aligned} &E[(B_H(t+h) - B_H(t))(B_H(s+h) - B_H(s))] \\ &= E[(B_H(t-s+h) - B_H(t-s))][(B_H(h) - B_H(0))] \\ &:= \rho_h(t-s). \end{aligned}$$

In contrast with sBm, the increments of fBm are correlated. They even display long-range dependence or long memory when $H > 1/2$, that is, for all $h \neq 0$,

$$\sum_{k \in \mathbb{Z}} |\rho_h(k)| = \infty.$$

More precisely, let $Y(j) = B_H(j+1) - B_H(j)$ denote the increments of fBm and $r(j) := E[Y(j)Y(0)] = \rho_1(j - 0)$ its correlation. Then the following is well-known:

• if $H = 1/2$, then the increments are independent;

• if $H > 1/2$, then $\sum_{k=-\infty}^{+\infty} |r(k)| = +\infty$, thus, we have long memory of the increments;

• if $H < 1/2$, then $\sum_{k=-\infty}^{+\infty} |r(k)| < +\infty$, thus, we have short memory of the increments.

3. Roughness of the path. In spite of its usefulness, fBm model has some limitations, an important one of them is that the roughness of its path remains everywhere the same.

In order to explain this important issue, based on Ayache and Lévy-Véhel [1], we introduce the notion of pointwise Hölder exponent which provides a measure of the local Hölder regularity of a process path in neighborhood of some fixed point *t*.

Let $(X(t))_{t\in\mathbb{R}}$ be a stochastic process whose paths are with probability 1 continuous and nowhere differentiable functions (this is the case of fBm paths). Let $\alpha \in [0, 1)$ and *t* be fixed. One says that a path $X(t, \omega)$ belongs to the pointwise Hölder space $C^{\alpha}(t)$, if for all $s \in \mathbb{R}$ small enough, one has

$$| X(t + s, \omega) - X(t, \omega) | \leq C(\omega) | s |^{\alpha}.$$

The *pointwise Hölder exponent* of the path $X(t, \omega)$ at the point *t*, is defined as

$$\alpha_X(t, \omega) = \sup\{\alpha \in [0, 1);\ X(\cdot, \omega) \in C^{\alpha}(t)\}.$$

Then the roughness of fBm path remains everywhere the same, that is,

$$P(\forall t,\ \alpha_{B_H(t)} = H) = 1.$$

From a geometrical point of view, *H* determines the (constant) roughness of the sample paths of the fBm and is linked to the *fractal* (or *Hausdorff*) *dimension D* of the graph by the simple relation $D = 2 - H$.

With regard to the construction of fBm, there are well-known results, one is nonanticipative stochastic integral representation and the other is spectral representation as follows:

1. Moving average representation. This was introduced by Mandelbrot and Van Ness [31] in 1968, and presented by Samorodnitsky and Taqqu [38, Chap. 14] in 1994, in the slightly modified form as follows:

$$B_H(t) = \frac{V_H^{1/2}}{\Gamma(H + 1/2)} \int_{\mathbb{R}} [(t - s)_+^{H-1/2} - (-s)_+^{H-1/2}] dB(s), \tag{1.1}$$

where for all reals x and θ,

$$(x)_+^{\theta} = \begin{cases} x^{\theta}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and $V_H := \Gamma(2H + 1)\sin(\pi H)$ is a normalizing factor such that $Var(B_H(1)) = 1$ and the kernel $(t - s)^{H-1/2}$ rules the dependence between the process increments. We note that

$$\frac{\sqrt{\Gamma(2H + 1)\sin(\pi H)}}{\Gamma(H + 1/2)} = \left\{ \int_0^{\infty} [(1 + s)^{H-1/2} - s^{H-1/2}]^2 ds + \frac{1}{2H} \right\}^{-1/2},$$

where $\Gamma(\cdot)$ is the Gamma function. In (1.1), $(B(t))_{t\in\mathbb{R}}$ denotes the ordinary two-sided Brownian motion, that is,

$$B(t) = \begin{cases} B_1(t), & \text{for } t \geq 0, \\ B_2(-t), & \text{for } t < 0, \end{cases}$$

where $B_1(t)$ and $B_2(t)$ are two independent Brownian motions for $t \geq 0$.

We can rewrite (1.1) as follows:

$$B_H(t) = \frac{V_H^{1/2}}{\Gamma(H + 1/2)} \left\{ \int_{-\infty}^{0} [(t - s)^{H-1/2} - (-s)^{H-1/2}] dB(s) \right.$$
$$\left. + \int_0^t (t - s)^{H-1/2} dB(s) \right\} \text{ for } t \geq 0, \tag{1.1$'$}$$

$$B_H(t) = \frac{V_H^{1/2}}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{t}[(t-s)^{H-1/2} - (-s)^{H-1/2}]dB(s)\right.$$

$$\left. - \int_t^0 (-s)^{H-1/2}dB(s)\right\} \text{ for } t < 0. \tag{1.1''}$$

2. Harmonizable representation. This was first defined by Kolmogorov [24] in 1940 as follows:

$$B_H(t) = \frac{1}{C(H)}\int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H+1/2}} d\hat{W}(\xi) \text{ for all } t \in \mathbb{R}, \tag{1.2}$$

where $d\hat{W}$ is "the Fourier transform of the white noise", that is, the unique complex-valued stochastic measure which satisfies, for all $f \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x)dW(x) = \int_{\mathbb{R}} \hat{f}(\xi)d\hat{W}(\xi), \text{ almost surely,} \tag{1.3}$$

dW being the usual real-valued white noise (i.e., a Brownian measure). Here $\hat{f}$ denotes the Fourier transform of f;

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)dx.$$

For all the technical details, we refer to Samorodnitsky and Taqqu [38, pp. 325-326]. Equation (1.3) implies that $B_H(t)$ is real-valued. The constant $C(H)$ in (1.2) is deduced from the requirement that $E[B_H(1)^2] = 1$, and defined by

$$C(H) := \left(\frac{2\cos(\pi H)\Gamma(2-2H)}{H(1-2H)}\right)^{1/2}$$

$$= \left(\frac{2\pi}{\Gamma(2H+1)\sin(\pi H)}\right)^{1/2}. \tag{1.4}$$

Here the last equality follows from Euler's reflection formula such that

$$\Gamma(z)\Gamma(1-z) = -z\Gamma(z)\Gamma(-z) = \frac{\pi}{\sin(\pi z)}$$

which is derived from Euler's expression for the trigonometric function $\sin(\pi z)$ in terms of infinite product. In fact, from the relation that $\Gamma(1+z) = z\Gamma(z)$, we have formal calculations, heuristically, as follows:

$$\frac{\pi}{\sin(\pi(2H))} = \Gamma(2H)\Gamma(1-2H)$$

$$\Leftrightarrow \frac{\pi}{\sin(\pi(2H))} = \frac{(2H)\Gamma(2H)(1-2H)\Gamma(1-2H)}{(2H)(1-2H)}$$

$$\Leftrightarrow \frac{\pi}{\sin(\pi(2H))} = \frac{\Gamma(2H+1)\Gamma(2-2H)}{(2H)(1-2H)}$$

$$\Leftrightarrow \sin(2(\pi H)) = \frac{2\pi H(1-2H)}{\Gamma(2H+1)\Gamma(2-2H)}$$

$$\Leftrightarrow 2\sin(\pi H)\cos(\pi H) = \frac{2\pi H(1-2H)}{\Gamma(2H+1)\Gamma(2-2H)}$$

$$\Leftrightarrow \frac{2\cos(\pi H)\Gamma(2-2H)}{H(1-2H)} = \frac{2\pi}{\Gamma(2H+1)\sin(\pi H)}.$$

It is known that the probability laws of the processes defined by (1.1) and (1.2) are equivalent. Hence, in (1.1) and (1.2), we used the same notation $(B_H(t))_{t\in\mathbb{R}}$ to denote the fBm.

2. Multifractional Brownian motion

Since the intensity of the long-range dependence depends on both Hurst parameter $H \in (0, 1)$ and the lag *h*, once fixed the lag, the autocorrelation only depends on the Hurst parameter. Hence, the most immediate generalization of the fBm can be obtained by allowing *H* to vary over time, that is, the constant Hurst parameter *H* in equations (1.1) and (1.2) will be substituted by a time-dependent Hurst exponent $h(t)$.

This idea was proposed by Lévy-Véhel [29] in 1995. In fact, in some real datasets there is evidence that the roughness of the sample path varies with location. In such cases, a single number, i.e., Hurst parameter H or fractal (Hausdorff) dimension D, may not provide an adequate global description of the roughness of the sample path and there is motivation for developing models which allow for varying roughness. Lévy-Véhel [29] has considered such datasets in Image Analysis and Signal Processing contexts, and these led him to consider a generalization of fBm which he calls multifractional Brownian motion.

The first representation is a mean average approach and was proposed by Peltier and Lévy-Véhel [35] in1995, subsequently to Lévy-Véhel [29]:

Definition 2.1. Let $h : [0, \infty) \to (0, 1)$ be Hölder continuous with exponent $\beta > 0$. Then, for each $t \geq 0$, relation $(1.1)'$ defines the value $B_H(t)$ with $H = h(t)$. For $t \geq 0$, the following random process is called *multifractional Brownian motion* (*mBm*) with Hurst function $h(t)$:

$$B_{h(t)}(t) := \frac{V_{h(t)}^{1/2}}{\Gamma(h(t) + 1/2)} \left\{ \int_{-\infty}^{0} [(t-s)^{h(t)-1/2} - (-s)^{h(t)-1/2}] dB(s) \right.$$

$$\left. + \int_0^t (t-s)^{h(t)-1/2} dB(s) \right\}. \tag{2.1}$$

The second representation is a spectral approach introduced by Benassi et al. [7].

Definition 2.2. Let $h : [0, \infty) \to (0, 1)$ be Hölder continuous with exponent $\beta > 0$. Then, for each $t \geq 0$, relations (1.2) and (1.3) define the real value $B_H(t)$ with $H = h(t)$. For $t \geq 0$, the following random process is called *multifractional Brownian motion* (*mBm*):

$$B_{h(t)}(t) := \frac{1}{C(h(t))} \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{h(t)+1/2}} d\hat{W}(\xi) \tag{2.2}$$

with $d\hat{W}(\xi)$ as appeared in (1.2), and $C(x)$ as given by (1.4), i.e., $C(x) = \left(\frac{2\pi}{\Gamma(2x+1)\sin(x\pi)}\right)^{1/2}$.

The processes $B_{h(t)}(t)$ in Definitions 2.1 and 2.2 are well defined (i.e., square integrable) if the function $h(\cdot)$ is Hölderian of order $0 < \beta \leq 1$ on $[0, 1]$. Cohen [13] proved the equality in distribution of both processes normalized in such a way that $E[B_{h(t)}(t)^2] = t^{2h(t)}$. From these definitions, it is easy to see that mBm is a zero mean Gaussian process whose increments are in general neither independent nor stationary; recall that fBm has stationary correlated increments for $H \neq 1/2$. When $h(t) = H$ for all $t \geq 0$, mBm is of course just fBm with constant Hurst parameter H.

For the sake of simplicity, the mBm with Hurst function $h(\cdot)$ defined by (2.1) or (2.2), is denoted by $(X(t))_{t\geq 0}$. Then we summarize as follows:

Remark 2.3. MBm $X(t)$ satisfies the following properties:

(i) $X(t)$ is a Gaussian process,

(ii) $X(0) = 0$ a.s.,

(iii) $E[X(t)] = 0,\ t \geq 0,$ that means the process is centered,

(iv) it follows from Ayache et al. [2, Proposition 4] that the autocovariance of $X(t)$ of the standard mBm, namely of an mBm with $E[X^2(1)] = 1$, is given by

$$E[X(t)X(s)] = D(h(t), h(s))[|t|^{h(t)+h(s)} + |s|^{h(t)+h(s)} - |t-s|^{h(t)+h(s)}],$$

where

$$D(x, y) = \frac{\sqrt{\Gamma(2x+1)\Gamma(2y+1)\sin(\pi x)\sin(\pi y)}}{2\Gamma(x+y+1)\sin(\pi(x+y)/2)}.$$

The mBm $X(t)$ is a continuous process for all $t \geq 0$ with probability one. This was shown in Peltier and Lévy-Véhel [35, Proposition 3], by the help of skilful splittings of the Ito integral representation, some fundamental inequalities and the Kolmogorov criterion.

Remark 2.4. At each point s, the mBm $X(t) := B_{h(t)}(t)$ is locally asymptotically self-similar with index $h(s)$ in the following sense: assume that $h(t)$ is Hölder continuous with exponent β and that $\sup_{t \geq 0} h(t) < \min(1, \beta)$. Consider $X(s + \rho u) - X(s)$, i.e., the increment process of the mBm at time s, and lag ρu. Then Benassi et al. [6] proved that

$$\lim_{\rho \to 0^+} \left(\frac{X(s + \rho u) - X(s)}{\rho^{h(s)}} \right)_{u \in \mathbb{R}_+} \overset{\text{law}}{\sim} (B_{h(s)}(u))_{u \in \mathbb{R}_+},$$

where $(B_{h(s)}(u))_{u \in \mathbb{R}_+}$ is an fBm with parameter $h(s)$ defined on $\mathbb{R}_+$. The above distributional equality states that at any point s, there exists an fBm with parameter $h(s)$ tangent to the mBm.

Remark 2.5. Unlike fBm's, the increments of mBm $X(t) := B_{h(t)}(t)$ are no longer stationary nor self-similar, and its path regularity explicitly varies with time. More precisely, the following properties of mBm are known by Ayache et al. [2, Propositions 1 and 2]: Assume that $\beta > \sup_{t \geq 0} h(t)$. Then

(i) With probability one, for each t_0, the Hölder exponent at point $t_0 \geq 0$ of mBm is $h(t_0)$; recall that the Hölder exponent of a process $X(t)$ at point s is defined as

$$\alpha_X(s, \omega) := \sup \left\{ \alpha;\ \lim_{h \to 0} \frac{X(s + h) - X(s)}{h^{\alpha}} = 0 \right\}.$$

A "large" $\alpha_X(s, \omega)$ means that X is smooth at s, while irregular behavior of X at s translates into α close to 0.

(ii) With probability one, for each interval $[a, b] \subset \mathbb{R}_+$, the graph of mBm $(X(t))_{t\in[a, b]}$ verifies the following property:

$$\dim_H \{X(t); t \in [a, b]\} = \dim_B \{X(t); t \in [a, b]\} = 2 - \min\{h(t); t \in [a, b]\},$$

where $\dim_H$ and $\dim_B$ denote the Hausdorff dimension and the box dimension, respectively.

We note that there is a Gaussian process generalizing the mBm and having the Hölder regularity that can be a very "irregular function" (see Ayache and Lévy-Véhel [1]).

We can refer statistical study of fBm and mBm, and modeling in finance, to Bertrand et al. [8] and Bianchi [11].

The following Sections 3-6 provide the necessary backgrounds on the *S*-transform and the white noise theory to define a stochastic integral and to handle stochastic differential equations (SDEs) driven by mBm.

3. The *S*-transform Approach

Since the fractional Brownian motion (fBm) with Hurst parameter $H \neq 1/2$ is not a semimartingale, the integration theory of the Ito type cannot be applied to this family of processes. Therefore, different extensions have been proposed. The first one was introduced by Lin [30], who proved that an interesting class of processes is integrable with respect to an fBm and derived an analogue of the Black-Scholes formula in financial market. In this case, the integration theory is based on the ordinary pathwise product in defining the integral:

$$\int_a^b \sigma(t, \omega)\delta B_H(t) := \lim_{|\Delta|\to 0} \sum_{k=0}^{n-1} \sigma(t_k, \omega)(B_H(t_{k+1}) - B_H(t_k))$$

for suitable integrand $\sigma(t, \omega)$. Here and hereafter, $\Delta : a = t_0 < t_1 < \cdots < t_n = b$, $|\Delta| = \max_{0\le k\le n-1}(t_{k+1} - t_k)$. However, this integral, in general, does not have expectation zero. Moreover, Rogers [36] and Dasgupta and Kallianpur [16] showed an arbitrage opportunity in the Black-Scholes model under the pathwise integral setting as described above.

Thus, the second integration theory for the fBm was introduced by Duncan et al. [17]. This integration is based on the method of the Wick product, that is, the Wick product $\lozenge$ is used instead of the ordinary product in Riemann sums:

$$\int_a^b \sigma(t, \omega) dB_H(t) := \lim_{|\Delta| \to 0} \sum_{k=0}^{n-1} \sigma(t_k, \omega) \lozenge (B_H(t_{k+1}) - B_H(t_k)),$$

for suitable integrand $\sigma(t, \omega)$. An important property is that this integral has expectation zero, i.e.,

$$E\left[\int_a^b \sigma(t, \omega) dB_H(t)\right] = 0.$$

Duncan et al. [17] began to define the Wick product of two exponential functions $\mathcal{E}(f) \lozenge \mathcal{E}(g) := \mathcal{E}(f + g)$ and also extended to define the Wick product of two functionals in the space of the linear span of the exponential functions which is dense in $L^p(\Omega)$ $(p \geq 1)$. Further, they extended to more general functionals, including the functionals of the form $\int_0^\infty f(t) dB_H(t)$ for suitably given *f*. This integral by the method of the Wick product has been further developed by Hu and ∅ksendal [22] in a fractional white noise setting. As an application, they obtained no-arbitrary result in the Black-Scholes model. However, in this setting, the underlying probability space depends on the Hurst parameter H of the fBm, i.e., one has to consider different spaces for different parameters. Moreover, $H > 1/2$ is assumed in constructing the appropriate spaces. Regarding this matter, Elliott and Van Der Hoek [18], in white noise setting, presented a new framework for fBm in which processes with all Hurst parameters in $(0, 1)$ could be considered under the same probability measure. As an application, they obtained the no-arbitrage result in the Black-Scholes model.

Stochastic integration theory with respect to fBms was also developed by Hu [21]. Hu mainly used the integral kernels $K_H(t, s)$ and $\eta_H(t, s)$ such that

$$B_H(t) = \int_0^t K_H(t, s)dB(s) \quad \text{and} \quad B(t) = \int_0^t \eta_H(t, s)dB_H(s)$$

with fBm $B_H(t)$ $(0 < H < 1)$ and sBm $B(t)$, and extended the correspondence between fBm and sBm to that between nonlinear functionals of fBm and nonlinear functionals of sBm. Further, Hu used Wiener chaos expansion and idea of creation operator from quantum field theory in order to define stochastic integral. In fact, he introduced 'algebraically integrable integrands' for which stochastic integral could be defined. This integral has expectation zero and relates to the method of the Wick product, Malliavin calculus and Skorohod integral; for more details, see Biagini et al. [10], Mishura [32] and Nualart [34].

On the other hand, Bender [3, 5] gave a motivation for a simple definition of the fractional Ito integral and generalized the models in Hu and ∅ksendal [22] and Elliott and Van Der Hoek [18]. This definition is based on the *S*-transform, an important tool in the white noise analysis, but carries over to an arbitrary probability space on which a two-sided Brownian motion lives. Concerning the definition of stochastic integrals, the *S*-transform approach is equivalent to the white noise definition as developed in Hu and ∅ksendal [22], Elliott and Van Der Hoek [18] and Bender [4], as long as we suppose the integrand and the integral to be $L^2(\Omega)$-valued. However, the *S*-transform approach is much simpler and does not make use of the complicated constructions from the white noise calculus.

In the following, we briefly present the basic idea of the concept of the *S*-transform according to Bender [3]; we also refer a summary to Rostek [37].

First, we have to introduce some notation. For $0 < \alpha < 1$, the *Riemann-Liouville fractional integrals* (the *fractional integrals of Weyl's type*) are defined by

$$I_{-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} f(t)(t-x)^{\alpha-1} dt = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} f(x+t) t^{\alpha-1} dt,$$

$$I_{+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1} dt = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} f(x-t) t^{\alpha-1} dt$$

if the integrals exist for almost all $x \in \mathbb{R}$. These fractional integrals are nothing but normalized blurred version of the function *f*, either averaging over future or over past function values. On the other hand, the *fractional derivatives of Marchaud's type* are given by

$$D_{\pm,\varepsilon}^{\alpha} f := \frac{\alpha}{\Gamma(1-\alpha)} \int_{\varepsilon}^{\infty} \frac{f(x) - f(x \mp t)}{t^{1+\alpha}} dt$$

and

$$D_{\pm}^{\alpha} f := \lim_{\varepsilon \to 0^{+}} D_{\pm,\varepsilon}^{\alpha} f,$$

if the limit exists in $L^{p}(\mathbb{R})$ for some $p > 1$. The notation $D_{\pm}^{\alpha} f \in L^{p}(\mathbb{R})$ indicates convergence in the $L^{p}(\mathbb{R})$-norm. Concerning the latter representation

$$D_{\pm}^{\alpha} f = \lim_{\varepsilon \to 0^{+}} \frac{\alpha}{\Gamma(1-\alpha)} \int_{\varepsilon}^{\infty} \frac{f(x) - f(x \mp t)}{t} t^{-\alpha} dt,$$

we can interpret this fractional derivative as a weighted sum, this time averaging difference quotients yielding a blurred version of the first derivative *f*.

Based on these definitions, for given Hurst parameter $H \in (0, 1)$, consider the fractional integrals $I_{\pm}^{\alpha} f$ with $\alpha = H - 1/2$ $(1/2 < H < 1)$, and the fractional derivatives $D_{\pm}^{\alpha} f$ with $\alpha = 1/2 - H$ $(0 < H < 1/2)$. Then the operators $M_{\pm}^{H}$ are defined as

$$M_{\pm}^{H} f = \begin{cases} K_H D_{\pm}^{-(H-1/2)} f, & 0 < H < \frac{1}{2}, \\ f, & H = \frac{1}{2}, \\ K_H I_{\pm}^{H-1/2}, & \frac{1}{2} < H < 1. \end{cases} \tag{3.1}$$

Here

$$K_H := \Gamma(H + 1/2)\left\{\int_0^\infty [(1+s)^{H-1/2} - s^{H-1/2}]^2 ds + \frac{1}{2H}\right\}^{-1/2}.$$

We notice that $K_H = V_H^{1/2}$ with $V_H = \Gamma(2H+1)\sin(\pi H)$ as given in (1.1).

We recall a construction of fBm starting from a Brownian motion.

Let $(\Omega, \mathcal{F}, P)$ be a probability space that carries a two-sided Brownian motion *B*; observe *B* in equation (1.1). For $a, b \in \mathbb{R}$, we define the indicator function

$$\mathbf{1}[a, b](t) = \begin{cases} 1, & \text{if } a \le t \le b, \\ -1, & \text{if } b \le t \le a, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

By Bender [3, Theorems 2.1 and 2.6], we note the properties of the operators $M_{\pm}^{H}$ as the following remarks:

Remark 3.1. The fBm can be represented in terms of the operators $M_{\pm}^{H}$ and the indicator function $\mathbf{1}[0, t]$: For $0 < H < 1$, let the operators $M_{\pm}^{H}$ be defined by (3.1). Then $M_{-}^{H}\mathbf{1}[0, t] \in L^2(\mathbb{R})$ and an fBm B_H is given by a continuous version of the Wiener integral

$$B_H(t) = \int_{\mathbb{R}} (M_{-}^{H}\mathbf{1}[0, t])(s)\, dB(s). \tag{3.3}$$

This is the well-known Mandelbrot and Van Ness representation (1.1); more details can be found in Bender [4].

Remark 3.2. Using the operators $M_{\pm}^{H}$, one can formulate the useful fractional integration by parts rule:

$$\int_{\mathbb{R}} f(s)(M_{-}^{H} g)(s)ds = \int_{\mathbb{R}} (M_{+}^{H} f)(s)g(s)ds \tag{3.4}$$

for rapidly decreasing functions *f* and *g* on $\mathbb{R}$.

The *S*-transform is an important tool in white noise analysis. Here we give a definition and state some results that do not depend on properties of the white noise space. We first introduce some notation. $I^{B}(f)$ denotes the Wiener integral $\int_{\mathbb{R}} f(s)dB(s)$ for function $f \in L^2(\mathbb{R})$; we notice that the underlying probability space is given by $(\Omega, \mathcal{F}, P)$ that carries a two-sided Brownian motion *B*. If there is no danger of confusion, then we shall drop the superscript *B*. $|f|_0$ is the usual $L^2(\mathbb{R})$-norm, and the corresponding inner product is denoted by $(f, g)_0$.

Let $\mathcal{G}$ be the σ-field generated by $\{I(f): f \in \mathbb{R}\}$. Then we define $(L^2) := L^2(\Omega, \mathcal{G}, P)$ and denote by $\|\Phi\|_0$ the (L^2) norm. We also denote by $S(\mathbb{R})$ the Schwartz space of smooth rapidly decreasing functions. Then, based on Bender [3, Definition 2.2], we define the *S*-transform on (L^2):

Definition 3.3. For $\Phi \in (L^2)$, the *S-transform* is defined by

$$S(\Phi)(\eta) \overset{\text{def}}{=} E[\Phi \cdot :e^{I(\eta)}:], \quad \eta \in S(\mathbb{R}). \tag{3.5}$$

Here the *Wick exponential* of $I(\eta)$ is defined by

$$:e^{I(\eta)}: \overset{\text{def}}{=} e^{I(\eta)-\frac{1}{2}|\eta|_0^2} = \exp\left(\int_{\mathbb{R}} \eta(s)dB(s) - \frac{1}{2}\int_{\mathbb{R}} \eta^2(s)ds\right). \tag{3.6}$$

Hence, (3.5) is rewritten as

$$S(\Phi)(\eta) = E\left[\Phi \cdot \exp\left(\int_{\mathbb{R}} \eta(s)dB(s) - \frac{1}{2}\int_{\mathbb{R}} \eta^2(s)ds\right)\right], \quad \eta \in S(\mathbb{R}).$$

From now on, the *S*-transform of an element Φ of (L^2) is noted by $S\Phi$ or $S[\Phi]$.

By Definition 3.3, for $\Phi \in (L^2)$ fixed, the *S-transform* is a functional from $S(\mathbb{R})$ to $\mathbb{R}$, i.e., $S(\Phi) : S(\mathbb{R}) \to \mathbb{R}$.

Remark 3.4. The mapping $\Phi \mapsto S(\Phi)$ is injective: If $S(\Phi)(\eta) = S(\Psi)(\eta)$ for all $\eta \in S(\mathbb{R})$, then $\Phi = \Psi$. This result is well known in the white noise setting; an elementary proof can be found in Bender [3, Theorem 2.2].

By Bender [3, Theorem 2.3], we can characterize (L^2) convergence in terms of the *S*-transform:

Remark 3.5. Let Φ_n be a sequence in (L^2) and $\Phi \in (L^2)$. Then the following assertions are equivalent:

(i) Φ_n (strongly) converges to $\Phi \in (L^2)$.

(ii) $\| \Phi_n \|_0 \to \| \Phi \|_0$, and for all $\eta \in S(\mathbb{R})$, $S(\Phi_n)(\eta) \to S(\Phi)(\eta)$.

It is easy to prove the following property of the Wick exponential:

Remark 3.6. Let $f, g \in L^2(\mathbb{R})$. Then

$$E[: e^{I(f)} : \cdot : e^{I(g)} :] = e^{(f, g)_0}.$$

In particular, since $(S : e^{I(f)} :)(\eta) = E[: e^{I(f)} : \cdot : e^{I(\eta)} :]$ for $\eta \in S(\mathbb{R})$, we have

$$(S : e^{I(f)} :)(\eta) = e^{(f, \eta)_0}, \quad \eta \in S(\mathbb{R}).$$

Further, by Remark 3.6, we have $E[: e^{I(f)} :] = 1$ for $f \in L^2(\mathbb{R})$. Hence, we can define a probability measure on $\mathcal{G}$ by

$$dQ_f \overset{\text{def}}{=} : e^{I(f)} : dP. \tag{3.7}$$

We notice that P and Q_f are equivalent. Thus, with the measures Q_η, $\eta \in S(\mathbb{R})$, we can rewrite the S-transform as

$$(S\Phi)(\eta) = E^{Q_\eta}[\Phi]. \tag{3.8}$$

In the following, we briefly summarize several properties of the S-transform according to Bender [3, Sections 2 and 3].

The S-transform of a simple Wiener integral $\int_a^b f(t)dB(t)$ for deterministic function f is

$$S\left(\int_a^b f(t)dB(t)\right)(\eta) = \int_a^b f(t)\eta(t)dt. \tag{3.9}$$

In fact, since $\tilde{B}(t) := B(t) - \int_0^t \eta(s)ds$ for $\eta \in S(\mathbb{R})$ is a two-sided Brownian motion under the measure Q_η, equation (3.8) yields

$$\begin{aligned} S\left(\int_a^b f(t)dB(t)\right)(\eta) &= E^{Q_\eta}\left[\int_a^b f(t)(d\tilde{B}(t) + \eta(t)dt)\right] \\ &= E^{Q_\eta}\left[\int_a^b f(t)\eta(t)dt\right] \\ &= \int_a^b f(t)\eta(t)dt. \end{aligned}$$

In particular, considering $f(t) \equiv 1$ in (3.9), we obtain

$$S(B(t))(\eta) = S\left(\int_0^t 1dB(s)\right)(\eta) = \int_0^t \eta(s)ds. \tag{3.10}$$

For simplicity, we interpret a stochastic process as an (L^2)-valued function. Then the notion of Pettis integrability in Bender [3, Definition 2.3] fits better than the pathwise integral:

Definition 3.7. Let $X : \mathcal{M} \to (L^2)$ ($\mathcal{M} \subset \mathbb{R}$ a Borel set). Then X is said to be *Pettis integrable* if $E[X\Psi] \in L^1(\mathcal{M})$ for any $\Psi \in (L^2)$. In that case there is a unique $\Phi \in (L^2)$ such that for all $\Psi \in (L^2)$,

$$E[\Phi\Psi] = \int_{\mathcal{M}} E[X(t)\Psi]dt.$$

Φ is called the *Pettis integral* of X and is denoted by $\int_{\mathcal{M}} X(t)dt$.

Note that by this definition we have, for a Pettis integrable X,

$$\int_{\mathcal{M}} E[X(t)\Psi]dt = E\left[\int_{\mathcal{M}} X(t)\Psi dt\right]$$

for all $\Psi \in (L^2)$. In particular, the Pettis integral interchanges with the S-transform; in fact, observe the case where $\Psi =: e^{I(\eta)} :$ with $\eta \in S(\mathbb{R})$.

We shall point out the relationship between the Pettis integral and the pathwise integral. Let $X : [a, b] \to \mathbb{R}$ be measurable and pathwise integrable such that the pathwise integral belongs to (L^2). If X is good enough to apply Fubini's theorem, then we can interchange the integrals:

$$E\left[\int_a^b X(t)dt \cdot \Psi\right] = \int_a^b E[X(t)\Psi]dt,$$

where the integral on the left-hand side is the ordinary pathwise integral. Hence, the Pettis integral defined in Definition 3.7 coincides with the pathwise integral in that case.

Now, according to Bender [3, Theorem 3.1], we describe the classical Ito integral and fractional Ito integral from an S-transform point of view.

Let $0 \le a \le b$ and $X : [a, b] \times \Omega \to \mathbb{R}$ be a progressively measurable (with respect to the filtration $\mathcal{F}_t$ generated by the Brownian motion $B(s)$, $0 \le s \le t$) process satisfying

$$E\left[\int_a^b |X(t)|^2 dt\right] < \infty.$$

Then the classical Ito integral $\int_a^b X(t)dB(t)$ with respect to the Brownian motion B exists. We calculate its S-transform in the following: let Q_η, $\eta \in S(\mathbb{R})$, be the measure defined by (3.7). Then, by the classical Girsanov theorem, $\tilde{B}(t) := B(t) - \int_0^t \eta(u)du$ is a two-sided Brownian motion under the measure Q_η, and $\int_a^s X(t)d\tilde{B}(t)$, $a \le s \le b$, is a Q_η-martingale with zero expectation. Then, by (3.8), (3.10) and Fubini's theorem, we obtain the following:

$$
\begin{aligned}
S\left(\int_a^b X(t)dB(t)\right)(\eta) &= E^{Q_\eta}\left[\int_a^b X(t)dB(t)\right] \\
&= E^{Q_\eta}\left[\int_a^b X(t)d\tilde{B}(t) + \int_a^b X(t)\eta(t)dt\right] \\
&= \int_a^b E^{Q_\eta}[X(t)]\eta(t)dt \\
&= \int_a^b (SX(t))(\eta)\eta(t)dt \\
&= \int_a^b S(X(t))(\eta)\frac{d}{dt}S(B(t))(\eta)dt.
\end{aligned}
$$

As the S-transform is injective, it also can be taken to define the above integrals. Thus, we can summarize as follows:

Theorem 3.8. (i) *Let* $0 \le a \le b$ *and* $X : [a, b] \times \Omega \to \mathbb{R}$ *be a progressively measurable process such that* $E\left[\int_a^b |X(t)|^2 dt\right] < \infty$. *Then the Ito integral* $\int_a^b X(t)dB(t)$ *is the unique element in* (L^2) *with S-transform given by*

$$S\left(\int_a^b X(t)dB(t)\right)(\eta) = \int_a^b (SX(t))(\eta)\eta(t)dt$$

$$= \int_a^b S(X(t))(\eta)\frac{d}{dt}S(B(t))(\eta)dt. \tag{3.11}$$

(ii) *The Wiener integral* $I(f) := \int_{\mathbb{R}} f(t)dB(t),\ f \in L^2(\mathbb{R}),$ *is the unique element in* (L^2) *with S-transform given by*

$$\int_{\mathbb{R}} f(t)\eta(t)dt. \tag{3.12}$$

Drawing the conclusion by analogy, one can accordingly define the fractional integral of Wick-Ito type $\int_a^b X(t)dB_H(t)$ to be the unique random variable with *S*-transform

$$S\left(\int_a^b X(t)dB_H(t)\right)(\eta) = \int_a^b S(X(t))(\eta)\frac{d}{dt}S(B_H(t))(\eta)dt. \tag{3.13}$$

If we recall (3.3) and apply the *S*-transform on the Wiener integral following (3.9) as well as the fractional integration by parts rule (3.4), then we receive

$$\frac{d}{dt}S(B_H(t))(\eta) = \frac{d}{dt}S\left(\int_{\mathbb{R}}(M_-^H\mathbf{1}[0, t])(s)dB(s)\right)(\eta)$$

$$= \frac{d}{dt}\int_{\mathbb{R}} M_-^H\mathbf{1}[0, t](s)\eta(s)ds$$

$$= \frac{d}{dt}\int_0^t (M_+^H\eta)(s)ds$$

$$= (M_+^H\eta)(t).$$

Thus, (3.13) can be reformulated and we arrive at the following definition:

Definition 3.9. Let $X : \mathcal{M} \to (L^2)$ ($\mathcal{M} \subset \mathbb{R}$ a Borel set). Then X is said to have a *fractional Ito integral* (S-transform approach) if $S(X(\cdot))(\eta)(M_+^H \eta)(\cdot) \in L^1(\mathcal{M})$ for any $\eta \in S(\mathbb{R})$ and there is a $\Phi \in (L^2)$ such that for all $\eta \in S(\mathbb{R})$,

$$S(\Phi)(\eta) = \int_{\mathcal{M}} S(X(t))(\eta)(M_+^H \eta)(t)dt. \tag{3.14}$$

In this case, Φ is uniquely determined by the injectivity of the S-transform (see Remark 3.4) and it is denoted by $\int_{\mathcal{M}} X(t)dB_H(t)$, i.e.,

$$S\left(\int_{\mathcal{M}} X(t)dB_H(t)\right)(\eta) = \int_{\mathcal{M}} S(X(t))(\eta)(M_+^H \eta)(t)dt.$$

Remark 3.10. The fractional Ito integral as defined by the S-transform approach in Definition 3.9 has expectation zero:

$$E\left[\int_a^b X(t)dB_H(t)\right] = 0.$$

In fact, for $\Phi = \int_a^b X(t)dB_H(t)$, the S-transform at $\eta = 0$ implies

$$E[\Phi] = E[\Phi \cdot : e^{I(\eta)} :]|_{\eta=0} = S(\Phi)(0) = \int_a^b S(X(t))(0)(M_+^H \cdot 0)(t)dt = 0.$$

Let us now introduce the Wick product in Bender [3, Definition 3.3].

Definition 3.11. Let $\Phi, \Psi \in (L^2)$ and assume that there is an element $\Phi \lozenge \Psi \in (L^2)$, that satisfies

$$S(\Phi \lozenge \Psi)(\eta) = S(\Phi)(\eta)S(\Psi)(\eta) \text{ for all } \eta \in S(\mathbb{R}).$$

Then $\Phi \lozenge \Psi$ is called the *Wick product* of Φ and Ψ.

The next result by Bender [3, Theorem 3.6] explores the relationship between the fractional Ito integral and the Wick product:

Theorem 3.12. *Let* $X : \mathbb{R} \to (L^2)$ *and* $Y \in (L^2)$. *Then*

$$Y \lozenge \int_{\mathbb{R}} X(s) dB_H(s) = \int_{\mathbb{R}} Y \lozenge X(s) dB_H(s)$$

in the sense that if one side is well-defined, then so is the other, and both coincide.

The straightforward proof can be carried out by calculating the *S*-transform of both sides. Theorem 3.12 leads us to the following:

(i) In particular, this result implies that, for good random variables *Y*,

$$Y \lozenge (B_H(b) - B_H(a)) = \int_{\mathbb{R}} \mathbf{1}[a, b](s) Y dB_H(s).$$

(ii) Together with the fractional Ito isometry (Bender [3, Corollary 3.5]), this shows that for sufficiently good processes *X*, the fractional Ito integral is an (L^2) limit of Wick-Riemann sums; for instance, see Duncan et al. [17].

(iii) Note that, in general, the Wick product does not coincide with the ordinary pathwise product.

We now turn to the white noise calculus approach. So we assume the underlying probability space to be the white noise space, that is, Ω is $S'(\mathbb{R})$, the space of tempered distributions. The general idea of the white noise approach is as follows: Although the fractional Brownian motion $B_H : \mathbb{R} \to (L^2)$ is not differentiable on almost every path, it has a derivative, if we look at $B_H : \mathbb{R} \to (S)^*$ (see Bender [4, Theorem 2.17]). Here $(S)^*$ denotes a space of generalized random variables, the so-called Hida distributions; $(L^2) \subset (S)^*$. For more information about Hida distributions, see Hida et al. [19] and Kuo [26]. Note that the *S*-transform can be extended to $(S)^*$. Then, by Bender [4, Theorem 3.7], since $S\left(\frac{d}{dt} B_H(t)\right)(\eta) = (M_+^H \eta)(t)$ for $\eta \in S(\mathbb{R})$, we get

$$S\left(X(t)\Diamond\frac{d}{dt}B_H(t)\right)(\eta) = S(X(t))(\eta)S\left(\frac{d}{dt}B_H(t)\right)(\eta)$$

$$= S(X(t))(\eta)(M_+^H\eta)(t), \quad \eta\in S(\mathbb{R}),$$

where the diamond $\Diamond$ denotes extension of the Wick product to $(S)^*$ such that

$$\Diamond : (\mathcal{S})^* \times (\mathcal{S})^* \to (\mathcal{S})^*, \text{ continuous mapping.}$$

This leads us to the white noise approach such that the fractional Ito integral can be defined as the $(S)^*$-valued Pettis integral of $X(t)\Diamond\frac{d}{dt}B_H(t)$, if it exists;

$$\int_a^b X(t)dB_H(t) = \int_a^b X(t)\Diamond\frac{d}{dt}B_H(t)dt. \tag{3.15}$$

In fact, a formal calculation shows that for $\eta\in S(\mathbb{R})$,

$$S\left(\int_a^b X(t)dB_H(t)\right)(\eta) = \int_a^b S\left(X(t)\Diamond\frac{d}{dt}B_H(t)\right)(\eta)dt$$

$$= \int_a^b S(X(t))(\eta)(M_+^H\eta)(t),$$

and hence (3.14) of Definition 3.9 will yield the required assertion.

4. The White Noise Setting

In the following, we present some background of the standard Gaussian white noise calculus which can be found in Hida et al. [19] and Kuo [26].

Let $S(\mathbb{R})$ be the Schwartz space, i.e., the space of smooth rapidly decreasing functions on $\mathbb{R}$. Let $S'(\mathbb{R})$ denote the space of tempered distributions, which is the dual space of $S(\mathbb{R})$. Consider the Gel'fand triple:

$$S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R})$$

on the real line $\mathbb{R}$. Then we assume the underlying probability space $(\Omega, \mathcal{F}, P)$ to be the white noise space, that is, Ω is the space $S'(\mathbb{R})$, $\mathcal{F}$ is the σ-algebra generated by the open sets in $S'(\mathbb{R})$ with respect to the weak* topology of $S'(\mathbb{R})$. The probability measure P is uniquely determined by the Bochner-Minlos theorem such that for all rapidly decreasing functions $f \in S(\mathbb{R})$,

$$\int_{S'(\mathbb{R})} \exp\{i\langle \omega, f\rangle\} dP(\omega) = \exp\left\{-\frac{1}{2}| f |_0^2\right\}, \quad i = \sqrt{-1}. \tag{4.1}$$

Here $\langle \omega, f\rangle$ denotes the dual action and $|\cdot|_0$ is the usual $L^2(\mathbb{R})$-norm. The corresponding inner product is denoted by $(\cdot, \cdot)_0$. Namely, for every $f \in S(\mathbb{R})$, the map $\langle \cdot, f\rangle : \Omega \to \mathbb{R}$ defined by $\langle \cdot, f\rangle(\omega) = \langle \omega, f\rangle$ (where $\langle \omega, f\rangle$ is by definition $\omega(f)$, i.e., the action of the distribution ω on the function f) is a centered Gaussian random variable with variance equal to $| f |_0^2$ under P.

From relation (4.1) and the isometry $E[\langle \cdot, f\rangle^2] = | f |_0^2$ for $f \in S(\mathbb{R})$, we can extend $\langle \cdot, g\rangle$ to $g \in L^2(\mathbb{R})$. Hence, for $f, g \in L^2(\mathbb{R})$,

$$E[\langle \cdot, f\rangle\langle \cdot, g\rangle] = (f, g)_0. \tag{4.2}$$

For $a, b \in \mathbb{R}$, let $\mathbf{1}[a, b](t)$ be the indicator function as defined by (3.2). Then a continuous version of $\langle \cdot, \mathbf{1}[0, t]\rangle$ is a classical Brownian motion on the white noise space. Hence, approximating with step functions yields

$$\langle \cdot, f\rangle = \int_{\mathbb{R}} f(t) dB(t), \tag{4.3}$$

where $\int_{\mathbb{R}} f(t) dB(t)$ denotes the classical *Wiener integral* of a function $f \in L^2(\mathbb{R})$.

Equation (3.3) of Remark 3.1 shows that an fBm with arbitrary Hurst parameter H is given by a continuous version of the Wiener integral of the Mandelbrot and Van Ness type as shown in (1.1). Then, by Bender [4, Theorem 2.2], we can describe this result in terms of the operators $M_{\pm}^{H}$ as follows:

Theorem 4.1. *Let* $0 < H < 1$. *Further, let* $M_{\pm}^{H}$ *be the operators as defined by* (3.1). *Then an fBm is given by a continuous version of* $\langle \cdot, M_{-}^{H}\mathbf{1}[0, t]\rangle$.

In the following, according to Bender [4, Theorems 2.4, 2.7 and Corollary 2.8], we summarize several properties of the operators $M_{\pm}^{H}$ for later use.

Theorem 4.2. M_{-}^{H} *and* M_{+}^{H} *are dual operators in a suitable sense, i.e., for nice functions, the following relation holds*:

$$(f, M_{-}^{H}g)_0 = (M_{+}^{H}f, g)_0.$$

This yields the following relation:

$$(f, M_{-}^{H}\mathbf{1}[0, t])_0 = \int_0^t (M_{+}^{H}f)(s)ds \text{ for } f \in S(\mathbb{R}) \text{ and } 0 < H < 1.$$

Theorem 4.3. *Let* $H \in (0, 1)$ *and* $f \in S(\mathbb{R})$. *Then*

(i) $M_{+}^{H}f$ *is continuous,*

(ii) $(f, M_{-}^{H}\mathbf{1}[0, t])_0$ *is differentiable and*

$$\frac{d}{dt}(f, M_{-}^{H}\mathbf{1}[0, t])_0 = M_{+}^{H}f(t).$$

As in the case of a standard Brownian motion $(H = 1/2)$, an fBm with Hurst parameter $0 < H < 1$ is nowhere differentiable on almost every path. However, we can show that B_H is differentiable as a mapping from $\mathbb{R}$ into a space of stochastic generalized functions, the so-called Hida distributions.

Hence, we obtain a representation of its derivative as generalized Wiener integral. In order to show these, we proceed to fundamental results, referring to Bender [4, Section 2.3], Corlay et al. [15, Section 5.1] and Lebovits et al. [28, Section 4.1].

Let $(L^2) := L^2(\Omega, \mathcal{G}, P)$, where $\mathcal{G}$ is the σ-field generated by $\langle \cdot, f \rangle_{f \in L^2(\mathbb{R})}$.

We recall the Wiener-Ito theorem (see Nualart [34, Theorem 1.1.2, p. 13]) which says that every $\Phi \in (L^2)$ can be uniquely decomposed as a sum of multiple Wiener integrals:

$$\Phi = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in \hat{L}^2(\mathbb{R}^n),$$

where $\hat{L}^2(\mathbb{R}^n)$ denotes the set of all symmetric functions f in $L^2(\mathbb{R}^n)$ and $I_n(f)$ denotes the nth multiple Wiener integral of f with respect to the Brownian motion, defined by

$$\begin{aligned} I_n(f) &:= \int_{\mathbb{R}^n} f(t) dB^n(t) \\ &= n! \int_{\mathbb{R}} \left(\int_{-\infty}^{t_n} \cdots \left(\int_{-\infty}^{t_2} f(t_1, \ldots, t_n) dB(t_1) \right) dB(t_2) \cdots dB(t_n) \right) \end{aligned}$$

with the convention that $I(f_0) = f_0$ for constants f_0. The above decomposition is called the *Wiener chaos* of Φ. Moreover, the (L^2)-norm $\| \Phi \|_0$ of Φ is given by

$$\| \Phi \|_0 := (E[\Phi^2])^{1/2} := \left(\sum_{n=0}^{\infty} n! | f_n |_0^2 \right)^{1/2},$$

where $| \cdot |_0$ denotes the $L^2(\mathbb{R}^n)$-norm for any n.

For $n = 0, 1, 2, \ldots,$ define

$$h_n(x) := (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2) \text{ (the } n\text{th Hermite polynomial)}$$

and

$$e_n(x) := (\pi)^{-1/4}(2^n n!)^{-1/2} \exp(-x^2/2) h_n(x) \text{ (the } n\text{th Hermite function)}$$

$$= (-1)^n (\pi)^{-1/4}(2^n n!)^{-1/2} \exp(x^2/2) \frac{d^n}{dx^n} \exp(-x^2).$$

Let A denote the operator $A = -\frac{d^2}{dx^2} + x^2 + 1.$ Then we notice that the Hermite functions form an orthonormal basis of $L^2(\mathbb{R})$ and that the Hermite functions are the eigenvectors of A, satisfying $Ae_n = (2n + 2)e_n$.

For any $\Phi := \sum_{n=0}^{\infty} I_n(f_n),$ the Wiener chaos expansion, satisfying the condition

$$\sum_{n=0}^{\infty} n! | A^{\otimes n} f_n |_0^2 < \infty,$$

defines the element $\Gamma(A)\Phi$ of (L^2) by

$$\Gamma(A)\Phi := \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n),$$

where $A^{\otimes n}$ denotes the nth tensor power of the operator A. Both operators, A and $\Gamma(A),$ are densely defined in $L^2(\mathbb{R})$ and $(L^2),$ respectively; they are invertible and the inverse operators A^{-1} and $\Gamma(A)^{-1}$ are bounded on $L^2(\mathbb{R})$ and $(L^2),$ respectively (see Kuo [26]).

Let us denote $\| \varphi \|_0 := \| \varphi \|_{(L^2)}$ for φ in $(L^2),$ i.e., the (L^2)-norm, and let $\mathbb{D}\text{om}(\Gamma(A)^n)$ be the domain of the nth iteration of $\Gamma(A).$ Define the family of norms $(\|\cdot\|)_{p\in\mathbb{Z}}$ by

$$\| \Phi \|_p := \| \Gamma(A)^p \Phi \|_0 = \| \Gamma(A)^p \Phi \|_{(L^2)},$$

$$\forall p \in \mathbb{Z}, \forall \Phi \in (L^2) \cap \mathbb{D}\mathrm{om}(\Gamma(A)^p).$$

For any $p \in \mathbb{N}$, let

$$(\mathcal{S}_p) := \{\Phi \in (L^2); \Gamma(A)^p \Phi \text{ exists and belongs to } (L^2)\}$$

and define $(\mathcal{S}_{-p})$ as the completion of the space (L^2) with respect to the norm $\|\cdot\|_{-p}$. As in Kuo [26], we let $(\mathcal{S})$ denote the projective limit of the sequence $((\mathcal{S}_p))_{p\in\mathbb{N}}$ and $(\mathcal{S})^*$ the inductive limit of the sequence $((\mathcal{S}_{-p}))_{p\in\mathbb{N}}$. Again this means that we have the equalities

$$(\mathcal{S}) = \bigcap_{p\in\mathbb{N}} (\mathcal{S}_p) \text{ resp. } (\mathcal{S})^* = \bigcup_{p\in\mathbb{N}} (\mathcal{S}_{-p})$$

and that convergence in $(\mathcal{S})$ (resp. in $(\mathcal{S})^*$) means convergence in $(\mathcal{S}_p)$ for every p in $\mathbb{N}$ (resp. convergence in $(\mathcal{S}_{-p})$ for some p in $\mathbb{N}$; by Kuo [26], this is equivalent to convergence in the weak* topology). Moreover, by Kuo [26, p. 21, pp. 28-29], $(\mathcal{S}_p)$ is a Hilbert space with norm $\|\cdot\|_p$ and $(\mathcal{S})$ is a countably Hilbert space.

Definition 4.4. The space $(\mathcal{S})$ is called the *space of stochastic test functions* and $(\mathcal{S})^*$ the *space of Hida distributions*.

By Kuo [26, p. 21], for any p in $\mathbb{N}$, the dual space $(\mathcal{S}_p)^*$ of $(\mathcal{S}_p)$ is $(\mathcal{S}_{-p})$. Thus, we can write $(\mathcal{S}_{-p})$, to denote the space $(\mathcal{S}_p)^*$. As the notation suggests, $(\mathcal{S})^*$ is the dual space of $(\mathcal{S})$. Thus, we have the Gel'fand triple:

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*.$$

We will note $\langle\langle\ ,\ \rangle\rangle$ the duality bracket between $(\mathcal{S})^*$ and $(\mathcal{S})$, that is, the bilinear pairing of $(\mathcal{S})^*$ and $(\mathcal{S})$ such that for $\Phi \in (\mathcal{S})^*$ and $\varphi \in (\mathcal{S})$, $\langle\langle \Phi, \varphi \rangle\rangle$ is the dual action. If Φ, φ belong to (L^2), then we have the equality

$$\langle\langle \Phi, \varphi \rangle\rangle = \langle \Phi, \varphi \rangle_{(L^2)} = E[\Phi \cdot \varphi];$$

see Bender [4, Section 2.3] and Lebovits and Lévy-Véhel [27, Section 2.3].

The concept of stochastic (Hida) test functions and stochastic distributions may seem rather abstract. The motivation for introducing these spaces, however, is very similar to that behind using Schwartz test functions and distributions.

Definition 4.5. Let $I \subset \mathbb{R}$ be an interval. Then a function $\Phi : I \to (\mathcal{S})^*$ is called a *stochastic distribution process*, or an $(\mathcal{S})^*$*-process*, or a *Hida process*.

Definition 4.6. Let $t_0 \in I$. A stochastic distribution process $\Phi : I \to (\mathcal{S})^*$ is said to be *differentiable* at t_0, if the limit $\lim_{h\to 0} \dfrac{\Phi(t_0 + h) - \Phi(t_0)}{h}$ exists in $(\mathcal{S})^*$. We note $\dfrac{d\Phi}{dt}(t_0)$ the $(\mathcal{S})^*$-derivative at t_0 of the stochastic distribution process Φ. Φ is said to be *differentiable over* I if it is differentiable at every t_0 in I; recall that convergence in $(\mathcal{S})^*$ means, that there exists $p \in \mathbb{N}$ such that we have convergence with respect to the norm $\|\cdot\|_{-p}$.

Now, the Pettis integrability of the (L^2)-valued function as introduced in Definition 3.7 is extended to that of the $(\mathcal{S})^*$-valued function:

Definition 4.7. Assume that $\Phi : \mathbb{R} \to (\mathcal{S})^*$ is weakly in $L^1(\mathbb{R}, dt)$, i.e., assume that for all φ in $(\mathcal{S})$, the mapping $u \mapsto \langle\langle \Phi, \varphi \rangle\rangle$ from $\mathbb{R}$ to $\mathbb{R}$ belongs to $L^1(\mathbb{R}, dt)$. Then there exists a unique element in $(\mathcal{S})^*$, noted

$\int_{\mathbb{R}} \Phi(u)du$ such that

$$\left\langle\left\langle \int_{\mathbb{R}} \Phi(u)du,\ \varphi \right\rangle\right\rangle = \int_{\mathbb{R}} \langle\langle \Phi(u),\ \varphi \rangle\rangle du \text{ for all } \varphi \text{ in } (\mathcal{S}).$$

We say in this case that Φ is $(\mathcal{S})^*$*-integrable* on $\mathbb{R}$ in the *Pettis sense*; see Corlay et al. [15, Theorem 5.1].

In the sequel, when we do not specify a name for the integral of an $(\mathcal{S})^*$-integrable process Φ on $\mathbb{R}$, we always refer to the integral of Φ in Pettis' sense; see Kuo [26] for more details. The useful criterion of the integrability in the Pettis sense will be given by Definition 4.21 and Theorem 4.22 in terms of the *S*-transform, later on.

Definition 4.8. For $\eta \in S(\mathbb{R})$, the *Wick exponential* of $\langle \cdot, \eta \rangle$, denoted $: e^{\langle \cdot, \eta \rangle} :$, is defined as the element of $(\mathcal{S})$ given by

$$: e^{\langle \cdot, \eta \rangle} : \overset{\text{def}}{=} \sum_{n=0}^{\infty} (n!)^{-1} I_n(\eta^{\otimes n}) \text{ (equality in } (L^2)).$$

More generally, for $f \in L^2(\mathbb{R})$, we can define $: e^{\langle \cdot, f \rangle} :$ as the (L^2) random variable equal to $e^{\langle \cdot, f \rangle - \frac{1}{2}| f |_0^2}$. We will sometimes note $\exp^{\Diamond}\langle \cdot, f \rangle$ instead of $: e^{\langle \cdot, f \rangle} :$. This random variable belongs to $L^p(\Omega, P)$ for every integer $p \geq 1$.

We now recall the definition of the *S*-transform of an element Φ of $(\mathcal{S})^*$, noted $S(\Phi)$, $S\Phi$, or $S[\Phi]$; see Bender [4, Definition 3.4], Lebovits et al. [28, Section 4.1] and Corlay et al. [15, Section 5.1].

Definition 4.9. The *S*-transform $S(\Phi)(\eta)$ of an element Φ of $(\mathcal{S})^*$ is defined as the function from $S(\mathbb{R})$ to $\mathbb{R}$ given by

$$S(\Phi)(\eta) \overset{\text{def}}{=} \langle\langle \Phi, : e^{\langle \cdot, \eta \rangle} : \rangle\rangle \text{ for every } \eta \in S(\mathbb{R}).$$

We notice that $S(\Phi)(\eta)$ is nothing but

$$E[\Phi : e^{\langle \cdot, \eta \rangle} :] = e^{-\frac{1}{2}| n |_0^2} E[\Phi e^{\langle \cdot, \eta \rangle}]$$

when Φ belongs to (L^2); we recall Definition 3.3.

Define for $\eta \in S(\mathbb{R})$ the probability measure Q_η by

$$\frac{dQ_\eta}{dP} \stackrel{\text{def}}{=} : e^{\langle \cdot, \eta \rangle} : .$$

Then the probability measures Q_η and P are equivalent. Hence, by definition

$$S(\Phi)(\eta) = E^{Q_\eta}[\Phi] \text{ for every } \Phi \in (L^2); \text{ we recall (3.8).}$$

The following lemma will be used later on.

Lemma 4.10. (i) *Let p be a positive integer and Φ be an element in $(\mathcal{S}_{-p})$. Then*

$$| S(\Phi)(\eta) | \leq \| \Phi \|_{-p} \exp\left(\frac{1}{2} | \eta |_p^2\right)$$

for any η in $S(\mathbb{R})$.

(ii) *Let $\Phi := \sum_{k=0}^{\infty} a_k \langle \cdot, e_k \rangle$ belong to $(\mathcal{S})^*$. Then the following equality holds for every η in $S(R)$:*

$$S(\Phi)(\eta) = \sum_{k=0}^{\infty} a_k \langle \eta, e_k \rangle_{L^2(\mathbb{R})}.$$

We note that (i) is proved in Kuo [26, p. 79] and (ii) is verified by (3.12) of Theorem 3.8 as follows: for $\eta \in S(\mathbb{R})$,

$$S(\Phi)(\eta) = \sum_{k=0}^{\infty} a_k S(\langle \cdot, e_k \rangle)(\eta)$$

$$= \sum_{k=0}^{\infty} a_k S\left(\int_{\mathbb{R}} e_k(t)dB(t)\right)(\eta)$$

$$= \sum_{k=0}^{\infty} a_k \langle \eta, e_k \rangle_{L^2(\mathbb{R})}.$$

Another useful tool in white noise analysis is the Wick product (Kuo [26, p. 92]):

Definition 4.11. For every $(\Phi, \Psi) \in (\mathcal{S})^* \times (\mathcal{S})^*$, there exists a unique element of $(\mathcal{S})^*$, called the *Wick product* of Φ and Ψ and noted $\Phi \lozenge \Psi$, such that for every $\eta \in S(\mathbb{R})$,

$$S(\Phi \lozenge \Psi)(\eta) = S(\Phi)(\eta)S(\Psi)(\eta).$$

For any $\Phi \in (\mathcal{S})^*$ and $k = 0, 1, 2, \ldots$, let $\Phi^{\lozenge k}$ denote the element $\overbrace{\Phi \lozenge \cdots \lozenge \Phi}^{k \text{ times}}$ of $(\mathcal{S})^*$. Then we can generalize the definition of $\exp^{\lozenge}$ to the case where Φ belongs to $(\mathcal{S})^*$.

Definition 4.12. For any $\Phi \in (\mathcal{S})^*$ such that $\sum_{k=0}^{+\infty} \frac{\Phi^{\lozenge k}}{k!}$ converges in $(\mathcal{S})^*$, define the element $\exp^{\lozenge} \Phi$ of $(\mathcal{S})^*$ by

$$\exp^{\lozenge} \Phi \overset{\text{def}}{=} \sum_{k=0}^{+\infty} \frac{\Phi^{\lozenge k}}{k!}.$$

This is called *Wick exponential* of Φ.

For *f* in $L^2(\mathbb{R})$ and $\Phi \overset{\text{def}}{=} \langle \cdot, f \rangle$, it is easy to verify that $\exp^{\lozenge} \Phi$ given by Definition 4.12 exists and coincides with $: e^{\langle \cdot,\, f \rangle} :$ given by Definition 4.8.

Remark 4.13. Let Φ be deterministic. Then, for all $\Psi \in (\mathcal{S})^*$, $\Phi \lozenge \Psi = \Phi\Psi$. Moreover, let $(X(t))_{t \in \mathbb{R}}$ be a Gaussian process and let $\mathcal{H}$ be the subspace of (L^2) defined by $\mathcal{H} := \overline{vect_{\mathbb{R}}\{X(t);\, t \in \mathbb{R}\}}^{(L^2)}$. If X and Y are two elements of $\mathcal{H}$, then $X \lozenge Y = XY - E[XY]$.

The following results on the S-transform will be used in the sequel; for proofs and a brief summary, see Bender [4, Theorem 3.6], Corlay et al. [15, Section 5.1], Kuo [26, p. 39] and Lebovits and Lévy-Véhel [27, Section 2.4].

Theorem 4.14. *The S-transform verifies the following properties*:

(i) *The map* $S : \Phi \mapsto S(\Phi)$, *from* $(\mathcal{S})^*$ *to* $(\mathcal{S})^*$, *is injective*.

(ii) *Let* $\Phi : \mathbb{R} \to (\mathcal{S})^*$ *be an* $(\mathcal{S})^*$*-process*. *If* Φ *is* $(\mathcal{S})^*$*-integrable over* $\mathbb{R}$, *then*

$$S\left(\int_{\mathbb{R}} \Phi(u)du\right)(\eta) = \int_{\mathbb{R}} S(\Phi(u))(\eta)du \text{ for all } \eta \in S(\mathbb{R}).$$

(iii) *Let* $\Phi : \mathbb{R} \to (\mathcal{S})^*$ *be an* $(\mathcal{S})^*$*-process differentiable over* $\mathbb{R}$. *Then*, *for every* $\eta \in S(\mathbb{R})$, *the map* $u \mapsto [S(\Phi)(u)](\eta)$ *is differentiable over* $\mathbb{R}$ *and verifies*

$$S\left[\frac{d\Phi}{dt}(t)\right](\eta) = \frac{d}{dt}[S[\Phi(t)](\eta)].$$

We recall that the sample paths of fBm are almost surely non-differentiable. However, fBm is differentiable as a stochastic distribution process as shown in the following.

First we shall find $\frac{d}{dt}M_-^H \mathbf{1}[0, t]$. This in turn depends on the following property of Hermite functions as given in Bender [4, Lemma 2.14]:

Lemma 4.15. *Assume* $H \in (0, 1)$ *and* e_n *is the nth Hermite function*. *Then there is a constant* $C_H > 0$ *such that*

$$\max_{x \in \mathbb{R}} |(M_+^H e_n)(x)| \le C_H (n+1)^{5/12}.$$

Then the following result in Bender [4, Lemma 2.15] is essential:

Lemma 4.16. *Let* $H \in (0, 1)$ *and* $f \in S(\mathbb{R})$. *Then* $M_{-}^{H}\mathbf{1}[0, \cdot] : \mathbb{R} \to S'(\mathbb{R})$ *is differentiable and*

$$\frac{d}{dt} M_{-}^{H}\mathbf{1}[0, t] = \sum_{k=0}^{\infty} (M_{+}^{H} e_k)(t) e_k, \tag{4.4}$$

where the limit is in $S'(\mathbb{R})$.

Before we prove Lemma 4.16, we recall that one can reconstruct $S(\mathbb{R})$ (resp. $S'(\mathbb{R})$), the space of Schwartz test functions (resp. the space of tempered distributions), as the projective limit (resp. the inductive limit), as follows: Observe that $Ae_k = (2k+2)e_k$, where $A = -\frac{d^2}{dx^2} + x^2 + 1$ and e_k are the Hermite functions which form an orthonormal basis of $L^2(\mathbb{R})$. Then, for $p \in \mathbb{Z}$, define

$$| f |_p^2 := | A^p f |_0^2 = \sum_{k=0}^{\infty} (2k+2)^{2p} (f, e_k)_0^2, \quad (p, f) \in \mathbb{Z} \times L^2(\mathbb{R}), \tag{4.5}$$

where the last equality follows from the fact that e_k is an eigenfunction of A with eigenvalue $(2k+2)$, and $|\cdot|_0$ denotes the usual $L^2(\mathbb{R})$-norm.

For $p \in \mathbb{N}$, define the spaces

$$S_p(\mathbb{R}) := \{ f \in L^2(\mathbb{R}); | f |_p < \infty \}$$

and $S_{-p}(\mathbb{R})$ as being the completion of the space $L^2(\mathbb{R})$ with respect to the norm $|\cdot|_{-p}$, that is,

$$| f |_{-p}^2 := | A^{-p} f |_0^2 = \sum_{k=0}^{\infty} (2k+2)^{-2p} (f, e_k)_0^2. \tag{4.6}$$

Then it is well-known that $S(\mathbb{R})$ is the projective limit of the sequence $(S_p(\mathbb{R}))_{p\in\mathbb{N}}$ and that $S'(\mathbb{R})$ is the inductive limit of the sequence $(S_{-p}(\mathbb{R}))_{p\in\mathbb{N}}$, that is,

$$S(\mathbb{R}) = \bigcap_{p\in\mathbb{N}} S_p(\mathbb{R}) \quad \text{and} \quad S'(\mathbb{R}) = \bigcup_{p\in\mathbb{N}} S_{-p}(\mathbb{R}).$$

Thus, convergence in $S(\mathbb{R})$ is nothing but convergence in $S_p(\mathbb{R})$ for every $p \in \mathbb{N}$ and that convergence in $S'(\mathbb{R})$ is convergence in $S_{-p}(\mathbb{R})$ for some $p \in \mathbb{N}$.

To summarize, we have a sequence of norms on $S(\mathbb{R})$:

$$\cdots \le |f|_{-p}^2 \le \cdots \le |f|_{-1}^2 \le |f|_0^2 \le |f|_1^2 \le \cdots \le |f|_p^2 \le \cdots.$$

The space $S(\mathbb{R})$ is topologised by an increasing sequence of norms, and hence it is a countably Hilbert space. Moreover, we note that for any $p \in \mathbb{N}$, the dual space $S'_p(\mathbb{R})$ of $S_p(\mathbb{R})$ is $S_{-p}(\mathbb{R})$; see Kuo [26, pp. 17-18], for more details.

Proof of Lemma 4.16. We refer the proof to Bender [4, p. 90]. We first mention that the Hermite functions e_k form an orthonormal basis in $L^2(\mathbb{R})$ and hence we can write

$$M_-^H \mathbf{1}[0, t] = \sum_{k=0}^{\infty} (M_-^H \mathbf{1}[0, t], e_k)_0 e_k = \sum_{k=0}^{\infty} \left(\int_0^t (M_+^H e_k)(s) ds \right) e_k,$$

where the last equality is a consequence of Theorem 4.2. Hence,

$$\left| \frac{M_-^H \mathbf{1}[0, t+h] - M_-^H \mathbf{1}[0, t]}{h} - \sum_{k=0}^{\infty} (M_+^H e_k)(t) e_k \right|_{-1}^2$$

$$= \left| \sum_{k=0}^{\infty} \left(\frac{1}{h} \int_t^{t+h} (M_+^H e_k)(s) ds - (M_+^H e_k)(t) \right) e_k \right|_{-1}^2$$

$$= \sum_{k=0}^{\infty} (2k+2)^{-2} \left(\frac{1}{h} \int_t^{t+h} (M_+^H e_k)(s) ds - (M_+^H e_k)(t) \right)^2. \qquad (4.7)$$

Here the last equality follows from (4.6) and from the fact that e_k are orthogonal, i.e., $(e_j, e_k)_0 = 0$ if $j \neq k$, which effectively removes the second summation. We next see that from Lemma 4.15, the right-hand side of (4.7) converges uniformly in h. Thus, we can pass the limit as $h \to 0$ under the summation. Finally,

$$\lim_{h \to 0} \left| \frac{M_-^H \mathbf{1}[0, t+h] - M_-^H \mathbf{1}[0, t]}{h} - \sum_{k=0}^{\infty} (M_+^H e_k)(t) e_k \right|_{-1}^2$$

$$= \sum_{k=0}^{\infty} (2k+2)^{-2} \lim_{h \to 0} \left(\frac{1}{h} \int_t^{t+h} (M_+^H e_k)(s) ds - (M_+^H e_k)(t) \right)^2 = 0,$$

because Theorem 4.3(i) tells us that $M_+^H e_k$ is continuous. Thus, with convergence in $|\cdot|_{-1}$ (for the norm, see (4.6)),

$$M_-^H \mathbf{1}[0, t] = \sum_{k=0}^{\infty} (M_-^H \mathbf{1}[0, t], e_k)_0 e_k = \sum_{k=0}^{\infty} \left(\int_0^t (M_+^H e_k)(s) ds \right) e_k$$

and since the Schwartz distributions $S'(\mathbb{R})$, that is continuous linear functionals on $S(\mathbb{R})$, is equal to the union over p of $S'_p(\mathbb{R}) := S_{-p}(\mathbb{R})$, i.e., $S'(\mathbb{R}) = \bigcup_{p \in \mathbb{N}} S_{-p}(\mathbb{R})$, we get the above equality with convergence in $S'(\mathbb{R})$. This completes the proof of Lemma 4.16.

Because of the identity $B_H(t) = \langle \cdot, M_-^H \mathbf{1}(0, t) \rangle$ ((3.3) of Remark 3.1), Lemma 4.16 might suggest that

$$\frac{d}{dt} B_H(t) = \left\langle \cdot, \sum_{k=0}^{\infty} (M_+^H e_k)(t) e_k \right\rangle.$$

But the integrand of this Wiener integral is not an element of $L^2(\mathbb{R})$, but a tempered distribution. So we need to extend the Wiener integral to tempered distributions.

Let $f \in L^2(\mathbb{R})$, $p \in \mathbb{N}$. Then we can use (4.3) and get

$$\| \langle \cdot, f \rangle \|_{-p} = \left\| \int_{\mathbb{R}} f(t) dB(t) \right\|_{-p} = \| I_1(f) \|_{-p};$$

in general, $I_n(f)$ means the *n*th multiple Wiener integral of *f* with respect to the Brownian motion $B(t)$. Further, by definition of the $\| \cdot \|_{-p}$-norm, i.e., $\| \Phi \|_{-p} = \| \Gamma(A)^{-p} \Phi \|_0 = \| \Gamma(A)^{-p} \Phi \|_{(L^2)}$, and due to the fact that $I_1(f)$ is a chaos expansion consisting of that single term, we obtain

$$\| \langle \cdot, f \rangle \|_{-p} = \| I_1(f) \|_{-p} = \| I_1(A^{-p} f) \|_0 = | A^{-p} f |_0 = | f |_{-p}. \quad (4.8)$$

In (4.8), we used (4.3) and Ito's isometry such that

$$\forall f \in L^2(\mathbb{R}), \quad E[\langle \cdot, f \rangle^2] = | f |_0^2.$$

Hence we have, for all $p \in \mathbb{N}$,

$$\| \langle \cdot, f \rangle \|_{-p} = \left\| \int_{\mathbb{R}} f(t) dB(t) \right\|_{-p} = | f |_{-p}; \quad (4.9)$$

for the norm $| \cdot |_{-p}$, see (4.6). Using (4.9), we can extend the Wiener integral to $f \in S'(\mathbb{R})$. However, as follows from Bender [4, p. 90], one has to be cautious, because when $\langle \cdot, f \rangle$ exists only as an element of $(\mathcal{S}_{-p})$ (and not (L^2)), then also $\int_{\mathbb{R}} f(t) dB(t)$ is an element of $(\mathcal{S})^*$ but not (L^2) and so it is a Hida distribution but not (necessarily) a random variable.

Recall that by (4.8), we can apply the Ito isometry not only to $L^2(\mathbb{R})$ functions but also to tempered distributions as follows: If we assume that $F : I \to S'(\mathbb{R})$ is differentiable, then we see that

$$\lim_{h\to 0}\left\| \left\langle \cdot,\ h^{-1}(F_{t+h} - F_t) - \frac{d}{dt}F_t \right\rangle \right\|_{-p}$$

$$= \lim_{h\to 0}\left| h^{-1}(F(t+h) - F(t)) - \frac{d}{dt}F(t) \right|_{-p}$$

$$= 0.$$

This yields the following theorem by Bender [4, Theorem 2.17], which allows us to calculate the derivative of B_H :

Theorem 4.17. *Let* $I \subset \mathbb{R}$ *be an interval and let* $F : I \to S'(\mathbb{R})$ *be differentiable. Then* $\langle \cdot, F(t)\rangle$ *is differentiable as a stochastic distribution process and*

$$\frac{d}{dt}\langle \cdot, F(t)\rangle = \left\langle \cdot, \frac{d}{dt}F(t)\right\rangle.$$

Hence, by Lemma 4.16, we obtain that B_H is differentiable for $0 < H < 1$ as a stochastic distribution process and

$$\frac{d}{dt}B_H(t) = \left\langle \cdot, \sum_{k=0}^{\infty}(M_+^H e_k)(t)e_k \right\rangle.$$

We can find even a simpler expression for $\frac{d}{dt}B_H$, though. For $t \in \mathbb{R}$, we define the distribution

$$\langle \delta_t \circ M_+^H,\ f\rangle := (M_+^H f)(t)$$

Then, by Bender [4, p. 91], we have the following:

$$\left| \sum_{k=0}^{\infty}(M_+^H e_k)(t)e_k - \delta_t \circ M_+^H \right|_{-1}^2$$

$$= \sum_{n=0}^{\infty}(2n+2)^{-2}\left\langle \sum_{k=0}^{\infty}(M_+^H e_k)(t)e_k - \delta_t \circ M_+^H, e_n \right\rangle^2$$

$$= \sum_{n=0}^{\infty} (2n+2)^{-2} \left(\sum_{k=0}^{\infty} (M_+^H e_k)(t)(e_k, e_n)_0 - (M_+^H e_n)(t) \right)^2 = 0.$$

Thus, we get the following corollary:

Corollary 4.18. *For* $0 < H < 1$, *fBm* B_H *is differentiable over* $\mathbb{R}$ *as a stochastic distribution process and*

$$\frac{d}{dt} B_H(t) = \langle \cdot, \delta_t \circ M_+^H \rangle.$$

Definition 4.19. For $0 < H < 1$, the derivative of B_H in $(\mathcal{S})^*$,

$$W_H(t) \coloneqq \langle \cdot, \delta_t \circ M_+^H \rangle$$

is called the *fractional white noise*.

Using the preceding results, we can obtain the S-transform of the fBm and the fractional white noise:

Theorem 4.20. *Let* $0 < H < 1$. *Then, for any* $\eta \in S(\mathbb{R})$,

(i) $S(B_H(t))(\eta) = (\eta, M_-^H \mathbf{1}(0, t))_0$,

(ii) $S(W_H(t))(\eta) = (M_+^H \eta)(t)$.

Proof of Theorem 4.20. Item (i) is obtained by the duality bracket between $(\mathcal{S})^*$ and $(\mathcal{S})$ (recall the description after Definition 4.4). In fact,

$$S(B_H(t))(\eta) = \langle\langle B_H(t), :e^{\langle \cdot, \eta \rangle}: \rangle\rangle = E[B_H(t) \cdot :e^{\langle \cdot, \eta \rangle}:] = (\eta, M_-^H \mathbf{1}[0, t])_0.$$

Let $\Phi : \mathbb{R} \to (\mathcal{S})^*$ be an $(\mathcal{S})^*$-process differentiable over $\mathbb{R}$. Notice Theorem 4.14(iii), that is, the S-transform of Φ satisfies, for every $\eta \in S(\mathbb{R})$,

$$S\left[\frac{d\Phi}{dt}(t)\right](\eta) = \frac{d}{dt}[S[\Phi(t)](\eta)].$$

Then item (ii) is obtained by Theorem 4.3(ii), because $W_H(t)$ is the derivative of $B_H(t)$. Hence, the proof is completed.

Now, the integrability of a stochastic distribution process X can be defined in terms of the S-transform:

Definition 4.21. A stochastic distribution process $X : I \to (\mathcal{S})^*$ is *integrable* in the white noise (Pettis) sense, if the following are satisfied:

(i) $S[X(\cdot)](\eta)$ is measurable for all $\eta \in S(\mathbb{R})$ and $S[X(\cdot)](\eta) \in L^1(I)$ for all $\eta \in S(\mathbb{R})$.

(ii) There exists $\Phi \in (\mathcal{S})^*$ such that $\int_I S[X(t)](\eta)dt = S[\Phi](\eta)$.

In this case, Φ is unique by injectivity of the S-transform (Theorem 4.14(i)). It is called the *white noise integral* of X and is denoted by $\Phi = \int_I X(t)dt$.

The following criterion by Kuo [26, Theorem 13.5] is useful for integrability in $(\mathcal{S})^*$:

Theorem 4.22. *Assume that* $X : I \to (\mathcal{S})^*$ *satisfies*:

(i) $S[X(\cdot)](\eta)$ *is measurable for all* $\eta \in S(\mathbb{R})$.

(ii) *There exist a natural integer p, a real a and a function* $L \in L^1(I, dt)$ *such that*

$$| S[X(t)](\eta) | \leq L(t)\exp(a| \eta |_p^2),$$

where $| \eta |_p^2 = | A^p\eta |_0^2$ *and* $| \cdot |_0$ *is the* $L^2(\mathbb{R})$*-norm* (*see* (4.5)).

Then X is $(\mathcal{S})^*$*-integrable* (*over I*) *in the white noise* (*Pettis*) *sense*.

For example, by Bender [4, Theorems 3.11 and 3.12], $\int_0^T W_H(t)dt$ and $\int_0^T B_H(t)\lozenge W_H(t)dt$ exist in the white noise (Pettis) sense. These examples provide motivation for the following definition of *fractional Ito integral*:

Definition 4.23. A stochastic (distribution) process $\Phi : [0, T] \to (\mathcal{S})^*$ is *fractional Ito integrable* provided that $\Phi \lozenge W_H$ is white noise (Pettis) integrable and we shall use the notation:

$$\int_0^T \Phi(t)dB_H(t) := \int_0^T \Phi(t) \lozenge W_H(t)dt.$$

When the stochastic distribution process is an (L^2)-valued process, the following criterion holds (see Bender [3, Theorem 2.8]):

Theorem 4.24. *Let* $X : \mathbb{R} \to (L^2)$ *such that* $t \mapsto S[X(t)](\eta)$ *is measurable for all* $\eta \in S(\mathbb{R})$ *and* $t \mapsto \| X(t) \|_0$ *is in* $L^1(\mathbb{R}, dt)$, *where* $\| \cdot \|_0$ *denotes the* (L^2) *norm*. *Then X is* $(\mathcal{S})^*$-*integrable in the white noise* (*Pettis*) *sense and*

$$\left\| \int_{\mathbb{R}} X(t)dt \right\|_0 \leq \int_{\mathbb{R}} \| X(t) \|_0 dt.$$

This result is based on the fact that the Pettis integral is an extension of the *Bochner integral* (see Kuo [26, p. 247]).

In the white noise setting, Lebovits and Lévy-Véhel [27] used the notion about the integral in the following Bochner sense:

Definition 4.25. Let I be a subset of $\mathbb{R}$ endowed with the Lebesgue measure. One says that $X : I \to (\mathcal{S})^*$ is *Bochner integrable* on I if it satisfies the two following conditions:

(i) X is weakly measurable on I, i.e., $u \mapsto \langle\langle X(u), \varphi \rangle\rangle$ is measurable on I for every $\varphi \in (\mathcal{S})$.

(ii) There exists $p \in \mathbb{N}$ such that $X(u) \in (\mathcal{S}_{-p})$ for almost every $u \in I$ and $u \mapsto \| X(t) \|_{-p}$ belongs to $L^1(I)$.

The Bochner integral of X on I is denoted $\int_I X(s)ds$.

The properties of the Bochner integral are given by Kuo [26] as follows:

Proposition 4.26. *If* $X : I \to (\mathcal{S})^*$ *is Bochner integrable on I, then the following hold*:

(i) *There exists an integer p such that*

$$\left\| \int_I X(s)ds \right\|_{-p} \leq \int_I \| X(s) \|_{-p} ds.$$

(ii) *X is also Pettis integrable on I and both integrals coincide on I.*

Remark 4.27. Proposition 4.26 shows that there is no risk of confusion by using the same notation for both the Bochner integral and the Pettis integral; see Lebovits and Lévy-Véhel [27, Appendix A].

5. The Operators M_H and their Derivatives

In the following, we shall generalize the previous stochastic integration with respect to fBm to the case of mBm. We first focus on the representation of fBm B_H with Hurst parameter $H \in (0, 1)$, the so-called harmonizable one ((1.2) and (1.3)). We next introduce the operator, denoted M_H, that will be useful for the definition of the integral with respect to fBm and mBm. Our description is based on the results in Corlay et al. [15, Section 5.1], Lebovits and Lévy-Véhel [27, Section 3.1] and Lebovits et al. [28, Section 4.1].

We note $\hat{u}$ or $\mathcal{F}(u)$ the Fourier transform of a tempered distribution u and we let $L^1_{loc}(\mathbb{R})$ denote the set of measurable functions which are locally integrable on $\mathbb{R}$. We also identify, here and in the sequel, any function f of $L^1_{loc}(\mathbb{R})$ with its associated distribution, also noted T_f.

We will say that a tempered distribution v is of *function type* if there exists a locally integrable function f such that $v = T_f$ (in particular, $\langle v, \phi \rangle = \int_{\mathbb{R}} f(t)\phi(t)dt$ for ϕ in $S(\mathbb{R})$).

Let H be a fixed real in $(0, 1)$. Then, following Elliott and Van Der Hoek [18, pp. 303-304], we define an operator, denoted M_H, which is specified in the Fourier domain by

$$\widehat{M_H(u)}(y) := \frac{\sqrt{2\pi}}{C(H)} |y|^{1/2-H} \hat{u}(y), \quad y \in \mathbb{R}. \tag{5.1}$$

Here and hereafter, $C(H)$ is the constant as given by (1.4), appearing in the harmonizable representation of the fBm. This operator is well defined on the homogeneous Sobolev space of order $\frac{1}{2} - H$, denoted $L_H^2(\mathbb{R})$ and defined by

$$L_H^2(\mathbb{R}) := \{u \in S'(\mathbb{R});\ \hat{u} = T_f,\ f \in L_{loc}^1(\mathbb{R}) \text{ and } \|u\|_H < \infty\}, \tag{5.2}$$

where $\|u\|_H^2 := \frac{1}{C(H)^2} \int_{\mathbb{R}} |\xi|^{1-2H} |\hat{u}(\xi)|^2 d\xi$ derives from the inner product on $L_H^2(\mathbb{R})$, defined by

$$\langle u, v \rangle_H := \frac{1}{C(H)^2} \int_{\mathbb{R}} |\xi|^{1-2H} \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi. \tag{5.3}$$

Then, by Lebovits and Lévy-Véhel [27, Lemma 3.1], we have the following:

Lemma 5.1. $(L_H^2(\mathbb{R}), \langle , \rangle_H)$ *is a Hilbert space. If* $H \in (0, 1/2]$, *then the space* $L_H^2(\mathbb{R})$ *is continuously embedded in* $L^{1/H}(\mathbb{R})$. *If* $H \in [1/2, 1)$, *then the space* $L^{1/H}(\mathbb{R})$ *is continuously embedded in* $L_H^2(\mathbb{R})$.

Since $\widehat{M_H(u)}$ *belongs to* $L^2(\mathbb{R})$ *for every u in* $L_H^2(\mathbb{R})$, M_H *is well defined as its inverse transform,*

$$M_H(u)(x) := \frac{1}{2\pi} \mathcal{F}(\widehat{M_H(u)})(-x), \text{ for almost every } x \in \mathbb{R}.$$

Further, by Lebovits and Lévy-Véhel [27, Proposition 3.2], we have the following:

Proposition 5.2. *The operator* M_H *is an isometry from* $(L_H^2(\mathbb{R}), \langle , \rangle_H)$ *to* $(L^2(\mathbb{R}), \langle , \rangle_{L^2(\mathbb{R})})$.

Let $\mathcal{E}(\mathbb{R})$ denote the space of simple functions on $\mathbb{R}$, which is the set of all finite linear combination of functions $\mathbf{1}[a, b](\cdot)$ with a and b in $\mathbb{R}$. It is easy to check that both $S(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$ are subsets of $L_H^2(\mathbb{R})$. It will be useful in the sequel to have an explicit expression for $M_H(f)$ when f is in $S(\mathbb{R})$ or in $\mathcal{E}(\mathbb{R})$.

For the indicator function $\mathbf{1}[a, b](t)$ as introduced in (3.2), it holds that

$$M_H(\mathbf{1}[a, b])(x) = \frac{\sqrt{2\pi}}{2C(H)\Gamma(H + 1/2)\cos\left(\frac{\pi}{2}(H - 1/2)\right)} \times \left[\frac{b - x}{|b - x|^{3/2-H}} - \frac{a - x}{|a - x|^{3/2-H}}\right].$$

Further, by Biagini et al. [9, Section 3] and Elliott and Van Der Hoek [18, p. 303], for f in $S(\mathbb{R})$, the following hold for almost every real x:

$$M_H(f)(x) = \gamma_H \int_{\mathbb{R}} \frac{f(x - t) - f(x)}{|t|^{3/2-H}} dt \quad (0 < H < 1/2), \tag{5.4}$$

$$M_H(f)(x) = f(x) \quad (H = 1/2), \tag{5.5}$$

$$M_H(f)(x) = \gamma_H \int_{\mathbb{R}} \frac{f(x)}{|t - x|^{3/2-H}} dt \quad (1/2 < H < 1), \tag{5.6}$$

where

$$\gamma_H := \frac{\sqrt{2\pi}}{2C(H)\Gamma(H - 1/2)\cos\left(\frac{\pi}{2}(H - 1/2)\right)} = \frac{(\Gamma(2H + 1)\sin(\pi H))^{1/2}}{2\Gamma(H - 1/2)\cos\left(\frac{\pi}{2}(H - 1/2)\right)}.$$

When f belongs to $S(\mathbb{R})$, it follows from Elliott and Van Der Hoek [18, Appendix] that

$$M_H(f)(x) = \alpha_H \frac{d}{dx}\left[\int_{\mathbb{R}} (t-x)|t-x|^{H-3/2} f(t)dt\right],$$

where

$$\alpha_H := \frac{-\gamma_H}{H-1/2} = \frac{-\sqrt{2\pi}}{2C(H)\Gamma(H+1/2)\cos\left(\frac{\pi}{2}(H-1/2)\right)}. \tag{5.7}$$

Remark 5.3. For the understanding of the construction of the stochastic integrals with respect to fBm and mBm, we refer to the following description in Lebovits and Lévy-Véhel [27, Proposition 3.3 and Remark 3.5]:

(i) In order to extend the Wiener integral with respect to fBm to an integral with respect to mBm, we will use the equality:

$$\overline{\mathcal{E}(\mathbb{R})}^{\langle,\ \rangle_H} = L^2_H(\mathbb{R}).$$

(ii) Because the space $S(\mathbb{R})$ is dense in $L^2_H(\mathbb{R})$ for the norm $\|\cdot\|_H$, it is also possible to define the operator M_H on the space $S(\mathbb{R})$ and extend it, by isometry, to all elements of $L^2_H(\mathbb{R})$. This is the approach of Elliott and Van Der Hoek [18] and Biagini et al. [9] (with a different normalization constant). This clearly yields the same operator as the one defined by (5.1). However, this approach does not lend itself to an extension to the case where the constant H is replaced by a Hurst function h, which is what we need for mBm. Therefore, it is possible to define the operator M_H on the space $\mathcal{E}(\mathbb{R})$ and extend it, by isometry, to all elements of $L^2_H(\mathbb{R})$. This extension coincides with (5.1).

By (5.3), we can get the following:

$$\begin{aligned} R_H(t, s) &:= \langle \mathbf{1}[0, t], \mathbf{1}[0, s]\rangle_H \\ &= \frac{1}{C^2(H)}\int_{\mathbb{R}} \frac{(e^{it\xi}-1)(e^{-is\xi}-1)}{|\xi|^{2H+1}} d\xi \end{aligned}$$

$$= \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}], \quad t, s \in \mathbb{R}.$$

Thus, as in the case of standard Brownian motion, i.e., in the white noise setting, we have that the process $(\widetilde{B}_H(t))_{t \in R}$, defined for all $(t, \omega) \in \mathbb{R} \times \Omega$ by

$$\widetilde{B}_H(t)(\omega) := \widetilde{B}_H(t, \omega) := \langle \omega, M_H \mathbf{1}[0, t] \rangle, \tag{5.8}$$

is a Gaussian process which admits a continuous version noted $B_H :=$ $(B_H(t))_{t \in \mathbb{R}}$. Indeed, under the probability measure P, the process B_H is a fractional Brownian motion since we have the equations:

$$\begin{aligned} E[B_H(t) B_H(s)] &= E[\langle \cdot, M_H \mathbf{1}[0, t] \rangle \langle \cdot, M_H \mathbf{1}[0, s] \rangle] \\ &= \langle M_H(\mathbf{1}[0, t]), M_H(\mathbf{1}[0, s]) \rangle_{L^2(\mathbb{R})} \\ &= \langle \mathbf{1}[0, t], \mathbf{1}[0, s] \rangle_H = R_H(t, s), \end{aligned} \tag{5.9}$$

where we used Proposition 5.2.

By (5.9), we observe that the constant $\frac{\sqrt{2\pi}}{C(H)}$ in formula (5.1) is given so that for all $H \in (0, 1)$, the process B_H defined by (5.8) is a normalized fBm.

The properties of the operator M_H are given by Lebovits and Lévy-Véhel [27, Theorem 3.7] as follows:

Theorem 5.4. (i) *For all* $H \in (0, 1)$, *the operator* M_H *is bijective from* $L^2_H(\mathbb{R})$ *into* $L^2(\mathbb{R})$.

(ii) *For all* $H \in (0, 1)$ *and* (f, g) *in* $(L^2(\mathbb{R}) \cap L^2_H(\mathbb{R}))^2$,

$$\langle f, M_H(g) \rangle_{L^2(\mathbb{R})} = \langle M_H(f), g \rangle_{L^2(\mathbb{R})}.$$

Moreover, the equality above remains true when f belongs to $L^1_{Loc}((R)) \cap$

$L_H^2(\mathbb{R})$ *and g belongs to* $S(\mathbb{R})$; *in this case, the equality reads*

$$\langle f, M_H(g)\rangle = \langle M_H(f), g\rangle_{L^2(\mathbb{R})},$$

where $\langle , \rangle$ *denotes the duality bracket between* $S'(\mathbb{R})$ *and* $S(\mathbb{R})$.

(iii) *There exists a constant D such that for every couple* (H, k) *in* $(0, 1)\times\mathbb{N}^*$,

$$\max_{x\in\mathbb{R}}|M_H(e_k)(x)| \le \frac{D}{C(H)}(k+1)^{2/3}.$$

In order to define the stochastic integral with respect to mBm, we shall consider the heuristic derivative of M_H with respect to H and use the operator $\frac{\partial M_H}{\partial H}$ later on. Following Lebovits and Lévy-Véhel [27, Section 3.2], define the operator $\frac{\partial M_H}{\partial H}$, specified in the Fourier domain, by

$$\widehat{\frac{\partial M_H}{\partial H}(u)}(y) := -(\beta_H + \log|y|)\frac{\sqrt{2\pi}}{C(H)}|y|^{1/2-H}\hat{u}(y), \quad y\in\mathbb{R},$$

where $\beta_H := C'(H)/C(H)$ with $C(H)$ as defined in (1.4) and $C'(H)$ denotes the derivative of the analytic map $H \mapsto C(H)$.

Equation (5.8) suggests that we can replace the constant H by a continuous deterministic function h, ranging in $(0, 1)$. Here, we recall Definition 2.2, that is, the definition of the mBm with Hurst function $h(t)$. Then, by Remark 2.3(iv), the covariance of the mBm reads

$$R_h(t, s) := \frac{C^2(h_{t,s})}{C(h(t))C(h(s))}\left[\frac{1}{2}\left(|t|^{2h_{t,s}} + |s|^{2h_{t,s}} - |t-s|^{2h_{t,s}}\right)\right], \quad (5.10)$$

where $h_{t,s} := \frac{h(t)+h(s)}{2}$ and $C(\cdot)$ has been given in (1.4).

As described in Lebovits and Lévy-Véhel [27, Sections 4 and 5], the operator M_H in (5.1) is defined on a distribution space, and hence we can not apply the considerations of Elliott and Van Der Hoek [18] about the links between the operator M_H and Riesz potential operator. However, it is crucial for our purpose that M_H is bijective from $L^2_H(\mathbb{R})$ into $L^2(\mathbb{R})$ (Theorem 5.4(i)).

Define the bilinear form $\langle , \rangle_h$ on $\mathcal{E}(\mathbb{R}) \times \mathcal{E}(\mathbb{R})$ by

$$\langle \mathbf{1}[0, t], \mathbf{1}[0, s] \rangle = R_h(t, s).$$

Then by Lebovits and Lévy-Véhel [27, Proposition 4.2], $\langle , \rangle_h$ is an inner product for every function *h*. Define the linear map M_h by

$$M_h : (\mathcal{E}(\mathbb{R}), \langle \ , \ \rangle_h) \to (L^2(\mathbb{R}), \langle \ , \ \rangle_{L^2(\mathbb{R})})$$

$$\mathbf{1}[0, t] \mapsto M_h(\mathbf{1}[0, t]) := M_{h(t)}(\mathbf{1}[0, t]) := M_H(\mathbf{1}[0, t])|_{H = h(t)}. \tag{5.11}$$

Define the process $\widetilde{B}^h(t) = \langle \cdot, M_h(\mathbf{1}[0, t]) \rangle, t \in \mathbb{R}.$ Then, by Kolmogorov's criterion, this process admits a continuous version which will be denoted $B^h(t)$ and called *multifractional Brownian motion* (*mBm*). Then, by Lebovits and Lévy-Véhel [27, Eq. (4.2) and Lemma 4.3], we summarize as follows:

Lemma 5.5. (i) *Almost surely, for every real t,*

$$B_h(t) = \langle \cdot, M_{h(t)}(\mathbf{1}[0, t]) \rangle = B_H(t)|_{H = h(t)}. \tag{5.12}$$

(ii) *The process* $B^h(t)$ *is a normalized mBm.*

(iii) *The map* M_h *is an isometry from* $(\mathcal{E}(\mathbb{R}), \langle , \rangle_h)$ *to* $(L^2(\mathbb{R}), \langle , \rangle_{L^2(\mathbb{R})})$.

Using (5.8) and the equality in Theorem 5.4(ii), we can write as follows: for every real t and almost surely,

$$
\begin{aligned}
B_H(t) &= \langle \cdot, M_H(\mathbf{1}[0, t])\rangle \\
&= \left\langle \cdot, \sum_{k=0}^{\infty} \langle M_H(\mathbf{1}[0, t]), e_k\rangle_{L^2(\mathbb{R})} e_k \right\rangle \\
&= \sum_{k=0}^{\infty} \langle \mathbf{1}[0, t], M_H(e_k)\rangle_{L^2(\mathbb{R})} \langle \cdot, e_k\rangle \\
&= \sum_{k=0}^{\infty} \left(\int_0^t M_H(e_k)(u)du \right) \langle \cdot, e_k\rangle.
\end{aligned}
$$

Thus, by Lebovits and Lévy-Véhél [27, Eq. (5.10)], we can write (5.12) under the following chaos decomposition of mBm:

Almost surely, for every real t,

$$
B_h(t) = \sum_{k=0}^{\infty} \left(\int_0^t M_{h(t)}(e_k)(s)ds \right) \langle \cdot, e_k\rangle. \tag{5.13}
$$

Moreover, by Lebovits and Lévy-Véhél [27, Eq. (5.11)], we can define multifractional white noise $W_h := (W_h(t))_{t\in\mathbb{R}}$ as the $(\mathcal{S})^*$-derivative of B_h, by

$$
W_h(t) := \sum_{k=0}^{\infty} \left[\frac{d}{dt} \left(\int_0^t M_{h(t)}(e_k)(s)ds \right) \right] \langle \cdot, e_k\rangle, \tag{5.14}
$$

assuming that h is differentiable. Then we summarize the result in Lebovits and Lévy-Véhel [27, Theorem-Definition 5.1] as follows:

Theorem 5.6. *Let $h : \mathbb{R} \to (0, 1)$ be a C^1 deterministic function such that its derivative function h' is bounded. Then the process W_h defined by* (5.14) *is an $(\mathcal{S})^*$-process which verifies, in $(\mathcal{S})^*$, the following equality*:

$$W_h(t) = \sum_{k=0}^{\infty} M_{h(t)}(e_k)(t)\langle \cdot, e_k \rangle$$

$$+ h'(t)\sum_{k=0}^{\infty}\left(\int_0^t \frac{\partial M_H}{\partial H}(e_k)(s)|_{H=h(t)}\, ds\right)\langle \cdot, e_k \rangle. \tag{5.15}$$

Moreover, the process B_h *is* $(\mathcal{S})^*$ *-differentiable and verifies*

$$\frac{dB_h}{dt}(t) = W_h(t) \text{ in } (\mathcal{S})^*; \tag{5.16}$$

this process is called multifractional white noise.

When the function h is constant, identically equal to H, we will write $W_H := (W_H(t))_{t \in \mathbb{R}}$ and call the $(\mathcal{S})^*$-process W_H a *fractional white noise*. This process was defined and studied in Elliott and Van Der Hoek [18, Eq. (4.2) and Proof of Theorem A.6] and Biagini et al. [9, Eqs. (3.21) and (3.22)]. By Lebovits et al. [28, Eq. (4.4)], we can rewrite (5.15), for every t, under the form:

$$W_h(t) = W_{h(t)}(t) + h'(t)\frac{\partial B_1}{\partial H}(t, h(t)) \text{ in } (\mathcal{S})^*, \tag{5.17}$$

where $B_1 := (B_1(t, H))_{(t, H) \in \mathbb{R} \times (0,1)}$ is a fractional Gaussian field, defined, for all $(t, H) \in \mathbb{R} \times (0,1)$ and all $\omega \in \Omega$, by $B_1(t, H)(\omega) := B_H(t, \omega) := \langle \omega, M_H(\mathbf{1}[0, t]) \rangle$ and where $W_{h(t)}(t)$ is nothing but $W_H(t)|_{H=h(t)}$.

The function g_f, introduced by Lebovits and Lévy-Véhel [27, Lemma 5.5], plays essential roles in the proof of Theorem 5.6 and properties of the S-transform of mBm (Theorem 6.1 in Section 6); it is instrumental to solve the SDE encountered later:

Lemma 5.7. *For* $H \in (0,1)$ *and* $f \in S(\mathbb{R})$, *define* $g_f : \mathbb{R} \times (0,1) \to \mathbb{R}$ *by*

$$g_f(t, H) := \int_0^t M_H(f)(x)\, dx. \tag{5.18}$$

Then:

(i) *The function* g_f *belongs to* $C^{\infty}(\mathbb{R} \times (0, 1), \mathbb{R})$.

(ii) $\forall x \in \mathbb{R},\ M_H(f)(x) = \alpha_H \int_0^{\infty} u^{H-1/2}(f'(x+u) - f'(x-u))du,$ *where* α_H *has been defined by* (5.7). *In particular, the function* $(x, H) \mapsto M_H(f)(x)$ *is differentiable on* $\mathbb{R} \times (0, 1)$.

(iii) *Assume that* $h : \mathbb{R} \to (0, 1)$ *is differentiable*. *Then, for any real* t_0,

$$\frac{d}{dt}[g_f(t, h(t))]|_{t=t_0} = M_{h(t_0)}(f)(t_0) + h'(t_0)\int_0^{t_0} \frac{\partial M_H}{\partial H}(f)(s)|_{H=h(t_0)}\ ds.$$

6. Stochastic Integral with Respect to mBm

The following theorem due to Lebovits and Lévy-Véhel [27, Theorem 5.12] makes explicit the *S*-transform of mBm, multifractional white noise and generalized functionals of mBm.

We denote by γ the heat kernel density on $\mathbb{R}_+ \times \mathbb{R}$, i.e.,

$$\gamma(t, x) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$$

if $t \neq 0$ and $\gamma(t, x) := 0$ if $t = 0$.

Theorem 6.1. *Let* $h : \mathbb{R} \to (0, 1)$ *be a* C^1 *function and* $(B_h(t))_{t\in\mathbb{R}}$ (*resp.* $(W_h(t))_{t\in\mathbb{R}}$) *be an mBm* (*resp. multifractional white noise*). *Then, for* $\eta \in S(\mathbb{R})$ *and* $t \in \mathbb{R}$, *the following hold*:

(i) $S[B_h(t)](\eta) = \langle \eta, M_h(\mathbf{1}[0, t])\rangle_{L^2(\mathbb{R})} = g_\eta(t, h(t)),$ *where* g_η *has been defined in Lemma* 5.7.

(ii)

$$S[W_h(t)](\eta) = \frac{d}{dt}[g_\eta(t,\ h(t))]$$

$$= M_{h(t)}(\eta)(t) + h'(t)\int_0^t \frac{\partial M_H}{\partial H}(\eta)(s)|_{H=h(t)}\, ds.$$

(iii) *For* $p \in \mathbb{N}$ *and* $F \in S_{-p}(\mathbb{R})$,

$$S[F(B_h(t))](\eta) = \left\langle F,\ \gamma\left(t^{2h(t)},\ \cdot - \int_0^t M_{h(t)}(\eta)(u)du\right)\right\rangle.$$

Proof. By (5.13), Lemma 4.10(ii), Theorem 5.4(ii) and (5.18), assertion (i) is verified as follows:

$$S(B_h(t))(\eta) = \sum_{k=0}^{\infty}\langle M_{h(t)}(\mathbf{1}[0,\ t]),\ e_k\rangle_{L^2(\mathbb{R})}\langle \eta,\ e_k\rangle_{L^2(\mathbb{R})}$$

$$= \langle M_{h(t)}(\mathbf{1}[0,\ t]),\ \eta\rangle_{L^2(\mathbb{R})}$$

$$= \langle \mathbf{1}[0,\ t],\ M_{h(t)}(\eta)\rangle_{L^2(\mathbb{R})} = g_\eta(t,\ h(t)).$$

Equation (5.17) and Theorem 4.14(iii) imply that

$$S[W_h(t)](\eta) = S\left[\frac{dB_h}{dt}(t)\right](\eta) = \frac{d}{dt}S[B_h(t)](\eta) = \frac{d}{dt}g_\eta(t,\ h(t)),$$

and hence assertion (ii) is verified by Lemma 5.7(iii). Further, assertion (iii) is verified by Theorem 7.3 in Kuo [26, p. 63] with $f = M_{h(t)}(\mathbf{1}[0,\ t])$ and by assertion (i) above. Thus, the proof is completed.

Now, we present the *multifractional Wick-Ito integral* with respect to mBm, according to Lebovits and Lévy-Véhel [27, Definition 5.1] and Corlay et al. [15, Definition 5.3].

From here to the last section, we will assume that *h* is in C^1 function on $\mathbb{R}$ with bounded derivative.

Then, recalling the definition of the $(\mathcal{S})^*$-integrability (Definition 4.7), we proceed to the following:

Definition 6.2. Let $Y : \mathbb{R} \to (\mathcal{S})^*$ be a process such that the process $t \mapsto Y(t) \lozenge W_h(t)$ is $(\mathcal{S})^*$-integrable on $\mathbb{R}$. Then the process Y is said to be $d^{\lozenge}B_h$*-integrable* on $\mathbb{R}$ or *integrable* on $\mathbb{R}$ with respect to mBm B_h. The integral of Y with respect to B_h is defined by

$$\int_{\mathbb{R}} Y(s) d^{\lozenge} B_h(s) := \int_{\mathbb{R}} Y(s) \lozenge W_h(s) ds.$$

For an interval I of $\mathbb{R}$, define

$$\int_I Y(s) d^{\lozenge} B_h(s) := \int_{\mathbb{R}} \mathbf{1}_I(s) Y(s) d^{\lozenge} B_h(s).$$

When $h(t) \equiv H \in (0, 1)$, the multifractional Wick-Ito integral coincides with the *fractional Ito integral* as defined in Elliott and Van Der Hoek [18], Biagini et al. [9] and Bender [3, 4]. In the particular case when $h(t) \equiv 1/2$, the multifractional Wick-Ito integral coincides with the classical Ito integral with respect to standard Brownian motion, if Y is Ito integrable.

As shown in Lebovits and Lévy-Véhel [27, Proposition 5.14] and Corlay et al. [15, Proposition 5.4], the multifractional Wick-ito integral satisfies the following properties:

Proposition 6.3. (i) *Let* (a, b) *in* $\mathbb{R}^2$, $a < b$. *Then* $\int_a^b 1 d^{\lozenge} B_h(t) = B_h(b) - B_h(a)$, *almost surely*.

(ii) *Let* $X : I \to (\mathcal{S})^*$ *be a* $d^{\lozenge}B_h$*-integrable process over I, an interval of* $\mathbb{R}$. *Assume that* $\int_I X(s) d^{\lozenge} B_h(s)$ *belongs to* (L^2). *Then*

$$E\left[\int_I X(s) d^{\lozenge} B_h(s)\right] = 0.$$

The assertion (ii) is verified as follows: Let $\int_I X(s) d^{\lozenge} B_h(s)$ be in (L^2). Consider the definition of the S-transform (Definition 3.3 and Definition 4.9), the definition of Wick product $\lozenge$ (Definition 3.11 and Definition 4.11), the interchangeability between the S-transform and the integration (Theorem 4.14(ii)), the S-transform of $W_h(t)$ (Theorem 6.1(ii)), and the property of the function g_f (Lemma 5.7). Then we obtain that

$$\begin{aligned} E\left[\int_I X(s)d^{\Diamond}B_h(s)\right] &= S\left(\int_I X(s)d^{\Diamond}B_h(s)\right)(0) \\ &= S\left(\int_I X(s)\Diamond W_h(s)\,ds\right)(0) \\ &= \int_I S(X(s))(0)S(W_h(s))(0)ds \\ &= \int_I S(X(s))(0)\frac{d}{ds}g_0(s,\, h(s)) \\ &= \int_I S(X(s))(0)\cdot 0 = 0. \end{aligned}$$

Example 6.4. Let $T > 0$ be fixed. Set $I := \int_0^T B_h(t)d^{\Diamond}B_h(t),$ that is,

$$I = \int_0^T W_h(t)\Diamond B_h(t)dt = \int_0^T \frac{dB_h(t)}{dt}\Diamond B_h(t)dt.$$

Define

$$B_h(T)^{\Diamond 2} := B_h(T)\Diamond B_h(T) = B_h(T)^2 - T^{2h(T)};$$

here the last equality follows from Remark 4.13. Then

$$I = \frac{1}{2}B_h(T)^{\Diamond 2} = \frac{1}{2}(B_h(T)^2 - T^{2h(T)}).$$

This is verified as follows: The definition of Wick product $\Diamond$ (Definition 3.11 and Definition 4.11), the interchangeability between the S-transform and the integration (Theorem 4.14(ii)), and the S-transforms of $B_h(t)$ and $W_h(t)$ (Theorem 6.1(i)-(ii)) yield the following: for η in $S(\mathbb{R})$,

$$
\begin{aligned}
& S\left(\int_0^T B_h(t)d^{\Diamond}B_h(t)\right)(\eta) \\
&= \int_0^T S(B_h(t))(\eta)S(W_h(t))(\eta)dt \\
&= \int_0^T g_\eta(t,\ h(t))\frac{d}{dt}[g_\eta(t,\ h(t))]dt \\
&= \frac{1}{2}g_\eta^2(T,\ h(T)) \\
&= \frac{1}{2}(S(B_h(T))(\eta))^2 \\
&= \frac{1}{2}S(B_h(T)\Diamond B_h(T))(\eta) \\
&= S\left(\frac{1}{2}(B_h(T)^2 - T^{2h(T)})\right)(\eta).
\end{aligned}
$$

Thus, the assertion results from the injectivity of the S-transform (Remark 3.4 and Theorem 4.14(i)).

Remark 6.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a deterministic function which belongs to $C^1(\mathbb{R};\ \mathbb{R})$ with bounded derivative. Then the process defined by

$$
Z(t) := \int_0^t f(s)d^{\Diamond}B_h(s)
$$

is Gaussian, and the following integration-by-parts formula holds:

$$
\int_0^t f(s)d^{\Diamond}B_h(s) \overset{(L^2)}{=} f(t)B_h(t) - \int_0^t f'(s)B_h(s)ds, \tag{6.1}
$$

which leads to

$$
\begin{aligned}
\mathbb{E}[Z(t)^2] = {} & f(t)^2t^{2h(t)} + \int_0^t\int_0^t f'(s)f'(u)R_h(s,\ u)dsdu \\
& - 2f(t)\int_0^t f'(s)R_h(t,\ s)ds.
\end{aligned} \tag{6.2}
$$

Equation (6.1) can be derived from the simple Ito formula concerning mBm $B_h(t)$ (Theorem 8.1 in the further Section 8) which is given by Lebovits and Lévy-Véhel [27, Theorem 6.9]. In fact, we apply Theorem 8.1 to the function $F(t, x) := f(t)x$ beforehand. Then, since

$$\frac{\partial F}{\partial t} = f'(t)x, \quad \frac{\partial F}{\partial x} = f(t) \quad \text{and} \quad \frac{\partial^2 F}{\partial x^2} = 0,$$

we can verify (6.1) by the following equalities:

$$\begin{aligned} dF(t, B_h(t)) &= \frac{\partial F}{\partial t}(t, B_h(t))dt + \frac{\partial F}{\partial x}(t, B_h(t))d^{\diamond}B_h(t) \\ &\quad + \frac{1}{2}\left(\frac{d}{dt}[R_h(t, t)]\right)\frac{\partial^2 F}{\partial x^2}(t, B_h(t))dt \\ &= f'(t)B_h(t)dt + f(t)d^{\diamond}B_h(t). \end{aligned}$$

7. SDE Driven by mBm

Let us consider the following *mixed multifractional stochastic differential equation* (*mixed multifractional SDE*):

$$dX(t) = [\alpha_1(t) + \alpha_2(t)X(t)]dt + [\beta_1(t) + \beta_2(t)X(t)]d^{\diamond}Z(t), \quad t \geq 0, \tag{7.1}$$

$$Z(t) := \gamma_1 B_{1/2}(t) + \gamma_2 B_h(t),$$

$$X(0) \in (\mathcal{S})^*.$$

Here γ_1 and γ_2 are positive constants, $B_{1/2}$ is a standard Brownian motion (sBm), i.e., fBm $B_H(t)$ with $H = 1/2$, and $B_h := (B_h(t))_{t\geq 0} := (B_{h(t)})_{t\geq 0}$ is a multifractional Brownian motion (mBm) with Hurst function $h(t)$. Further, α_i and β_i, $i = 1, 2$, are deterministic functions such that

$$\alpha_i, \beta_i : [0, \infty) \to \mathbb{R}, \text{ continuously differentiable.}$$

We call $Z(t)$ *mixed multifractional Brownian motion* (*mixed mBm*). Precisely, equation (7.1) reads as follows:

$$\begin{aligned}
dX(t) &= [\alpha_1(t) + \alpha_2(t)X(t)]dt + \beta_1(t)[\gamma_1 d^{\Diamond}B_{1/2}(t) + \gamma_2 d^{\Diamond}B_h(t)] \\
&\quad + \beta_2(t)X(t)[\gamma_1 d^{\Diamond}B_{1/2}(t) + \gamma_2 d^{\Diamond}B_h(t)] \\
&= [\alpha_1(t) + \alpha_2(t)X(t)]dt \\
&\quad + [\gamma_1\beta_1(t)d^{\Diamond}B_{1/2}(t) + \gamma_2\beta_1(t)d^{\Diamond}B_h(t)] \\
&\quad + [\gamma_1\beta_2(t)X(t)d^{\Diamond}B_{1/2}(t) + \gamma_2\beta_2(t)X(t)d^{\Diamond}B_h(t)], \quad t \geq 0.
\end{aligned}$$

Equation (7.1) can be written by the integral equation:

$$X(t) = X(0) + \int_0^t [\alpha_1(s) + \alpha_2(s)X(s)]ds + \int_0^t [\beta_1(s) + \beta_2(s)X(s)]d^{\Diamond}Z(s),$$

where the equality holds in $(\mathcal{S})^*$. Rewriting the equation in terms of derivatives in $(\mathcal{S})^*$, we get the equation:

$$\frac{dX(t)}{dt} = [\alpha_1(t) + \alpha_2(t)X(t)] + [\beta_1(t) + \beta_2(t)X(t)] \Diamond \frac{dZ(t)}{dt}, \quad t \geq 0. \quad (7.2)$$

If we notice useful properties of Wick product, which are summarized in Remark 7.4 at the end of this section, equation (7.2) reads as follows:

$$\begin{aligned}
\frac{dX(t)}{dt} &= [\alpha_1(t) + \alpha_2(t)X(t)] + [\beta_1(t) + \beta_2(t)X(t)] \Diamond (\gamma_1 W_{1/2}(t) + \gamma_2 W_h(t)) \\
&= [\alpha_1(t) + \alpha_2(t)X(t)] + \beta_1(t) \Diamond (\gamma_1 W_{1/2}(t) + \gamma_2 W_h(t)) \\
&\quad + \beta_2(t)X(t) \Diamond (\gamma_1 W_{1/2}(t) + \gamma_2 W_h(t)) \\
&= [\alpha_1(t) + \alpha_2(t)X(t)] + \gamma_1\beta_1(t) \Diamond W_{1/2}(t) + \gamma_2\beta_1(t) \Diamond W_h(t) \\
&\quad + \gamma_1\beta_2(t)X(t) \Diamond W_{1/2}(t) + \gamma_2\beta_2(t)X(t) \Diamond W_h(t) \\
&= [\alpha_1(t) + \alpha_2(t)X(t)] + \gamma_1\beta_1(t) \cdot W_{1/2}(t) + \gamma_2\beta_1(t) \cdot W_h(t)
\end{aligned}$$

$$+\gamma_1\beta_2(t)X(t)\Diamond W_{1/2}(t)+\gamma_2\beta_2(t)X(t)\Diamond W_h(t),\quad t\ge 0,$$

$$X(0)\in(\mathcal{S})^*.$$

Here $W_h := \dfrac{dB_h}{dt}(t)$ is the multifractional white noise defined by (5.14)-(5.17) $(W_{1/2}$ is the standard white noise as derived from $W_h|_{h(t)\equiv 1/2})$. In the last equality of equations above, since $\beta_i(t)\gamma_j,\ i,\ j=1,\ 2,$ are deterministic, we used the property such that the Wick product $\Diamond$ coincides with the ordinary product (see Remark 7.4(ii) at the end of this section):

$$\gamma_1\beta_1(t)\Diamond W_{1/2}(t)+\gamma_2\beta_1(t)\Diamond W_h(t)=\gamma_1\beta_1(t)\cdot W_{1/2}(t)+\gamma_2\beta_1(t)\cdot W_h(t).$$

Theorem 7.1. *The* $(\mathcal{S})^*$*-process* $(X(t))_{t\ge 0}$ *defined by*

$$X(t)=\exp^{\Diamond}(A(t))\Diamond\left\{\int_0^t \widetilde{A}(s)\Diamond\exp^{\Diamond}(-A(s))ds+X(0)\right\} \tag{7.3}$$

is the unique solution of SDE (7.1) *in* $(\mathcal{S})^*$. *Here*

$$A(t):=\int_0^t \alpha_2(s)ds+\int_0^t \beta_2(s)d^{\Diamond}Z(s), \tag{7.4}$$

that is,

$$A(t)=\int_0^t \alpha_2(s)ds+\int_0^t \beta_2(s)(\gamma_1 d^{\Diamond}B_{1/2}(s)+\gamma_2 d^{\Diamond}B_h(s)),$$

$$=\int_0^t \alpha_2(s)ds+\gamma_1\int_0^t \beta_2(s)d^{\Diamond}B_{1/2}(s)+\gamma_2\int_0^t \beta_2(s)d^{\Diamond}B_h(s).$$

Further,

$$\widetilde{A}(t):=\alpha_1(t)+\beta_1(t)(\gamma_1 W_{1/2}(t)+\gamma_2 W_h(t)), \tag{7.5}$$

that is,

$$\widetilde{A}(t)=\alpha_1(t)+\beta_1(t)\Diamond(\gamma_1 W_{1/2}(t)+\gamma_2 W_h(t)),$$

since $\beta_i(t)\gamma_j,\ i,\ j=1,\ 2,$ *are deterministic* (*see Remark* 7.4(ii)).

Proof. We notice that for fixed $\eta \in S(\mathbb{R})$, the S-transform $S[\Phi](\eta)$ is a linear functional of $\Phi \in (\mathcal{S})^*$:

$$S[a\Phi + b\Psi](\eta) = aS[\Phi](\eta) + bS[\Psi](\eta), \quad a, b \in \mathbb{R}$$

(Definition 3.3, Definition 4.9).

Our proof is proceeded by using the following properties of the S-transform: for every $\eta \in S(\mathbb{R})$,

$$S[\Phi \diamond \Psi](\eta) = S[\Phi](\eta)S[\Psi](\eta) \text{ (Definition 3.11, Definition 4.11),}$$

$$S[\Phi](\eta) = S[\Psi](\eta) \Rightarrow \Phi = \Psi \text{ (Remark 3.4, Theorem 4.14(i)),}$$

$$S\left[\int_{\mathbb{R}} \Phi(u)du\right](\eta) = \int_{\mathbb{R}} S[\Phi(u)](\eta)du \text{ (Theorem 4.14(ii)),}$$

$$\frac{d}{dt} S[X(t)](\eta) = S\left[\frac{dX(t)}{dt}\right](\eta) \text{ (Theorem 4.14(iii)),}$$

$$S[W_h(t)](\eta) = \frac{d}{dt} g_\eta(t, h(t)) \text{ (Lemma 5.7 and Theorem 6.1(ii)),}$$

$$S[W_{1/2}(t)](\eta) = M_{1/2}(\eta)(t) \text{ (Theorem 6.1(ii) with } h(t) \equiv 1/2).$$

First of all, applying the S-transform to both sides of equation (7.2) and denoting by y_η the map $t \mapsto S[X(t)](\eta)$ for every $\eta \in S(\mathbb{R})$, we get the following:

$$\begin{aligned} y'_\eta(t) &= [\alpha_1(t) + \alpha_2(t) y_\eta(t)] \\ &\quad + [\beta_1(t) + \beta_2(t) y_\eta(t)]\left(\gamma_1 M_{1/2}(\eta)(t) + \gamma_2 \frac{d}{dt}[g_\eta(t, h(t))]\right) \\ &= \left[\alpha_2(t) + \beta_2(t)\left(\gamma_1 M_{1/2}(\eta)(t) + \gamma_2 \frac{d}{dt}[g_\eta(t, h(t))]\right)\right] y_\eta(t) \\ &\quad + \left[\alpha_1(t) + \beta_1(t)\left(\gamma_1 M_{1/2}(\eta)(t) + \gamma_2 \frac{d}{dt}[g_\eta(t, h(t))]\right)\right]. \end{aligned}$$

Define

$$P(t) := \alpha_2(t) + \beta_2(t)\left(\gamma_1 M_{1/2}(\eta)(t) + \gamma_2 \frac{d}{dt}[g_\eta(t, h(t))]\right),$$

$$Q(t) := \alpha_1(t) + \beta_1(t)\left(\gamma_1 M_{1/2}(\eta)(t) + \gamma_2 \frac{d}{dt}[g_\eta(t, h(t))]\right).$$

Then we have the linear differential equation of the first-order:

$$y'_\eta(t) = P(t)y_\eta(t) + Q(t), \quad y_\eta(0) = S[X(0)](\eta), \tag{7.6}$$

and hence

$$y_\eta(t) = e^{\int_0^t P(s)\,ds}\left\{\int_0^t Q(s)e^{-\int_0^s P(u)\,du}ds + y_\eta(0)\right\}. \tag{7.7}$$

We shall find an explicit expression for $y_\eta(t)$ in the following steps:

Step 1. Recalling the linearity of $S[\Phi](\cdot)$ with respect to Φ, we rewrite $Q(t)$ as follows:

$$\begin{aligned} Q(t) &= \alpha_1(t) + \beta_1(t)(\gamma_1 S[W_{1/2}(t)](\eta) + \gamma_2 S[W_h(t)](\eta)) \\ &= S[\alpha_1(t) + \beta_1(t)(\gamma_1 W_{1/2}(t) + \gamma_2 W_h(t))](\eta) \\ &= S[\tilde{A}(t)](\eta) \end{aligned} \tag{7.8}$$

with $\tilde{A}(t)$ as given by (7.5). In equations above, by Remark 3.6 and Definition 4.9, we used the property such that $S[\alpha_i(t)](\eta) = \alpha_i(t)$, $i = 1, 2$; notice that $\alpha_i(t)$ are deterministic and that $E[: e^{I(\eta)} :] = 1$ for $I(\eta) = \int_{\mathbb{R}} \eta(s)\,dB(s)$.

Step 2. Recalling the linearity of $S[\Phi](\cdot)$ with respect to Φ, again, we rewrite $P(t)$ as follows:

$$\begin{aligned} P(t) &= \alpha_2(t) + \beta_2(t)(\gamma_1 S[W_{1/2}(t)](\eta) + \gamma_2 S[W_h(t)](\eta)) \\ &= S[\alpha_2(t) + \beta_2(t)(\gamma_1 W_{1/2}(t) + \gamma_2 W_h(t))](\eta). \end{aligned}$$

Therefore, the interchangeability between the S-transform and the integration (Theorem 4.14(ii)) yields the following:

$$\begin{aligned}
\int_0^t P(s)ds &= \int_0^t S[\alpha_2(s) + \beta_2(s)(\gamma_1 W_{1/2}(s) + \gamma_2 W_h(s)](\eta)ds \\
&= S\left[\int_0^t \{\alpha_2(s) + \beta_2(s)(\gamma_1 W_{1/2}(s) + \gamma_2 W_h(s))\} ds\right](\eta) \\
&= S\left[\int_0^t \alpha_2(s)ds + \int_0^t \beta_2(s)(\gamma_1 W_{1/2}(s) + \gamma_2 W_h(s))ds\right](\eta) \\
&= S\left[\int_0^t \alpha_2(s)ds + \int_0^t \beta_2(s)\lozenge(\gamma_1 W_{1/2}(s) + \gamma_2 W_h(s))ds\right](\eta) \\
&= S\left[\int_0^t \alpha_2(s)ds + \int_0^t \beta_2(s)(\gamma_1 d^{\lozenge} B_{1/2}(s) + \gamma_2 d^{\lozenge} B_h(s))\right](\eta) \\
&= S\left[\int_0^t \alpha_2(s)ds + \int_0^t \beta_2(s)d^{\lozenge} Z(s)\right](\eta) \\
&= S[A(t)](\eta) \qquad (7.9)
\end{aligned}$$

with the function $A(t)$ as given by (7.4). In the fourth equality of equations above, we used the property that if $\beta_i(t)\gamma_j,\ i, j = 1, 2,$ are deterministic, then the Wick product $\lozenge$ coincides with the ordinary product (see Remark 7.4(ii)):

$$\beta_2(s)\lozenge(\gamma_1 W_{1/2}(s) + \gamma_2 W_h(s)) = \beta_2(s)(\gamma_1 W_{1/2}(s) + \gamma_2 W_h(s)).$$

Step 3. Equation (7.9) implies the following:

$$\begin{aligned}
e^{\int_0^t P(s)ds} &= e^{S[A(t)](\eta)} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}(S[A(t)](\eta))^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} S[A(t)^{\lozenge k}](\eta)
\end{aligned}$$

$$= S\left[\sum_{k=0}^{\infty}\frac{1}{k!}A(t)^{\Diamond k}\right](\eta)$$

$$= S[\exp^{\Diamond}(A(t))](\eta). \tag{7.10}$$

Here, for $k \geq 1$, $\Phi^{\Diamond k}$ denotes the kth Wick power of $\Phi \in (\mathcal{S})^*$ and $\exp^{\Diamond}(\Phi)$ the Wick exponential (Definition 4.12). By the same argument as taken in (7.9) and (7.10), the linearity of $S(\Phi)$ with respect to $\Phi \in (\mathcal{S})^*$ yields that for $\eta \in S(\mathbb{R})$, $-\int_0^t P(s)ds = S[-A(t)](\eta)$ and

$$e^{-\int_0^t P(s)ds} = S[\exp^{\Diamond}(-A(t))](\eta). \tag{7.11}$$

Thus, considering (7.8) and (7.11), by the interchangeability between the S-transform and the integration, we get the following:

$$\int_0^t Q(s)e^{-\int_0^s P(u)du}ds = \int_0^t S[\tilde{A}(s)](\eta)S[\exp^{\Diamond}(-A(s))](\eta)ds$$

$$= \int_0^t S[\tilde{A}(s)\Diamond\exp^{\Diamond}(-A(s))](\eta)ds$$

$$= S\left[\int_0^t \tilde{A}(s)\Diamond\exp^{\Diamond}(-A(s))ds\right](\eta). \tag{7.12}$$

Therefore, by (7.7), (7.10) and (7.12), we obtain the expression for $y_\eta(t)$:

$$y_\eta(t) = e^{\int_0^t P(s)ds}\left\{\int_0^t Q(s)e^{-\int_0^s P(u)du}ds + y_\eta(0)\right\}$$

$$= S[\exp^{\Diamond}(A(t))](\eta)\left\{S\left[\int_0^t \tilde{A}(s)\Diamond\exp^{\Diamond}(-A(s))ds\right](\eta) + S[X(0)](\eta)\right\}$$

$$= S\left[\exp^{\Diamond}(A(t))\Diamond\left\{\int_0^t \tilde{A}(s)\Diamond\exp^{\Diamond}(-A(s))ds + X(0)\right\}\right](\eta).$$

By the injectivity of the S-transform defined by $X(t) \mapsto y_\eta(t) = S(X(t))(\eta)$ for every $\eta \in S(\mathbb{R})$, we conclude the required expression (7.3) for $X(t)$. The uniqueness of the solution of (7.1) follows from the uniqueness of the solution of the linear differential equation (7.6) and the injectivity of the S-transform. Hence, the proof is completed.

Example 7.2. Consider the equation:

$$\begin{aligned} dX(t) &= X(t)d^{\Diamond}Z(t) \\ &= X(t)(\gamma_1 d^{\Diamond}B_{1/2}(t) + \gamma_2 d^{\Diamond}B_h(t)), \qquad (7.13) \\ X(0) &= x_0 \in \mathbb{R}. \end{aligned}$$

This is the mixed multifractional SDE (7.1) with coefficients $\alpha_1(t) = \alpha_2(t) = \beta_1(t) \equiv 0$ and $\beta_2(t) \equiv 1$. Then, by (7.4) and (7.5), we notice that $A(t) = \gamma_1 B_{1/2}(t) + \gamma_2 B_h(t)$ and $\tilde{A}(t) \equiv 0$. Moreover, in order to rewrite (7.3), we notice the distributive law of the operation $\Diamond$ (Remark 7.4(i)) and consider that $X(0) = x_0 \in \mathbb{R}$, that is, $X(0)$ is deterministic. Then we can replace the Wick product by the ordinary product, and hence $\exp^{\Diamond}(A(t)) \Diamond X(0) = x_0 \exp^{\Diamond}(A(t))$. Thus, by (7.3), we obtain the expression for the $(\mathcal{S})^*$-process X as

$$X(t) = x_0 \exp^{\Diamond}(\gamma_1 B_{1/2}(t) + \gamma_2 B_h(t)). \qquad (7.14)$$

This $X(t)$ is called *geometric mixed multifractional Brownian motion* (*geometric mixed mBm*).

Remark 7.3. Using Janson [23, equality 3.16], we see that X in (7.14) is an (L^2)-valued process that may be represented as

$$X(t) = x_0 \exp\left(\gamma_1 B_{1/2}(t) + \gamma_2 B_h(t) - \frac{1}{2}(\gamma_1^2 t + \gamma_2^2 t^{2h(t)})\right). \qquad (7.15)$$

Example 7.2 is the same as Corlay et al. [15, Theorem 5.7]; it is also a consequence of Holden et al. [20, Theorem 3.1.2].

For simplicity of consideration, we shall verify (7.15) in the particular case of $\gamma_1 = 0$: Taking the S-transforms of both sides of (7.14) with $\gamma_1 = 0$, we have that for $\eta \in S(\mathbb{R})$,

$$\begin{aligned} S[X(t)](\eta) &= x_0 S\left[\sum_{k=0}^{\infty} \frac{1}{k!}(\gamma_2 B_h(t))^{\Diamond k}\right](\eta) \\ &= x_0 \sum_{k=0}^{\infty} \frac{1}{k!}[S(\gamma_2 B_h(t))(\eta)]^k \\ &= x_0 \sum_{k=0}^{\infty} \frac{1}{k!}[\gamma_2 g_\eta(t, h(t))]^k \\ &= x_0 \exp(\gamma_2 g_\eta(t, h(t))). \end{aligned} \tag{7.16}$$

Here we used the property that $S[B_h(t)](\eta) = g_\eta(t, h(t))$ (Theorem 6.1(i)). On the other hand, since $B_h(t) = B_H(t)|_{H=h(t)}$ a.s., it is easy to see (in view of Bender [3, p. 978] and Lebovits and Lévy-Véhel [27, lines 4-6, p. 34, Proof of Theorem 6.9]) that $B_h(t)$ is a Gaussian variable with mean equal to $\int_0^t M_{h(t)}(\eta)(u)du = g_\eta(t, h(t))$ and variance equal to $t^{2h(t)}$ under the probability measure Q_η; Q_η is defined by (3.7) such that $\widetilde{B}(t) := B_{1/2}(t) - \int_0^t \eta(s)ds$, $\eta \in S(\mathbb{R})$, is a (two-sided) Brownian motion under the measure Q_η. Recall the S-transform as rewritten by (3.8), i.e., $(S\Phi)(\eta) = E^{Q_\eta}[\Phi]$, $\Phi \in (L^2)$. Notice that the right-hand side of (7.15) with $\gamma_1 = 0$ is the (L^2)-valued process. Then, taking the S-transform and considering the moment generating function for $B_h(t)$, we get that for $\eta \in S(\mathbb{R})$,

$$S[X(t)](\eta) = x_0 E^{Q_\eta}\left[\exp\left(\gamma_2 B_h(t) - \frac{1}{2}\gamma_2^2 t^{2h(t)}\right)\right]$$

$$= x_0 \exp\left(\gamma_2 g_\eta(t, h(t)) + \frac{1}{2}\gamma_2^2 t^{2h(t)}\right)\exp\left(-\frac{1}{2}\gamma_2^2 t^{2h(t)}\right)$$

$$= x_0 \exp(\gamma_2 g_\eta(t, h(t))). \tag{7.17}$$

Equation (7.17) coincides with (7.16). Therefore, the injectivity of the S-transform implies the required expression (7.15) when $\gamma_1 = 0$.

Example 7.3. Consider the equation:

$$dU(t) = \alpha(m - U(t))dt + d^\Diamond Z(t)$$

$$= \alpha(m - U(t))dt + (\gamma_1 d^\Diamond B_{1/2}(t) + \gamma_2 d^\Diamond B_h(t)), \tag{7.18}$$

$$U(0) = u_0 \in \mathbb{R}.$$

Here $(B_{1/2}(t))_{t\in\mathbb{R}}$ and $(B_h(t))_{t\in\mathbb{R}}$ are assumed to be independent. This is the mixed multifractional SDE (7.1) with coefficients $\alpha_1(t) \equiv \alpha m$, $\alpha_2(t) \equiv -\alpha$, $\beta_1(t) \equiv 1$ and $\beta_2(t) \equiv 0$, where α and m are constants, and $\alpha > 0$. Then, by Theorem 7.1, we obtain the unique solution $U(t)$ as

$$U(t) = u_0 e^{-\alpha t} + m(1 - e^{-\alpha t})$$

$$+ \gamma_1 \int_0^t e^{-\alpha(t-s)} d^\Diamond B_{1/2}(s) + \gamma_2 \int_0^t e^{\alpha(t-s)} d^\Diamond B_h(s). \tag{7.19}$$

This $U(t)$ is called *mixed multifractional Ornstein-Uhlenbeck process* (*mixed mOU process*). In fact, by (7.4) and (7.5), we observe that $A(t) = -\alpha t$ and $\tilde{A}(t) = \alpha m + (\gamma_1 W_{1/2}(t) + \gamma_2 W_h(t))$. Therefore, by (7.3), we get the following:

$$U(t) = \exp^\Diamond(-\alpha t) \Diamond \left\{\int_0^t (\alpha m + \gamma_1 W_{1/2}(s) + \gamma_2 W_h(s)) \Diamond \exp^\Diamond(\alpha s) ds + u_0\right\}$$

$$= \exp(-\alpha t)\left\{\int_0^t \exp(\alpha s)(\alpha m + \gamma_1 W_{1/2}(s) + \gamma_2 W_h(s))ds + u_0\right\}$$

$$= e^{-\alpha t}\left\{\int_0^t e^{\alpha s}(\alpha m)ds + \gamma_1 \int_0^t e^{\alpha s} \lozenge W_{1/2}(s)ds + \gamma_2 \int_0^t e^{\alpha s} \lozenge W_h(s)ds + u_0\right\}$$

$$= e^{-\alpha t}\left\{m(e^{\alpha t} - 1) + \gamma_1 \int_0^t e^{\alpha s} d^{\lozenge} B_{1/2}(s) + \gamma_2 \int_0^t e^{\alpha s} d^{\lozenge} B_h(s) + u_0\right\}. \quad (7.20)$$

Hence, expression (7.19) holds. In the third equality of equations above, we used the property such that the Wick product coincides with the ordinary product, since α and γ_j, $j = 1, 2,$ are deterministic (see Remark 7.4(ii) below).

Example 7.3 is the same as Corlay et al. [15, Theorem 5.9].

At the end of this section, according to Holden et al. [20, Section 2.4 and Chapter 3], we summarize familiar properties of Wick product as the following remark; these play roles in the proofs of Sections 7 and 8.

Remark 7.4. (i) In Wick product, commutative law, associative law and distributive law hold:

$$F \lozenge G = G \lozenge F,$$

$$F \lozenge (G \lozenge H) = (F \lozenge G) \lozenge H,$$

$$F \lozenge (A + B) = F \lozenge A + F \lozenge B.$$

(ii) If at least one of F and G is deterministic, e.g., $F = a_0 \in \mathbb{R}$, then the Wick product coincides with the ordinary product in the deterministic case:

$$F \lozenge G = F \cdot G, \text{ in particular, if } F = 0, \text{ then } F \lozenge G = 0.$$

(iii) When applied to ordinary stochastic differential equations, derivative product rule holds as in the case of ordinary calculus:

$$\frac{d}{dt}(U \lozenge V) = \frac{dU}{dt} \lozenge V + U \lozenge \frac{dV}{dt}.$$

These are verified by taking the S-transforms of both sides of the equality and considering the injectivity of the S-transform.

8. Ito Formula

By the S-transform approach, Lebovits and Lévy-Véhel [27, Theorem 6.9] showed the simple Ito formula in (L^2) for $C^{1,2}$ functions with sub-exponential growth as follows:

Theorem 8.1. *Let $T > 0$ and $h : \mathbb{R} \to (0, 1)$ be a C^1 function such that h' is bounded on $\mathbb{R}$. Let f be a $C^{1,2}([0, T] \times \mathbb{R}, \mathbb{R})$ function. Furthermore, assume that there are constants $C \geq 0$ and $\lambda < 1/(4 \max_{t \in [0,T]} t^{2h(t)})$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$,*

$$\max_{t \in [0,T]} \left\{ | f(t, x) |, \left| \frac{\partial f}{\partial t}(t, x) \right|, \left| \frac{\partial f}{\partial x}(t, x) \right|, \left| \frac{\partial^2 f}{\partial x^2}(t, x) \right| \right\} \leq C e^{\lambda x^2}.$$

Then, for all $t \in [0, T]$, the following holds in (L^2):

$$f(T, B_h(T)) = f(0, 0) + \int_0^T \frac{\partial f}{\partial t}(t, B_h(t)) dt + \int_0^T \frac{\partial f}{\partial x}(t, B_h(t)) d^{\Diamond} B_h(t)$$
$$+ \frac{1}{2} \int_0^T \left(\frac{d}{dt}[R_h(t, t)] \right) \frac{\partial^2 f}{\partial x^2}(t, B_h(t)) dt. \tag{8.1}$$

Here $R_h(t, s)$ denotes the covariance function of mBm B_h, that is, $R_h(t, s) = E[B_h(t) B_h(s)]$ (see Remark 2.3(iv) and (5.10)); we observe that

$$\frac{d}{dt}[R_h(t, t)] = 2t^{2h(t)-1}(h'(t) t \log t + h(t)) = \frac{d}{dt}[t^{2h(t)}].$$

The simple Ito formula (8.1) can be rewritten by the stochastic differentials:

$$df(t, B_h(t)) = \frac{\partial f}{\partial t}(t, B_h(t)) dt + \frac{\partial f}{\partial x}(t, B_h(t)) d^{\Diamond} B_h(t)$$
$$+ \frac{1}{2} \left(\frac{d}{dt}[R_h(t, t)] \right) \frac{\partial^2 f}{\partial x^2}(t, B_h(t)) dt.$$

Proposition 8.2. *Let* $h : \mathbb{R} \to (0, 1)$ *be a* C^1 *function such that* h' *is bounded on* $\mathbb{R}$. *Consider the equation*:

$$dX(t) = \mu X(t)dt + \sigma X(t)d^{\Diamond}B_h(t), \quad t \geq 0,$$
$$X(0) = x_0 \in \mathbb{R}, \tag{8.2}$$

where μ *and* $\sigma > 0$ *are constants*. *Recall that* (8.2) *can be rewritten in terms of derivatives in* $(\mathcal{S})^*$:

$$\frac{dX}{dt}(t) = \mu X(t) + \sigma X(t) \Diamond W_h(t) = (\mu + \sigma W_h(t)) \Diamond X(t),$$
$$X(0) = x_0 \in \mathbb{R}.$$

Then the unique solution of (8.2) *is given by*

$$X(t) = x_0 \exp^{\Diamond}(\mu t + \sigma B_h(t)), \quad t \geq 0. \tag{8.3}$$

This $X(t)$ *is called multifractional Wick exponential or geometric multifractional Brownian motion* (*geometric mBm*). *Further*, $X(t)$ *is an* (L^2) *-valued process with expression such that*

$$X(t) = x_0 \exp\left(\mu t + \sigma B_h(t) - \frac{1}{2}\sigma^2 t^{2h(t)}\right), \quad t \geq 0. \tag{8.4}$$

This is analogous to the formula in Biagini et al. [9, (3.31)] *and Elliott and Van Der Hoek* [18, (4.9)] *where the case of the constant Hurst parameter* $H \in (0, 1)$ *is discussed*.

Proof. Equation (8.2) is the special case of SDE (7.1) with coefficients such that

$$\gamma_1 = 0, \; \gamma_2 = 1,$$
$$\alpha_1(t) \equiv 0, \; \alpha_2(t) \equiv \mu, \; \beta_1(t) \equiv 0, \; \beta_2(t) \equiv \sigma.$$

Equations (7.4) and (7.5) in Theorem 7.1 imply that $A(t) = \mu t + \sigma B_h(t)$ and

$\widetilde{A}(t) \equiv 0;$ by (6.1) in Remark 6.5, observe that $\int_0^t \sigma d^{\lozenge} B_h(t) = \sigma B_h(t).$ Hence, (8.3) follows from (7.3). The uniqueness is guaranteed by Theorem 7.1. The expression (8.4) for the solution $X(t)$ can be verified as follows:

1. Proof by the *S*-transform. We first notice the Wick exponential defined by the Wick power (Definition 4.12). Then, by the *S*-transform, we get the following: for $\eta \in S(\mathbb{R})$,

$$S[\exp^{\lozenge}(B_h(t))](\eta) = \sum_{k=0}^{\infty} \frac{1}{k!} S[B_h(t)^{\lozenge k}](\eta)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (S[B_h(t)](\eta))^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (g_\eta(t, h(t)))^k = \exp(g_\eta(t, h(t))),$$

since $S[B_h(t)](\eta) = g_\eta(t, h(t))$ (Theorem 6.1(i)). Thus, taking the *S*-transforms of the both sides of (8.3), we get that for $\eta \in S(\mathbb{R})$,

$$S[X(t)](\eta) = x_0 \exp(\mu t + \sigma g_\eta(t, h(t))).$$

On the other hand, for $\eta \in S(\mathbb{R})$, let Q_η be the probability measure as defined by (3.8). Recall the *S*-transform under Q_η, i.e., $(S\Phi)(\eta) = E^{Q_\eta}[\Phi]$, $\Phi \in (L^2)$. Further, notice that $B_h(t)$ is a Gaussian variable with mean equal to $g_\eta(t, h(t))$ and variance equal to $t^{2h(t)}$ under Q_η. Take the *S*-transforms of the both sides of (8.4) under Q_η and consider the moment generation function for $B_h(t)$. Then, by the same argument as in the proof of (7.17), we get the following: for $\eta \in S(\mathbb{R})$,

$$S[X(t)](\eta) = x_0 E^{Q_\eta}\left[\exp\left(\mu t + \sigma B_h(t) - \frac{1}{2}\sigma^2 t^{2h(t)}\right)\right]$$

$$= x_0 \exp\left(\mu t + \sigma g_h(t, h(t)) + \frac{1}{2}\sigma^2 t^{2h(t)} - \frac{1}{2}\sigma^2 t^{2h(t)}\right)$$

$$= x_0 \exp(\mu t + \sigma g_h(t, h(t))).$$

Therefore, the injectivity of the *S*-transform implies that $X(t)$ of the form (8.3) has the expression (8.4).

2. Proof by the Ito formula. Define the function $f(t, x)$ by

$$f(t, x) := x_0 \exp\left(\mu t + \sigma x - \frac{1}{2}\sigma^2 t^{2h(t)}\right).$$

Let $R_h(t, t)$ be the covariance function of mBm $B_h(t)$. Then, since

$$\frac{d}{dt}[R_h(t, t)] = t^{2h(t)}(2h'(t)\log t + 2h(t)/t) = \frac{d}{dt}[t^{2h(t)}],$$

we have

$$\frac{\partial f}{\partial t} = f(t, x)\left\{\mu - \frac{1}{2}\sigma^2 \frac{d}{dt}[R_h(t, t)]\right\}.$$

Further, we have

$$\frac{\partial f}{\partial x} = f(t, x)\sigma, \ \frac{\partial^2 f}{\partial x^2} = f(t, x)\sigma^2.$$

Hence, Theorem 8.1 on Ito formula applied to $f(t, B_h(t))$ yields

$$df(t, B_h(t)) = f(t, B_h(t))\left\{\mu - \frac{1}{2}\sigma^2 \frac{d}{dt}[R_h(t, t)]\right\}dt + f(t, B_h(t))\sigma d^{\diamond} B_h(t)$$

$$+ \frac{1}{2}\frac{d}{dt}[R_h(t, t)]\, f(t, B_h(t))\sigma^2 dt$$

$$= \mu f(t, B_h(t))dt + \sigma f(t, B_h(t))d^{\diamond} B_h(t).$$

Thus, $f(t, B_h(t))$ satisfies SDE (8.2). By the uniqueness of the solution of (8.2), we obtain that $X(t) = f(t, B_h(t))$, completing the proof.

The following Ito formula for geometric mBm is a consequence of Theorem 8.1.

Theorem 8.3. *Let* $T > 0$ *and* $h : \mathbb{R} \to (0, 1)$ *be a* C^1 *function such that* h' *is bounded on* $\mathbb{R}$. *Consider SDE* (8.2). *Let* F *be a* $C^{1,2}([0, T] \times \mathbb{R}, \mathbb{R})$ *function with the sub-exponential growth as given in Theorem* 8.1. *Then, for all* $t \in [0, T]$, *the following equality holds in* (L^2):

$$dF(t, X(t)) = \left\{ \frac{\partial F}{\partial t}(t, X(t)) + \frac{\partial F}{\partial X}(t, X(t)) X(t) \mu \right\} dt$$
$$+ \frac{\partial F}{\partial X}(t, X(t)) X(t) \sigma d^{\diamond} B_h(t)$$
$$+ \frac{1}{2} \left(\frac{d}{dt} [R_h(t, t)] \right) \frac{\partial^2 F}{\partial X^2}(t, X(t)) X^2(t) \sigma^2 dt. \tag{8.5}$$

If we define

$$d^{\diamond} X(t) := \mu X(t) dt + \sigma X(t) d^{\diamond} B_h(t),$$

then (8.5) *can be simplified as*

$$dF(t, X(t)) = \frac{\partial F}{\partial t}(t, X(t)) dt + \frac{\partial F}{\partial X}(t, X(t)) d^{\diamond} X(t)$$
$$+ \frac{1}{2} \left(\frac{d}{dt} [R_h(t, t)] \right) \frac{\partial^2 F}{\partial X^2}(t, X(t)) X^2(t) \sigma^2 dt.$$

Proof. For $t \geq 0$ and $x \in \mathbb{R}$, define

$$f(t, x) := x_0 \exp\left(\mu t + \sigma x - \frac{1}{2} \sigma^2 t^{2h(t)} \right).$$

Set

$$X := f(t, x), \quad G(t, x) := F(t, f(t, x)) = F(t, X).$$

Then

$$\frac{\partial G}{\partial t}(t, x) = \frac{\partial F}{\partial t}(t, X) + \frac{\partial F}{\partial X}(t, X) \frac{\partial f}{\partial t}(t, x),$$

$$\frac{\partial G}{\partial x}(t, x) = \frac{\partial F}{\partial X}(t, X)\frac{\partial f}{\partial x}(t, x),$$

$$\frac{\partial^2 G}{\partial x^2}(t, x) = \frac{\partial^2 F}{\partial X^2}(t, X)\left(\frac{\partial f}{\partial x}\right)^2(t, x) + \frac{\partial F}{\partial X}(t, X)\frac{\partial^2 f}{\partial x^2}(t, x).$$

Let $X(t)$ be the solution of SDE (8.2). Then, by (8.4) of Proposition 8.2, we have

$$X(t) = f(t, B_h(t)), \; G(t, B_h(t)) = F(t, f(t, B_h(t))) = F(t, X(t)).$$

Further, by applying Theorem 8.1 to $G(t, B_h(t))$, we get the following:

$$\begin{aligned}
dG(t, B_h(t)) &= \frac{\partial G}{\partial t}(t, B_h(t))dt + \frac{\partial G}{\partial x}(t, B_h(t))d^{\Diamond}B_h(t) \\
&\quad + \frac{1}{2}\left(\frac{d}{dt}[R_h(t, t)]\right)\frac{\partial^2 G}{\partial x^2}(t, B_h(t))dt \\
&= \left\{\frac{\partial F}{\partial t}(t, X(t)) + \frac{\partial F}{\partial X}(t, X(t))\frac{\partial f}{\partial t}(t, B_h(t))\right\}dt \\
&\quad + \frac{\partial F}{\partial X}(t, X(t))\frac{\partial f}{\partial x}(t, B_h(t))d^{\Diamond}B_h(t) \\
&\quad + \frac{1}{2}\left(\frac{d}{dt}[R_h(t, t)]\right)\left\{\frac{\partial^2 F}{\partial X^2}(t, X(t))\left(\frac{\partial f}{\partial x}\right)^2(t, B_h(t))\right. \\
&\quad \left. + \frac{\partial F}{\partial X}(t, X(t))\frac{\partial^2 f}{\partial x^2}(t, B_h(t))\right\}dt. \qquad (8.6)
\end{aligned}$$

Substitute the following equations to (8.6):

$$\frac{\partial f}{\partial t}(t, x) = f(t, x)\left(\mu - \frac{1}{2}\sigma^2\frac{d}{dt}[R_h(t, t)]\right),$$

$$\frac{\partial f}{\partial x}(t, x) = f(t, x)\sigma, \; \frac{\partial^2 f}{\partial x^2}(t, x) = f(t, x)\sigma^2.$$

Then we get the following:

$$
\begin{aligned}
& dG(t,\ B_h(t)) \\
&= \left\{\frac{\partial F}{\partial t}(t,\ X(t)) + \frac{\partial F}{\partial X}(t,\ X(t))\, f(t,\ B_h(t))\left(\mu - \frac{1}{2}\sigma^2 \frac{d}{dt}[R_h(t,\ t)]\right)\right\} dt \\
&\quad + \frac{\partial F}{\partial X}(t,\ X(t))\, f(t,\ B_h(t))\sigma d^{\Diamond} B_h(t) \\
&\quad + \frac{1}{2}\left(\frac{d}{dt}[R_h(t,\ t)]\right)\left\{\frac{\partial^2 F}{\partial X^2}(t,\ X(t))\, f^2(t,\ B_h(t))\sigma^2\right. \\
&\quad \left. + \frac{\partial F}{\partial X}(t,\ X(t))\, f(t,\ B_h(t))\sigma^2\right\} dt \\
&= \left\{\frac{\partial F}{\partial t}(t,\ X(t)) + \frac{\partial F}{\partial X}(t,\ X(t))\, f(t,\ B_h(t))\mu\right\} dt \\
&\quad + \frac{\partial F}{\partial X}(t,\ X(t))\, f(t,\ B_h(t))\sigma d^{\Diamond} B_h(t) \\
&\quad + \frac{1}{2}\left(\frac{d}{dt}[R_h(t,\ t)]\right)\frac{\partial^2 F}{\partial X^2}(t,\ X(t))\, f^2(t,\ B_h(t))\sigma^2 dt. \qquad (8.7)
\end{aligned}
$$

Since $F(t, X(t)) = G(t, B_h(t))$ and $X(t) = f(t, B_h(t))$, equation (8.7) is equivalent to the following:

$$
\begin{aligned}
dF(t,\ X(t)) = dG(t,\ B_h(t)) &= \left\{\frac{\partial F}{\partial t}(t,\ X(t)) + \frac{\partial F}{\partial X}(t,\ X(t))X(t)\mu\right\} dt \\
&\quad + \frac{\partial F}{\partial X}(t,\ X(t))X(t)\sigma dB_h^{\Diamond}(t) \\
&\quad + \frac{1}{2}\left(\frac{d}{dt}[R_h(t,\ t)]\right)\frac{\partial^2 F}{\partial X^2}(t,\ X(t))X^2(t)\sigma^2 dt.
\end{aligned}
$$

Therefore, we obtain the formula (8.5), completing the proof.

9. Multifractional Black-Scholes Equation

In this section, we shall derive the governing partial differential equation (PDE) for the price of a European call, that is, a multifractional version of the Black-Scholes equation based on mBm. The derivation results from the approach used by Black and Scholes [12] under the following assumptions on the financial market:

(i) Trading takes place continuously in time.

(ii) The riskless interest rate r is known and constant over time.

(iii) The asset pays no dividend.

(iv) There are no transaction costs in buying or selling the asset or the option, and no taxes.

(v) The assets are perfectly divisible.

(vi) There are no penalties to short selling and the full use of proceeds is permitted.

(vii) There are no riskless arbitrage opportunities.

The evolution of the asset price X at time t is assumed to follow the geometric mBm as described by SDE (8.2), i.e.,

$$dX(t) = \mu X(t)dt + \sigma X(t)d^{\Diamond}B_h(t), \quad X(0) = x_0 > 0,$$

where μ is the expected rate of return, σ is the volatility and $B_h(t)$ is the mBm with Hurst function $h(t)$. Both μ and σ are assumed to be constants, and $h(t) : \mathbb{R} \to (0, 1)$ is assumed to be a C^1 function such that h' is bounded.

We shall follow the *Delta hedging method* as taken in Kwok [25, pp. 101-103].

Consider a portfolio which involves short selling of one unit of a European call option and long holding of $\Delta(t)$ units of the underlying asset.

The value of the portfolio $\Pi(t, X(t))$ is given by

$$\Pi = -F + \Delta(t) X(t),$$

where $F = F(t, X(t))$ denotes the call price. Note that $\Delta(t)$ changes with time *t*, reflecting the dynamic nature of hedging. Then Kwok [25, Remarks, p. 103] cited Carr and Bandyopadhyay [14] to describe the notion of financial gain on the hedged portfolio as follows: The number of units of the underlying asset in the hedged portfolio is assumed to be constant, that is, $-dF + \Delta(t) dX(t)$ is seen to be the differential financial gain on the portfolio over *dt* as the *self-financing portfolio*. Thus, we take the setting such that the differential change of portfolio value Π to be

$$d\Pi = -dF + \Delta(t) dX(t).$$

Since *F* is a stochastic function of $X(t)$, we can apply Theorem 8.3 on Ito formula to compute its differential. Hence,

$$\begin{aligned} d\Pi &= -dF(t, X(t)) + \Delta(t) dX(t) \\ &= -\left\{\frac{\partial F}{\partial t} + \frac{\partial F}{\partial X}\mu X(t) + \frac{1}{2}\left(\frac{d}{dt}[R_h(t, t)]\right)\frac{\partial^2 F}{\partial X^2} X^2(t)\sigma^2\right\} dt \\ &\quad - \frac{\partial F}{\partial X}\sigma X(t) d^{\lozenge} B_h(t) + \Delta(t)\{\mu X(t) dt + \sigma X(t) d^{\lozenge} B_h(t)\} \\ &= \left\{-\frac{\partial F}{\partial t} - \frac{1}{2}\left(\frac{d}{dt}[R_h(t, t)]\right)\frac{\partial^2 F}{\partial X^2} X^2(t)\sigma^2 + \left(\Delta(t) - \frac{\partial F}{\partial X}\right)\mu X(t)\right\} dt \\ &\quad + \left(\Delta(t) - \frac{\partial F}{\partial X}\right)\sigma X(t) d^{\lozenge} B_h(t). \end{aligned}$$

The cumulative financial gain on the portfolio at time *t*, denoted by $G(\Pi(t, X(t)))$, is given by

$$\begin{aligned} &G(\Pi(t, X(t))) \\ &= \int_0^t (-dF) + \int_0^t \Delta(u) dX(u) \end{aligned}$$

$$= \int_0^t \left\{ -\frac{\partial F}{\partial u} - \frac{1}{2}\left(\frac{d}{du}[R_h(u, u)]\right)\frac{\partial^2 F}{\partial X^2} X^2(u)\sigma^2 + \left(\Delta(u) - \frac{\partial F}{\partial X}\right)\mu X(u)\right\} du$$

$$+ \int_0^t \left(\Delta(u) - \frac{\partial F}{\partial X}\right)\sigma X(u) d^\diamond B_h(u).$$

The stochastic component of the portfolio gain stems from the last term: $\int_0^t \left(\Delta(u) - \frac{\partial F}{\partial X}\right)\sigma X(u) d^\diamond B_h(u).$ If we choose the dynamic hedging strategy by choosing $\Delta(u) = \frac{\partial F}{\partial X}(u, X(u))$ at all times $u < t,$ then the financial gain becomes deterministic at all times. By no riskless arbitrage opportunities, the financial gain should be the same as the gain from investing on the risk free asset with dynamic position whose value equals $-F + X(u)\frac{\partial F}{\partial X}(u, X(u)).$ The deterministic gain from this dynamical position of the riskless asset is given by

$$M(t) := \int_0^t r\left(-F + X(u)\frac{\partial F}{\partial X}(u, X(u))\right) du.$$

By equating these two deterministic gains, $G(\Pi(t, X(t)))$ and $M(t),$ we get

$$-\frac{\partial F}{\partial u}(u, X(u)) - \frac{1}{2}\left(\frac{d}{du}[R_h(u, u)]\right)\frac{\partial^2 F}{\partial X^2}(u, X(u)) X^2(u)\sigma^2$$

$$= r\left(-F(u, X(u)) + X(u)\frac{\partial F}{\partial X}(u, X(u))\right), \quad u < t.$$

Rearranging, we obtain

$$\frac{\partial F}{\partial u}(u, X(u)) + \frac{1}{2}\sigma^2\left(\frac{d}{du}[R_h(u, u)]\right)\frac{\partial^2 F}{\partial X^2}(u, X(u)) X^2(u)$$

$$+ r\frac{\partial F}{\partial X}(u, X(u)) X(u) - rF(u, X(u)) = 0, \quad u < t.$$

This is satisfied for any asset price X if $F(t, X)$ satisfies the equation

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2\left(\frac{d}{dt}[R_h(t, t)]\right)\frac{\partial^2 F}{\partial X^2}X^2 + r\frac{\partial F}{\partial X}X - rF = 0.$$

Note that the parameter μ, which is the expected rate of return of the asset, does not appear in the equation above.

In conclusion, we arrive at the following result:

Theorem 9.1. *The no-arbitrage price of a European call is given by* $F(t, X(t))$, *where* $F(t, X)$ *is the solution of the following PDE*:

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2\left(\frac{d}{dt}[R_h(t, t)]\right)\frac{\partial^2 F}{\partial X^2}X^2 + r\frac{\partial F}{\partial X}X - rF = 0, \tag{9.1}$$

where r denotes the riskless interest rate. At expiry, the payoff of the European call is given by

$$F(T, X) = \max\{X - K, 0\}, \tag{9.2}$$

where T is the time of expiration and K is the strike price; *this is the terminal payoff condition*.

If $h(t) \equiv H \in (0, 1)$, *then*

$$\frac{1}{2}\left(\frac{d}{dt}[R_h(t, t)]\right)|_{h(t)\equiv H} = \frac{1}{2}(2t^{2h(t)-1}(h'(t)t\log t + h(t)))|_{h(t)\equiv H}$$

$$= \frac{1}{2}\frac{d}{dt}[t^{2h(t)}]|_{h(t)\equiv H} = Ht^{2H-1},$$

and hence (9.1) *coincides with the fractional result of Necula* [33] *where the case of the Hurst parameter* $H \in (1/2, 1)$ *is investigated*.

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