POLYGAMMA FUNCTIONS, THE ZETA FUNCTIONS, AND EULER NUMBERS

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Abstract

A relation is shown between polygamma functions, expressed as c.f.s, and the Riemann zeta function. Also, the integral formulation of polygamma functions is shown to be related to Euler numbers.

1. Introduction

For the polygamma functions, Shenton and Bowman [3] give the following results:

$$\psi_m(z) = (-1)^{m+1}\left(\frac{(m-1)!}{z^m} + \frac{m!}{2z^{m+1}}\right) + \frac{(-1)^{m+1}}{z^m} C_m(z^2) \quad (\Re(z) > 0)$$

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for $m = 0, 1, ...$; also for $m = 0$, $\dfrac{(-1)!}{z^m}$ is replaced by $-\ln(z)$. Here the polygamma function is

$$\psi_m(z) = \frac{d^{m+1}}{dz^{m+1}} \ln \Gamma(z),$$

$\psi(z) = \psi_0(z)$ being the psi (digamma) function. Moreover there is the c.f.

$$C_m(z^2) = \frac{c_0^{(m)}}{z^2 +} \frac{c_1^{(m)}}{1 +} \frac{c_2^{(m)}}{z^2 +} \frac{c_3^{(m)}}{1 +} \cdots,$$

where $c_0^{(m)} = (m+1)!/12$, and $c_s^{(m)}$ is given to $s = 5$; further values could be found although there are complications (see Shenton and Bowman [3, p. 553]). The c.f.s are

$$\psi(z) = \ln z - \frac{1}{2z} - \left\{ \frac{\frac{1}{12}}{z^2 +} \frac{\frac{1}{10}}{1 +} \frac{\frac{79}{210}}{z^2 +} \frac{\frac{1205}{1659}}{1 +} \frac{\frac{262445}{209429}}{z^2 +} \frac{\frac{2643428417511}{1429053441530}}{1 +} \cdots \right\}$$

$$\psi_1(z) = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{z} \left\{ \frac{1/6}{z^2 +} \frac{1/5}{1 +} \frac{18/35}{z^2 +} \frac{20/21}{1 +} \frac{50/33}{z^2 +} \frac{315/143}{1 +} \cdots \right\}$$

$$\psi_3(z) = \frac{2}{z^3} + \frac{3}{z^4} + \frac{1}{z^3} \left\{ \frac{2}{z^2 +} \frac{1/2}{1 +} \frac{5/6}{z^2 +} \frac{22/15}{1 +} \frac{116/55}{z^2 +} \frac{942/319}{1 +} \cdots \right\}$$

$$\psi_4(z) = -\frac{6}{z^4} - \frac{12}{z^5} - \frac{1}{z^4} \left\{ \frac{10}{z^2 +} \frac{\frac{7}{10}}{1 +} \frac{\frac{71}{70}}{z^2 +} \frac{\frac{870}{497}}{1 +} \frac{\frac{15092}{6177}}{z^2 +} \frac{\frac{11140610}{3329403}}{1 +} \cdots \right\}$$

$$\psi_5(z) = \frac{24}{z^5} + \frac{60}{z^6} + \frac{1}{z^5} \left\{ \frac{60}{z^2 +} \frac{\frac{14}{15}}{1 +} \frac{\frac{127}{105}}{z^2 +} \frac{\frac{3645}{1778}}{1 +} \frac{\frac{21971635}{7838694}}{z^2 +} \frac{\frac{145316386786}{38746664420}}{1 +} \cdots \right\}$$

$$\psi_6(z) = -\frac{120}{z^6} - \frac{360}{z^7} - \frac{1}{z^6} \left\{ \frac{420}{z^2 +} \frac{\frac{6}{5}}{1 +} \frac{\frac{149}{105}}{z^2 +} \frac{\frac{14795}{6258}}{1 +} \frac{\frac{2811525}{881782}}{z^2 +} \frac{\frac{271790214451}{65240845725}}{1 +} \cdots \right\}$$

$$\psi_2(z) = -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{2z^2} \left\{ \frac{1/3}{z^2 +} \frac{2/3}{1 +} \frac{6/5}{z^2 +} \frac{9/5}{1 +} \frac{18/7}{z^2 +} \frac{24/7}{1 +} \cdots \right\}.$$

Now $\psi_2(z)$ is given by Stieltjes [6, p. 385],

$$\psi_2(z) = -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^2}\left\{\frac{1/2}{z^2} + \frac{p_1}{1} + \frac{q_1}{z^2} + \frac{p_2}{1} + \frac{q_2}{z^2} + \cdots\right\},$$

where $p_s = \dfrac{s^2(s+1)}{4s+2}$, $q_s = \dfrac{s(s+1)^2}{4s+2}$, $s = 1, 2, \ldots$. As far as we know this is the only case for which all partial numerators are known. Indeed, for the general case from Shenton and Bowman [3, p. 553],

$$(-1)^{m+1}\psi_m(z) = \frac{(m-1)!}{z^m} + \frac{m!}{2z^{m+1}} + \frac{(m+1)!}{12z^m}\left\{\frac{1}{z^2+b_1^{(m)}} - \frac{a_1^{(m)}}{z^2+b_2^{(m)}} - \cdots\right\}$$

$$= \frac{(m-1)!}{z^m} + \frac{m!}{2z^{m+1}} + \frac{(m+1)!}{12z^m}\left\{\frac{1}{z^2} + \frac{c_1^{(m)}}{1} + \frac{c_2^{(m)}}{z^2} + \frac{c_3^{(m)}}{1} + \cdots\right\},$$

$m = 1, 2, \ldots$, where

$$a_1^{(m)} = \frac{(m+2)(m+3)\pi_2(m)}{5 \cdot 7!}, \quad a_2^{(m)} = \frac{(m+4)(m+5)\pi_6(m)}{11088\pi_2^2(m)};$$

$$b_1^{(m)} = \frac{(m+2)(m+3)}{60}, \quad b_2(m) = \frac{m^4 + 60m^3 + 762m^2 + 3455m + 5226}{30\pi_2(m)},$$

$$b_3^{(m)} = \frac{\pi_{10}(m)}{780\pi_2(m)\pi_6(m)}; \quad c_1^{(m)} = b_1^{(m)},$$

$$c_2^{(m)} = \frac{\pi_2(m)}{420}, \quad c_3^{(m)} = \frac{(m+4)(m+5)\pi_2^*(m)}{84\pi_2(m)},$$

$$c_4^{(m)} = \frac{\pi_6(m)}{132\pi_2(m)\pi_2^*(m)}, \quad c_5^{(m)} = b_3^{(m)} - c_4^{(m)};$$

and

$$\pi_2(m) = 3m^2 + 55m + 158, \quad \pi_2^*(m) = m^2 + 93m + 482,$$

$$\pi_6(m) = m^6 + 723m^5 + 22659m^4 + 339313m^3$$

$$+ 2581460m^2 + 934168m + 12597360,$$

$$\pi_{10}(m) = 10m^{10} + 40518m^9 + 270571m^8$$

$$+ 89788404m^7 + 1794433554m^6 + 22947479418m^5$$

$$+ 190840554986m^4 + 1020244044156m^3 + 3360583420456m^2$$

$$+ 6173053021104m + 4817280991680.$$

The fifth partial numerator involves $\pi_6(m)$, not an attractive prospect. Another example arises from the well known expression

$$\ln \Gamma(z) = \left(z - \frac{1}{2} \right) \ln(z) - z + \frac{1}{2} \ln(2\pi) + J(z), \quad (\Re(z) > 0),$$

where

$$J(z) = \frac{1}{\pi} \int_0^\infty \ln\left(\frac{1}{1 - e^{-2\pi t}} \right) \frac{z dt}{(z^2 + t^2)} = \frac{a_1}{z +} \frac{a_2}{z +} \frac{a_3}{z +} \frac{a_4}{z +} \frac{a_5}{z +} \cdots,$$

and

$$a_1 = 1/12, \quad a_2 = 1/30, \quad a_3 = 53/210,$$

$$a_4 = 195/371, \quad a_5 = 22999/22737,$$

the a_i being positive. This was given by Stieltjes ([5, p. 351], letter of 1889). Wall [7] gives a_6 and a_7. These partial numerators do not seem to subscribe to simple closed forms. But as already noted, by contrast, for $\psi_2(z)$ all partial numerators are known in simple closed forms. Perhaps J. B. S. Haldane would have remarked "Mathematics is not only curious but more curious than we can think".

In this paper we use the recurrence $\Gamma(x + 1) = x\Gamma(x)$ and its derivatives to establish relations between polygamma functions and the Riemann zeta function.

2. Polygamma Functions and the Zeta Function

We have

$$\psi(z + n) - \psi(z) = \sum_{s=0}^{n-1} \frac{1}{z + s} \quad (\Re(z) > 0, \ n = 1, 2, ...)$$

which is derived from

$$\psi(z+1) - \psi(z) = \frac{1}{z}.$$

Differentiating m times, we have

$$-\psi_m(z) = (-1)^m m! \sum_{s=0}^{n-1} \frac{1}{(z+s)^{m+1}} - \psi_m(z+n)$$

$$= (-1)^m m! \sum_{s=0}^{n-1} \frac{1}{(z+s)^{m+1}}$$

$$+ (-1)^m \left\{ \frac{(m-1)!}{(z+n)^{m+1}} + \frac{m!}{2(z+n)^{m+1}} + \frac{C_m\{(z+n)^2\}}{(z+n)^m} \right\}$$

$$(m = 0, 1, ...; \ n = 1, 2, ...; \ \Re(z) > 0).$$

If $m = 0, \ z = 1,$ then

$$-\psi(1) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) + \frac{1}{2(n+1)} - \ln(n+1) + C_0(n+1)^2, \ (n \to \infty).$$

Now $C_0(n+1)^2 = \dfrac{\dfrac{1}{12}}{N^2 +} \dfrac{\dfrac{1}{10}}{1 +} \dfrac{\dfrac{79}{210}}{N^2 +} \dfrac{\dfrac{1205}{1659}}{1 +} \cdots, \ N = n+1$ from Section 1.

This we set up in the tabulation of approximants to γ.

Table 1. Bounds for Euler's Constant with $n = 1$ and
successive terms of the c.f.

s	Using s partial numerators	$\mid$ Error $\mid$
1	0.5776861528	0.0004704879
2	0.5771780227	0.0000376422
3	0.5772207272	0.0000050623
4	0.5772146373	0.0000010276
5	0.5772159268	0.0000002619
6	0.5772155823	0.0000000826

The first approximant is $5/4 - \ln 2 + 1/48$, second is $5/4 - \ln 2 + 5/246$.

Clearly the c.f. approximants are quite sharp, the odd convergents providing a set of decreasing bounds, the even a set of increasing bounds. Note that in the 1971 paper we have provided that all partial numerators are positive. The general expression for $\psi_m(1)$ may be used to set up similar bounds for (i) $\psi_1(1)$ which is related to the zeta function $\zeta(2) = \pi^2/6$, (ii) $\psi_3(1)$ which is related to $\pi^4/90$, and (iii) $\psi_{2s-1}(1)$ which is related to $\zeta(2s) = \dfrac{(2\pi)^{2s}}{2(2s)!}\mid B_{2s}\mid$.

Adams [1], computed γ, Euler's constant to 264 decimal places. His result being

$$E = .57721\ 56649\ 01532\ 86060\ 65120\ 90082\ 40243\ 10421\ 59335\ 93992$$
$$35988\ 05767\ 23488\ 48677\ 26777\ 66467\ 09369\ 47063\ 29174\ 67495$$
$$14631\ 44724\ 98070\ 82480\ 96050\ 40144\ 86542\ 83622\ 41739\ 97644$$
$$92353\ 62535\ 00333\ 74293\ 73377\ 37673\ 94279\ 25952\ 58247\ 09491$$
$$60087\ 35203\ 94816\ 56708\ 53233\ 15177\ 66115\ 28621\ 19950\ 15079$$
$$84793\ 74508\ 5697 \qquad\qquad (E = \gamma)$$

Adams had to compute the Bernoulli number B_{62}; he gives an interesting account of Euler's use of divergent series. His value for γ is correct up to 261 digits according to the value computed on Maple.

In our 1971 paper on derivatives of the psi function we give the Stieltjes transforms

$$\psi(z) = \ln z - \frac{1}{2z} - \int_0^\infty \frac{dx}{(y-1)(z^2+x)},$$

$$\psi_1(z) = \frac{1}{z} + \frac{1}{2z^2} + \left(\frac{2\pi}{z}\right)\int_0^\infty \frac{y\sqrt{x}\,dx}{(y-1)^2(z^2+x)},$$

$$\psi_2(z) = -\frac{1}{z^2} - \frac{1}{z^3} - \left(\frac{2\pi}{z}\right)^2\int_0^\infty \frac{(y+y^2)dx}{(y-1)^3(z^2+x)},$$

$$\psi_3(z) = \frac{2}{z^3} + \frac{3}{z^4} + \left(\frac{2\pi}{z}\right)^3\int_0^\infty \frac{(y+4y^2+y^3)x^{3/2}dx}{(y-1)^4(z^2+x)}, \quad (\Re(z) > 0). \qquad (1)$$

The general case will be

$$\psi_m(z) = (-1)^{m+1}\left\{\frac{(m-1)!}{z^m} + \frac{m!}{2z^{m+1}} + \left(\frac{2\pi}{z}\right)^m \int_0^\infty \frac{x^{m/2}\Theta_m(y)dx}{(z^2+x)(y-1)^{m+1}}\right\}, \quad (2)$$

where $y = \exp(2\pi\sqrt{x})$, and $\Theta_m(y)$ is perhaps a polynomial in y of degree m.

Now there is the asymptotic expansion

$$\psi_m(z) \sim (-1)^{m+1}\left\{\frac{(m-1)!}{z^m} + \frac{m!}{2z^{m+1}} + \sum_{k=1}^\infty \frac{(2k+n-1)!}{(2k)!\,z^{2k+m}} B_{2k}\right\}$$

for $z \to \infty$ in $|\arg z| < \pi$. Therefore, the coefficient of $1/z^{2k+m}$ is

$$(-1)^{m+1}\frac{(2k+m-1)!\,B_{2k}}{(2k)!}.$$

This implies that

$$(2\pi)^m \int_0^\infty \frac{x^{\frac{m}{2}+k-1}\Theta_m(y)dx}{(y-1)^{m+1}} = \frac{(2k+m-1)!\,|B_{2k}|}{(2k)!},$$

$$(m = 1, 2, ...; \ k = 1, 2, \cdots), \quad (3)$$

where B_k is a Bernoulli number.

3. An Expression for $\Theta_m(y)$ Involving Euler Numbers

Now in connection with a related problem, Shenton and Bowman [4] used the formula

$$\Theta_m(t) = (1-e^{-t})^{m+1}\left(\frac{d}{dt}\right)^m \frac{1}{1-e^{-t}}, \quad (t > 0, \ m = 0, 1, ...).$$

For example

$$\Theta_0(t) = 1, \qquad\qquad \Theta_1(t) = e^{-t},$$

$$\Theta_2(t) = e^{-t} + e^{-2t}, \qquad \Theta_3(t) = e^{-t} + 4e^{-2t} + e^{-3t}.$$

Now with $y = e^{-t}$ in the integrals of (3), there is agreement corresponding

to the forms in (1). Hence

$$\frac{\Theta_m(t)}{(1 - e^{-t})^{m-1}} = \left(\frac{d}{dt}\right)^m \frac{1}{1 - e^{-t}} = 1^m e^{-t} + 2^m e^{-2t} + 3^m e^{-3t} + \cdots.$$

Returning to (3), the left side becomes, using $1/y = e^{-t}$,

$$= \frac{2}{(2\pi)^{2k}} (2k + m - 1)! \left\{1 + \frac{2^m}{2^{m+2k}} + \frac{3^m}{3^{m+2k}} + \frac{4^m}{4^{m+2k}} + \cdots\right\},$$

$$= \frac{2}{(2\pi)^{2k}} (2k + m - 1)! \, \zeta(2k),$$

$$= \frac{2}{(2\pi)^{2k}} (2k + m - 1)! \, \frac{(2\pi)^{2k}}{2(2k)!} \mid B_{2k} \mid,$$

$$= \frac{(2k + m - 1)! \mid B_{2k} \mid}{(2k)!},$$

as it should.

The $\Theta_m(e^{-t})$ polynomials turn up in expressions for the cumulants of a geometric random variable.

4. Conclusion

At least formally when $\psi_m(z)$ is expressed in terms of a Stieltjes integral (expression (1)), we have shown that the integrand has a polynomial component whose coefficients are Euler numbers, such as

$$k_1^{(m)} = 1^{(m)},$$

$$k_2^{(m)} = 2^m - (m + 1)1^m,$$

$$k_3^{(m)} = 3^m - \binom{m + 1}{1}2^m + \binom{m + 1}{2}1^m,$$

$$k_s^{(m)} = \sum_{u=0}^{s-1} (-1)^u \binom{m + 1}{u} (s - u)^m, \quad (m = 1, 2, ...; \, s = 1, 2, ..., m),$$

Riordan [2].

Acknowledgement

References

[1]   J. C. Adams, On the value of Euler's constant, Proc. Roy. Soc. 27 (1878), 92-94.

[2]   J. Riordan, Introduction to Combinatorial Analysis, John Wiley & Sons, New York, 1958.

[3]   L. R. Shenton and K. O. Bowman, Continued fractions for the psi function and its derivatives, SIAM J. Appl. Math. 20(4) (1971), 547-554.

[4]   L. R. Shenton and K. O. Bowman, The geometric distribution, cumulants, Eulerian numbers, and the logarithmic function, Far East J. Theo. Stat. 5(1) (2001), 113-142.

[5]   T. J. Stieltjes, Correspondance D'Hermite et de Stieltjes, Gauthier Villars, Paris, 1905.

[6]   T. J. Stieltjes, Oeuvres Complètes, Tome II. P. Noordhoff, Groningen, 1918.

[7]   H. Wall, Analytic Theory of Continued Fractions, Chelsea Publishing Company, New York, 1948.

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