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EXTENDED PRÉKOPA-LEINDLER INEQUALITY FOR GEOMETRIC MEANS

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Abstract

Using the method of "marginals of geometric inequality", we obtain the extended Prékopa-Leindler inequality for the geometric mean.

1. Introduction

The method of "marginals of geometric inequality" is very effective to obtain the functional inequalities from different types of geometric inequalities, and has been extensively applied in Functional Analysis and Convex Geometry (see [2-8], [10]).

This method can be simply explained as follows. Given a compact set $K \subset \mathbb{R}^n$ and a k-dimensional subspace $E \subset \mathbb{R}^n$, the marginal of K on the subspace E is the functional $f_{K,E} : E \to [0, \infty)$ defined as

$$f_{K,E}(x) = Vol_{n-k}(K \cap [x + E^{\perp}]),$$

where $E^{\perp}$ is the orthogonal complement to E in $\mathbb{R}^n$, and Vol_{n-k} is the

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Lebesgue measure on the affine subspace $x + E^{\perp}$. A trivial observation is that

$$vol_n(A) \geq vol_n(B) \Rightarrow \int_E f_{A,E} dx \geq \int_E f_{B,E} dx.$$

Thus, geometric inequalities give rise to certain functional inequalities in lower dimension.

Applying the method of marginals, Klartag [11] proved some well-known functional inequalities, such as the L_p logarithmic Sobolev inequality, the Prékopa-Leindler inequality, and the functional version of Minkowski inequality and the Alexandrov-Fenchel inequality. Another application of the "marginals of geometric inequality" is the functional version of Blaschke-Santaló inequality and its inverse. Appropriately taking marginals of both sides of Blaschke-Santaló inequality [12], the following inequalities are established (see [1, 7, 8, 10]): There exist universal constants $c, C > 0$ such that for any dimension *n* and for any $f : \mathbb{R}^n \to [0, \infty)$, an even log-concave function with $0 < \int_{\mathbb{R}^n} f dx < \infty$, we have

$$c < \left(\int_{\mathbb{R}^n} f dx \int_{\mathbb{R}^n} f^{\circ} dx\right)^{\frac{1}{n}} \leq C,$$

where f° is the polar of *f* defined by

$$f^{\circ} = \inf_{y \in \mathbb{R}^n} [e^{-\langle x, y \rangle} / f(y)].$$

The right equality holds if and only if *f* is a certain Gaussian function.

Using the method of "marginals of geometric inequality", in this paper, we will establish the extended Prékopa-Leindler inequality for the geometric mean.

Theorem 1. *Let* $f_i : \mathbb{R}_+^n \to [0, \infty)$, $0 \leq i \leq m$ *be integrable functions and* $\mu_i > 0$ *such that* $\sum_{i=1}^m \mu_i = 1$. *If*

$$\prod_{i=1}^{m} f_i(x_{i1}, \ldots, x_{in})^{\mu_i} \le f_0\left(\prod_{i=1}^{m} x_{i1}^{\mu_i}, \ldots, \prod_{i=1}^{m} x_{in}^{\mu_i}\right),$$

then

$$\prod_{i=1}^{m}\left(\int_{\mathbb{R}_+^n} f_i dx\right)^{\mu_i} \le \int_{\mathbb{R}_+^n} f_0 dx.$$

Next, we have another version of the extended Prékopa-Leindler inequality.

Corollary 1.1. *Let* $\tilde{f}_i : \mathbb{R}_+^n \to [0, \infty)$ *be integrable functions, and* $\mu_i > 0$ *such that* $\sum_{i=1}^{m} \mu_i = 1$. *Define* $\tilde{h} : \mathbb{R}_+^n \to [0, \infty)$ *by*

$$\tilde{h}(x) = \sup_{x=\prod x_i^{\mu_i}} \prod_{i=1}^{m} \tilde{f}_i^{\mu_i}(x_i)$$

for all $x \in \mathbb{R}_+^n$. *Then we have*

$$\int_{\mathbb{R}_+^n} \tilde{h}(x) \Big/ \prod_{j=1}^{n} x_j dx \ge \prod_{i=1}^{m}\left(\int_{\mathbb{R}_+^n} \tilde{f}_i(x) \Big/ \prod_{j=1}^{n} x_j dx\right)^{\mu_i}.$$

2. Proofs of the Main Results

Let $n, m, s > 0$ be integers, and let $f : \mathbb{R}^n \to [0, \infty)$ be a function. The support of *f*, denoted by Supp(*f*), is the closure of $\{x \in \mathbb{R}^n;\ f(x) > 0\}$. We say *f* is *s*-concave if Supp(*f*) is compact, convex and $f^{\frac{1}{s}}$ is concave on Supp(*f*). Note that an *s*-concave function is continuous in the interior of its support [13].

The classical Brunn-Minkowski inequality (see [14]) states that for any non-empty compact sets $A, B \subset \mathbb{R}^m$,

$$Vol_m(A+B)^{\frac{1}{m}} \geq Vol_m(A)^{\frac{1}{m}} + Vol_m(B)^{\frac{1}{m}}, \tag{2.1}$$

where $A+B$ is the Minkowski sum defined by $A+B=\{a+b : a\in A, b\in B\}$.

For any function $f : \mathbb{R}^n \to [0, \infty)$, define

$$\mathcal{K}_f = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^s : x \in Supp(f), |y| \leq \kappa_s^{-\frac{1}{s}} f^{\frac{1}{s}}(x)\right\}, \tag{2.2}$$

where $\kappa_s = \dfrac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}$ is the volume of the s-dimensional Euclidean unit ball. If the function f is measurable, so is the set $\mathcal{K}_f$. In addition, the set $\mathcal{K}_f$ is convex if and only if f is s-concave. From the definition of $\mathcal{K}_f$, we have

$$Vol_{n+s}(\mathcal{K}_f) = \int_{\mathbb{R}^n} \kappa_s \left(\kappa_s^{-\frac{1}{s}} f^{\frac{1}{s}}(x)\right)^s dx = \int_{\mathbb{R}^n} f dx. \tag{2.3}$$

For functions $f_i : \mathbb{R}^n \to [0, \infty), 1 \leq i \leq m, \lambda > 0,$ we define

$$[\lambda \times_s f](x) = \lambda^s f\left(\frac{x}{\lambda}\right), \tag{2.4}$$

$$\left[\sum_{i=1}^{m} \oplus_s f_i\right](x) = [f_1 \oplus_s \cdots \oplus_s f_m](x) = \left(\sup_{\substack{x_i \in Supp(f_i) \\ x=\sum x_i}} \sum_{i=1}^{m} f_i(x_i)^{\frac{1}{s}}\right)^s \tag{2.5}$$

whenever $x \in \sum \text{Supp}(f_i)$. It is easy to verify that

$$\mathcal{K}_{\lambda \times_s f} = \lambda \mathcal{K}_f = \{\lambda y : y \in \mathcal{K}_f\} \tag{2.6}$$

and

$$\mathcal{K}_{\sum_{i=1}^m \oplus_s f_i} = \sum_{i=1}^{m} \mathcal{K}_{f_i}. \tag{2.7}$$

Lemma 2.1 (Hölder's inequality [9]). *Let* $\mu_i > 0, 1 \le i \le m,$ *and* $1 < p_j < \infty$ *such that* $\sum_{j=1}^{q} \frac{1}{p_j} = 1.$ *Then for* $a_{ij} \in \mathbb{R},$

$$\sum_{i=1}^{m}\left(\mu_i \prod_{j=1}^{q} | a_{ij} |\right) \le \prod_{j=1}^{q}\left(\sum_{i=1}^{m} \mu_i | a_{ij} |^{p_j}\right)^{\frac{1}{p_j}}.$$

Taking marginals of both sides of the Brunn-Minkowski inequality (2.1), we obtain the following lemma.

Lemma 2.2. *For* $1 \le i \le m,$ *let* $f_i, h : \mathbb{R}^n \to [0, \infty)$ *be integrable functions, and s,* $\mu_i > 0$ *be real numbers. Assume that for any* $x_i \in \mathbb{R}^n,$

$$h\left(\sum_{i=1}^{m} \mu_i x_i\right) \ge \left(\sum_{i=1}^{m} \mu_i f_i(x)^{\frac{1}{s}}\right)^{s}. \tag{2.8}$$

Then

$$\left(\int_{\mathbb{R}^n} h dx\right)^{\frac{1}{n+s}} \ge \sum_{i=1}^{m} \mu_i \left(\int_{\mathbb{R}^n} f_i dx\right)^{\frac{1}{n+s}}. \tag{2.9}$$

Proof. Assume that s is an integer. By (2.6) and (2.7), the Brunn-Minkowski inequality (2.1) for $(n + s)$-dimensional sets implies that

$$Vol_{n+s}^{*}\left(\mathcal{K}_{\sum_{i=1}^{m} \oplus_s [\mu_i \times_s f_i]}\right)^{\frac{1}{n+s}} \ge \sum_{i=1}^{m} \mu_i Vol_{n+s}(\mathcal{K}_{f_i})^{\frac{1}{n+s}},$$

where Vol_{n+s}^{*} is outer Lebesgue measure. Using (2.3), we obtain that

$$\left(\int_{\mathbb{R}^n}^{*} \sum_{i=1}^{m} \oplus_s [\mu_i \times_s f_i] dx\right)^{\frac{1}{n+s}} \ge \sum_{i=1}^{m} \mu_i \left(\int_{\mathbb{R}^n} f_i dx\right)^{\frac{1}{n+s}}, \tag{2.10}$$

where $\int^*$ is the outer integral. Since $h \geq \sum_{i=1}^{m} \oplus_s [\mu_i \times_s f_i]$ pointwise, it follows that $\int_{\mathbb{R}^n} h dx = \int^*_{\mathbb{R}^n} h dx$. Thus, this proves the inequality (2.9) for integer s.

Next, assume that $s = \frac{p}{q}$ is a rational number, and $p, q > 0$ are integers. By Lemma 2.1 and (2.8), for any $x_{ij} \in \mathbb{R}^n, 1 \leq i \leq m, 1 \leq j \leq q,$ we have

$$\sum_{i=1}^{m} \left(\mu_i \prod_{j=q}^{q} f_i(x_{ij})^{\frac{1}{qs}} \right) \leq \prod_{j=1}^{q} \left(\sum_{i=1}^{m} \mu_i f_i(x_{ij})^{\frac{1}{qs}} \right)^{\frac{1}{q}} \leq \prod_{j=1}^{q} h\left(\sum_{i=1}^{m} \mu_i x_{ij} \right)^{\frac{1}{qs}}.$$

Since qs is an integer, the above argument implies that

$$\begin{aligned} \left(\int_{\mathbb{R}^n} h\left(\sum_{i=1}^{m} \mu_i x_i \right) dx \right)^{\frac{1}{n+s}} &= \left(\int_{\mathbb{R}^{nq}} \prod_{j=1}^{q} h\left(\sum_{i=1}^{m} \mu_i x_{ij} \right) dx_j \right)^{\frac{1}{q(n+s)}} \\ &\geq \sum_{i=1}^{m} \left(\mu_i \int_{\mathbb{R}^{nq}} \prod_{j=1}^{q} f_i(x_{ij}) dx_{ij} \right)^{\frac{1}{q(n+s)}} \\ &= \sum_{i=1}^{m} \mu_i \left(\int_{\mathbb{R}^n} f_i(x_i) dx_i \right)^{\frac{1}{n+s}}. \end{aligned}$$

This completes the proof. □

If $\sum \mu_i = 1$, letting s tend to infinity, then we obtain the extended Prékopa-Leindler inequality as follows.

Lemma 2.3. *For* $1 \leq i \leq m$, *let* $f_i, h : \mathbb{R}^n \to [0, \infty)$ *be integrable functions, and* $\mu_i > 0$ *such that* $\sum_{i=1}^{m} \mu_i = 1$. *If*

$$h\left(\sum_{i=1}^{m}\mu_i x_i\right) \geq \prod_{i=1}^{m} f_i(x_i)^{\mu_i},$$

then

$$\int_{\mathbb{R}^n} hdx \geq \prod_{i=1}^{m}\left(\int_{\mathbb{R}^n} f_i dx\right)^{\mu_i}.$$

Proof. Let $M > 1.$ The basic observation is that

$$\left(\sum_{i=1}^{m}\mu_i f_i(x_i)^{\frac{1}{s}}\right)^s \overset{s\to\infty}{\to} \prod_{i=1}^{m} f_i(x_i)^{\mu_i}$$

uniformly for each $\frac{1}{M} < f_i(x_i) < M$, $i = 1, \ldots, m.$ Therefore, for any $\varepsilon > 0,$ there exists an $s_0(\varepsilon, M),$ such that whenever $s > s_0(\varepsilon, M)$ and $\frac{1}{M} < f_i(x_i) < M$ for all i,

$$h\left(\sum_{i=1}^{m}\mu_i x_i\right) + \varepsilon \geq \left(\sum_{i=1}^{m}\mu_i f_i(x_i)^{\frac{1}{s}}\right)^s.$$

Denote

$$K_{f_i}^M = \left\{x_i \in \mathbb{R}^n : \frac{1}{M} < f_i(x_i) < M\right\}.$$

Then Lemma 2.2 implies that for $\varepsilon > 0,$ $s > s_0(\varepsilon, M),$

$$\int_{\sum \mu_i K_{f_i}^M} (h(x) + \varepsilon)dx \geq \left(\sum_{i=1}^{m}\mu_i\left(\int_{K_{f_i}^M} f_i dx\right)^{\frac{1}{n+s}}\right)^{n+s}$$

$$\geq \prod_{i=1}^{m}\left(\int_{K_{f_i}^M} f_i dx\right)^{\mu_i}.$$

Since f_i are integrable, the sets $K_{f_i}^M \subset \mathbb{R}^n$ are bounded, and so is $\sum \mu_i K_{f_i}^M$. Letting ε tend to zero, and then M tends to infinity. Then we have

$$\int_{\mathbb{R}^n} hdx \geq \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} f_i dx \right)^{\mu_i}. \qquad \square$$

The following theorem can be viewed as extended Prékopa-Leindler inequality for the geometric mean.

Theorem 2.4. *Let* $f_i : \mathbb{R}_+^n \to [0, \infty)$, $0 \leq i \leq m$ *be integrable functions and* $\mu_i > 0$ *such that* $\sum_{i=1}^m \mu_i = 1$. *If*

$$\prod_{i=1}^{m} f_i(x_{i1}, \ldots, x_{in})^{\mu_i} \leq f_0\left(\prod_{i=1}^{m} x_{i1}^{\mu_i}, \ldots, \prod_{i=1}^{m} x_{in}^{\mu_i} \right),$$

then

$$\prod_{i=1}^{m} \left(\int_{\mathbb{R}_+^n} f_i dx \right)^{\mu_i} \leq \int_{\mathbb{R}_+^n} f_0 dx.$$

Proof. For $t_i = (t_{i1}, \ldots, t_{in}) \in \mathbb{R}^n$, define

$$g_i(t_{i1}, \ldots, t_{in}) = f_i(e^{t_{i1}}, \ldots, e^{t_{in}}) e^{\sum_{j=1}^n t_{ij}}, \quad 0 \leq i \leq m.$$

Therefore, we get

$$\int_{\mathbb{R}^n} g_i(t_{i1}, \ldots, t_{in}) dt_i = \int_{\mathbb{R}^n} f_i(e^{t_{i1}}, \ldots, e^{t_{in}}) e^{\sum_{j=1}^n t_{ij}} dt_i$$

$$= \int_{\mathbb{R}_+^n} f_i(x_{i1}, \ldots, x_{in}) dx_i.$$

Moreover,

$$\prod_{i=1}^{m} g_i(t_{i1}, \ldots, t_{in})^{\mu_i} = \prod_{i=1}^{m} f_i(e^{t_{i1}}, \ldots, e^{t_{in}})^{\mu_i} e^{\mu_i \sum_{j=1}^{n} t_{ij}}$$

$$\leq f_0\left(e^{\sum_{i=1}^{m} \mu_i t_{i1}}, \ldots, e^{\sum_{i=1}^{m} \mu_i t_{in}}\right) e^{\sum_{i=1}^{m} \sum_{j=1}^{n} \mu_i t_{ij}}$$

$$= f_0\left(e^{\sum_{i=1}^{m} \mu_i t_{i1}}, \ldots, e^{\sum_{i=1}^{m} \mu_i t_{in}}\right) e^{\sum_{j=1}^{m}\left(\sum_{i=1}^{m} \mu_i t_{ij}\right)}$$

$$= g_0\left(\sum_{i=1}^{m} \mu_i t_{i1}, \ldots, \sum_{i=1}^{m} \mu_i t_{in}\right).$$

Hence, the results follow from Lemma 2.3. □

For any functions $f_i : \mathbb{R}^n \to [0, \infty), 1 \leq i \leq m,$ we define their Asplund product as (see [1])

$$\left[\prod_{i=1}^{m} \star f_i\right](x) = (f_1 \star \cdots \star f_m)(x) = \sup_{x = \sum x_i} \prod_{i=1}^{m} f_i(x_i).$$

Define

$$\lambda \cdot f(x) = f^{\lambda}\left(\frac{x}{\lambda}\right).$$

Then Lemma 2.3 can be read as follows: For $1 \leq i \leq m,$ let $f_i : \mathbb{R}^n \to [0, \infty)$ be integrable functions, and $\mu_i > 0$ such that $\sum_{i=1}^{m} \mu_i = 1.$ Then

$$\int_{\mathbb{R}^n} \left[\prod_{i=1}^{m} \star(\mu_i \cdot f_i)\right](x) dx \geq \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} f_i dx\right)^{\mu_i}. \tag{2.11}$$

For $x \in \mathbb{R}^n,$ let

$$f_i(x) = \tilde{f}(e^{-x_1}, \ldots, e^{-x_n}), \quad 1 \leq i \leq m.$$

Then for every $t \in \mathbb{R}_+^n$, we have

$$\left[\prod_{i=1}^{m} \star(\mu_i \cdot \tilde{f}_i)\right](t) = \sup_{t=\prod t_i^{\mu_i}} \prod_{i=1}^{m} \tilde{f}_i^{\mu_i}(t_i).$$

In (2.11), setting $y_j = e^{-x_j}$ for $j = 1, \ldots, n$, we obtain

Corollary 2.5. *Let* $\tilde{f}_i : \mathbb{R}_+^n \to [0, \infty)$ *be integrable functions, and* $\mu_i > 0$ *such that* $\sum_{i=1}^{m} \mu_i = 1$. *Define* $\tilde{h} : \mathbb{R}_+^n \to [0, \infty)$ *by*

$$\tilde{h}(x) = \sup_{x=\prod x_i^{\mu_i}} \prod_{i=1}^{m} \tilde{f}_i^{\mu_i}(x_i)$$

for all $x \in \mathbb{R}_+^n$. *Then we have*

$$\int_{\mathbb{R}_+^n} \tilde{h}(x) \Big/ \prod_{j=1}^{n} x_j dx \geq \prod_{i=1}^{m} \left(\int_{\mathbb{R}_+^n} \tilde{f}_i(x) \Big/ \prod_{j=1}^{n} x_j dx \right)^{\mu_i}.$$

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