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APPLYING MULTIPLICITY OF ROOTS TO GRAPHS OF RATIONAL FUNCTIONS

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Abstract

This paper introduces and discusses how the concept of multiplicity can be used to aid in the construction or analysis of the graphs of rational functions. Many textbooks for college algebra or precalculus present a procedure that eventually involves plotting points. The insight provided here is based on the concepts ranging from factoring to multiplicity of roots of a polynomial. This makes the process of analyzing graphs of rational functions cumulative as well as brief.

1. Introduction

Rational functions in the real plane can be a source of difficulty for students in college algebra and precalculus classes, in part due to continuing difficulty with rational expressions. The analysis of these functions in such classes can also be computationally exhausting. Here, a presentation of results to help ease this computational burden is given. These results build upon pre-existing results from polynomials and functions. This lends credence to deeper connections between polynomial functions and rational functions. These connections can also provide solidity and application of students' understanding of polynomial function ideas.

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The main purpose of this paper is to demonstrate how the multiplicity of the roots of polynomials can be used to aid in sketching graphs of rational functions. Nonlinear asymptotes and holes are also part of the analysis. Several claims on the graphs of rational functions in the real plane are stated and proved. This in turn enhances the comprehension of concepts involving polynomials.

Section 2 of this paper introduces the theorems that we will apply to aid in the sketching of the graphs. Many popular textbooks such as [1] and [4] propose guidelines for graphing these functions that usually involves plotting points or using a symmetry argument. With the increased use of graphing calculators in algebra and precalculus courses, these guidelines are falling out of favor. There is also a tendency of students to view rational functions as a completely separate type of function from polynomials. The upcoming theorems and application of mutliplicity will demonstrate a strong connection between these two types of functions as well as provide the instructor with applications of multiplicity. The third section of this paper demonstrates the use of these ideas with several examples. The last section of the paper has concluding remarks.

2. Theorems

A rational function is a function formed by the division of two polynomials. The numerator polynomial $n(x)$ is assumed to be non-zero and the denominator polynomial $d(x)$ is also assumed to be at least a first-degree polynomial to avoid degeneracies. We will make this assumption here as well. With rational functions, as with any other function, a breakdown and analysis usually follows the introduction. The domain and range of rational functions is usually discussed as well intercepts. Asymptotes are usually discussed after these topics. In many textbooks, they omit holes and nonlinear oblique asymptotes in the discussion. This paper does not omit these topics.

The Sign Preserving Property of Continuous Functions from calculus is very important to our analysis here. It is stated as follows: Let *f* be a function defined on an interval (a, b) and suppose that $f(c) \neq 0$ at some $c \in (a, b)$ where *f* is continuous. Then there is an interval $(c - \delta, c + \delta) \in (a, b)$ about *c*, where *f* has the same sign as $f(c)$ for some $\delta > 0$ [3]. This means that if we know a function is continuous and we know the sign of an output value, then there is an interval around this input value where all the function values share the same sign. This is usually proven in calculus or analysis courses, but explains why test points work when determining the sign of continuous functions between roots. The following ideas are also noteworthy:

- If a rational function has a horizontal asymptote, then it cannot have an oblique asymptote. Also, if a rational function has an oblique asymptote, then it cannot have a horizontal asymptote. This follows directly from the Trichotomy Principle of Real Numbers.
- A rational function has exactly one horizontal or exactly one oblique asymptote. This follows from the uniqueness of quotients in the Division Algorithm.

A hole may occur in a graph of a rational function when it is not in lowest terms. If the rational function has a common factor in the numerator and denominator, then usually it is canceled and the analysis proceeds. When graphing such rational functions, though, extra care must be taken to preserve the domain of the *original* rational function. A rigorous definition of a hole, specific to rational functions and without the use of limits, can be stated as a point (a, b) that occurs when $x = a$ is a root of both the numerator polynomial and the denominator polynomial such that the multiplicity of this root in the numerator is greater than or equal to the multiplicity of this root in the denominator.

In a rational function, the *x*-coordinate of the hole is the root and the *y*-coordinate is the result of substituting the root *a* into the *reduced* form of the rational function. Now, if the multiplicity of a root is greater in the denominator than the same root in the numerator, then the root represents the location of a vertical asymptote. It is not the *x*-coordinate of the location of a hole in the graph. The use of multiplicity when determining holes or vertical asymptotes is done after the rational function is reduced.

The last topic to introduce is the notion of critical numbers. Here we define a critical number as an input value where a change in sign of the output value may occur. It constitutes a root to a function, but whether the graph crosses the *x*-axis or not is not determined. This is a common notion from calculus courses when applying the first derivative test. Now, we introduce some lemmas that will aid us in establishing the connection between multiplicity of roots and the graphs of rational functions.

First Signs of Rational Functions Lemma. *Let* $R(x) = \dfrac{1}{d(x)}$, *where* $d(x)$ *is a non-constant polynomial. Then* $d(x)$ *and* $R(x)$ *have the same sign over the same intervals on the real axis.*

Proof. Since $d(x)$ is a polynomial, we may conclude that between real solutions $d(x)$ is of one sign. This statement follows directly from the Sign Preserving Property of Continuous Functions. Since taking reciprocals does not change the sign nor the multiplicity of roots for $d(x)$, where $d(x)$ is positive, $\dfrac{1}{d(x)}$ will also be positive and where $d(x)$ is negative, $\dfrac{1}{d(x)}$ is also negative. If $d(x)$ has no real solutions, then $d(x)$ is either entirely positive or entirely negative and the same result holds. □

Second Signs of Rational Functions Lemma. *Let* $R(x) = \dfrac{n(x)}{d(x)}$ *with* $d(x) \neq 0$ *and* $R(x)$ *is in lowest terms. It is only necessary to determine the multiplicity of the real solutions to* $n(x)$, *the multiplicity of the real solutions to* $d(x)$ *and determine the sign of* $R(x)$ *in one interval between critical numbers of either* $n(x)$ *or* $d(x)$ *to determine the sign of* $R(x)$ *in all other intervals.*

Proof. Here, we begin with factoring $n(x)$ and $d(x)$. By the Factoring Polynomials Over Real Numbers Theorem, we may rewrite these polynomials factored over the real numbers, so all real solutions are determined along with their multiplicity. Dividing $n(x)$ by $d(x)$ where $d(x)$ has no common factors with $n(x)$ puts $R(x)$ in lowest terms. But this division does not change the solutions to $n(x)$ nor does it change the multiplicity of the roots of $n(x)$. As for the sign of $R(x)$ in one interval between critical numbers, we have the following situations:

1. The sign of $R(x)$ has been determined to be positive in a specified interval and the right endpoint of the interval is a root to $n(x)$.
2. The sign of $R(x)$ has been determined to be positive in a specified interval and the right endpoint of the interval is a root to $d(x)$.
3. The sign of $R(x)$ has been determined to be negative in a specified interval and the right endpoint of the interval is a root to $n(x)$.
4. The sign of $R(x)$ has been determined to be negative in a specified interval and the right endpoint of the interval is a root to $d(x)$.

In situation 1, the sign of $d(x)$ is the same between its solutions, so if the multiplicity of the root of $n(x)$ is odd, then there is a sign change in the output values of $R(x)$ across that root. If the multiplicity of the root of $n(x)$ is even, then there is no sign change in the output values of $R(x)$ across the interval between critical points. In situation 2, the sign of $n(x)$ is the same between its solutions, so if the multiplicity of the root of $d(x)$ is odd, then there is a sign change in the output values of $R(x)$ across the root. If the multiplicity of the root of $d(x)$ is even, then there is no sign change across the interval. Similarly, for situations 3 and 4. The same argument holds for the left endpoint of the interval for which the sign of $R(x)$ is determined. This concludes the proof. □

What these two lemmas enable us to do is to avoid using test numbers to determine the signs of $R(x)$ in each interval between critical numbers for the numerator or denominator and therefore avoids computations. Here is where instructors may ask more critical or analytical questions concerning rational functions such as the following:

- Why did (or did not) the rational function's y-values change sign around the vertical asymptote?
- Why is there a hole in the graph at (x, y) or $(x, 0)$?

Current technology cannot provide answers to these questions and this also reinforces rigorous understanding of multiplicity of roots.

Signs of Rational Functions Theorem. *Let* $R(x) = \dfrac{n(x)}{d(x)}$ *and* $d(x) \neq 0$ *but* $R(x)$ *is not necessarily in lowest terms. To determine the signs of* $R(x)$ *between critical numbers it is necessary only to determine the sign of* $R(x)$ *in one interval and the multiplicity of the roots of* $n(x)$ *and* $d(x)$.

Proof. We will assume that $R(x)$ is not in lowest terms to begin. There are two cases to consider:

- Case I. Multiplicity of the root in $d(x)$ is greater than in $n(x)$: If the root in the denominator has a higher multiplicity than the same root in the numerator, then the reduction process would show this is still a vertical asymptote. The multiplicity of the root is then examined after this reduction process is complete. After the reduction process is completed, the Second Signs of Rational Functions Lemma is applicable.
- Case II. Multiplicity of the root in $n(x)$ is greater than or equal to that in $d(x)$: If this is the case, then the rational function has a hole in its graph. The hole is either on the x-axis or not. If it is not on the x-axis, then the factor has been divided completely out of the numerator and denominator. This means when we reduce the rational function, these holes are not critical numbers of the reduced rational function and the y-coordinate will be either positive or negative. We

may choose an input value in a neighborhood of the root since the function is continuous to determine the sign. If the hole occurs on the x-axis, then it would be an endpoint of an interval. So, it is treated as a critical number since in the reduced format it still appears as a factor of $n(x)$. The rest now follows from the Second Signs of Rational Functions Lemma. □

Our last theorem answers the question of rational functions which intersect horizontal or oblique asymptotes. Since horizontal and oblique asymptotes serve as approximations to rational functions when input values are far from the origin and are not a local phenomenon, we entertain the application of multiplicity of roots to analyze possible intersections of graphs with these asymptotes. Now, if the remainder is zero, then $n(x)$ is a multiple of $d(x)$. The remaining factors in the numerator polynomial constitute the asymptote. This implies that the rational function and the quotient are identical over the domain of the rational function *before* the reduction process takes place.

Intersection with Asymptotes Theorem. *Let $R(x)$ be a rational function such that $n(x)$ is not a multiple of $d(x)$. The real roots and their multiplicity in the remainder polynomial of $R(x)$ inform us of the x-coordinates of intersections between the rational function and the asymptote as well as whether the rational function will cross or bounce off the asymptote.*

Proof. The Division Algorithm allows us to conclude that a rational function $R(x)$ can be rewritten as a quotient polynomial and a remainder polynomial. The resulting quotient is the asymptote because the remainder will eventually become negligible. This occurs because the degree of the remainder must be strictly less than the degree of the denominator polynomial and by the End Behavior Theorem this term approaches zero. The remainder is either a constant or another polynomial of degree at least one. If the remainder is constant and nonzero, then the graph of the rational function does not intersect the asymptote since there is an additional nonzero term required to have equality. If the remainder is a polynomial of degree at least one, there is a root in the remainder. If the remainder equals zero at a specified input value from the domain of $R(x)$, then the quotient and the rational function are identical. Thus, the asymptote and the rational function intersect at the root of the remainder. Since the remainder is a polynomial, the graphical interpretation of multiplicity applies. □

3. Examples

Let us consider some examples for demonstrating the theorems in analyzing rational functions.

Example 1. Answer the following questions concerning the graph of $R(x) = \dfrac{3x^2 + 2x - 1}{x + 1}$:

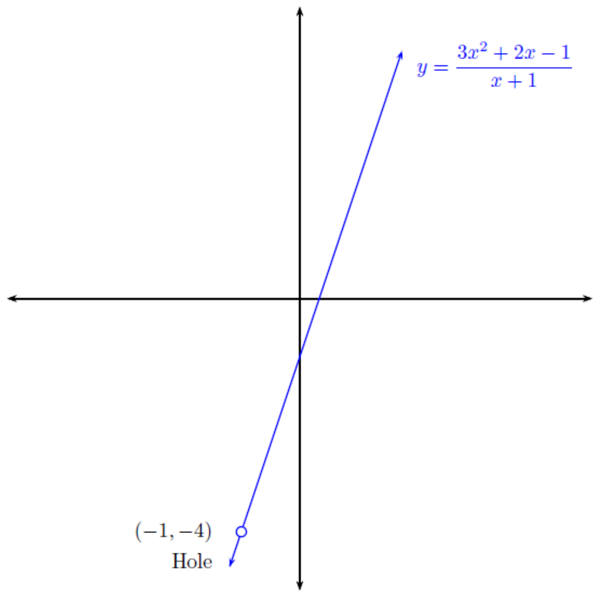
a. Determine the domain.

b. Determine the vertical asymptotes or location of holes in the graph.

Answer. If we begin by finding the domain of the function, then we have the domain to be all real numbers except $x = -1$. If we put -1 in for x, then we end up with $0/0$ which means that -1 may not be a vertical asymptote. Since we have a $0/0$ case, we may reduce the original form of $R(x)$ and obtain the following:

$$R(x) = \frac{3x^2 + 2x - 1}{x + 1} = \frac{(3x - 1)(x + 1)}{(x + 1)} = 3x - 1 \text{ if } x \neq -1.$$

Now, because we were able to reduce $R(x)$ down to a simpler form does not mean we begin again. We start with our reduced form and continue the analysis. The domain of $R(x)$ is still all real numbers except $x = -1$. We now must find the location of the hole. This is accomplished by substituting -1 into the reduced form of $R(x)$ and evaluating. In this case, the location of the hole will be $(-1, -4)$. The graph must have an open circle at this point to insure that the reader knows the graph "skips" this point. This is shown below:



The next example demonstrates several questions that are typically asked when covering the topic of graphing rational functions in college algebra or pre-calculus courses.

Example 2. Answer the following questions concerning the rational function $R(x) = \dfrac{6x^4 - 19x^3 - 14x^2 + 41x + 30}{x-1}$:

a. Determine the domain of $R(x)$.

b. Determine the vertical asymptotes or holes of $R(x)$.

c. Determine the intercepts of $R(x)$.

d. Determine the oblique asymptote.

e. Determine the intervals where $R(x)$ is positive and where it is negative.

Answer. Let us rewrite this rational function in factored form

$$R(x) = \frac{6x^4 - 19x^3 - 14x^2 + 41x + 30}{x-1} = \frac{(x+1)(x-2)(x-3)(6x+5)}{x-1}.$$

Now let us examine each individual question:

a. The domain of this function is $\{x : x \in \mathbb{R}\backslash\{1\}\}$.

b. We can see in the factorization that $x = 1$ is not a root of the numerator, so $x = 1$ is a vertical asymptote. The root $x = 1$ is a root of odd multiplicity so there will be a sign change across this vertical asymptote.

c. The x-intercepts are $(-1, 0)$, $(2, 0)$, $(3, 0)$ and $\left(-\frac{5}{6}, 0\right)$ all of multiplicity one. The y-intercept is $(0, -30)$.

d. The degree is greater in the numerator than in the denominator, so there is an oblique asymptote. Using synthetic division

$$\begin{array}{r|rrrrr} & 6 & -19 & -14 & 41 & 30 \\ 1 & & 6 & -13 & -27 & 14 \\ \hline & 6 & -13 & -27 & 14 & 44 \end{array}$$

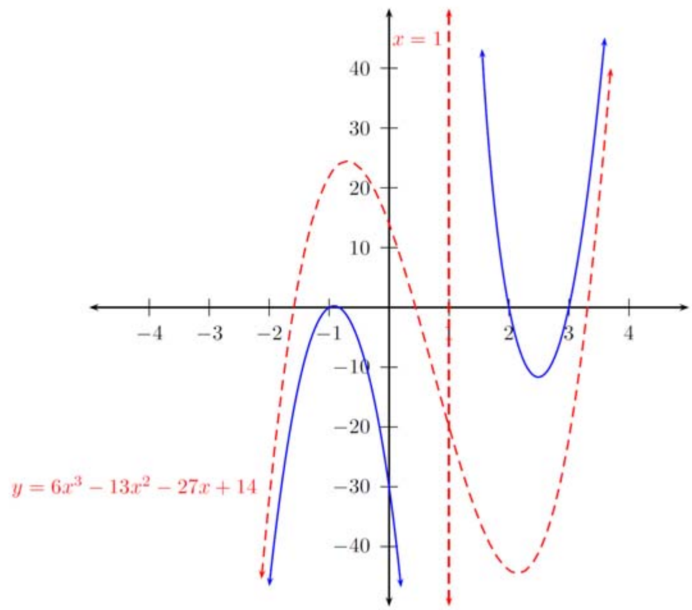
we determine the oblique asymptote to be

$$y = 6x^3 - 13x^2 - 27x + 14.$$

What this implies is that as $|x| \to \infty$ the graph of $R(x)$ will become closer to the graph of $6x^3 - 13x^2 - 27x + 14$.

e. Now, $x = 0$ is in the interval $\left(-\frac{5}{6}, 1\right)$ and we know the y-value is negative. Using the Signs of Rational Functions Theorem, we may conclude $R(x)$ will be positive over the intervals $\left(-1, -\frac{5}{6}\right) \cup (1, 2) \cup (3, \infty)$ and it will be negative over the intervals $(-\infty, -1) \cup \left(-\frac{5}{6}, 1\right) \cup (2, 3)$.

Examining the graph supports our answers to the questions:



One thing to notice here is that all signs were determined without the use of test points after the *y*-intercept was discovered. Another thing to notice in this graph, the rational function does not intersect the asymptote. This is due to the Intersection with Asymptotes Theorem. Our remainder was the constant 44. Since this is never zero, the remainder does not vanish. Thus, the rational function does not intersect the asymptote.

Our next example is one where a root has a higher multiplicity in the denominator than in the numerator. It also demonstrates that rational functions can intersect horizontal asymptotes. Here, we ask specifically about intersections and our Intersection with Asymptotes Theorem can be used to answer the question and aid in the graphing of the rational function along with the asymptotes.

Example 3. Sketch the graph of $R(x) = \dfrac{x^2 - x - 2}{x^3 - x^2 - 5x - 3}$ by first determining the following:

a. The domain of $R(x)$.

b. The vertical asymptotes and holes of $R(x)$, if any.

c. The intercepts of $R(x)$.

d. The horizontal asymptote of $R(x)$.

e. The intervals where $R(x)$ is positive and where $R(x)$ is negative.

f. Does the graph intersect the horizontal asymptote? If so, give the x-coordinate of this location.

Answer. Let us rewrite this rational function in factored form

$$R(x) = \frac{x^2 - x - 2}{x^3 - x^2 - 5x - 3} = \frac{(x+1)(x-2)}{(x+1)^2(x-3)} = \frac{x-2}{(x+1)(x-3)}.$$

a. The domain of the function is $\{x : x \in \mathbb{R} \backslash \{-1, 3\}\}$.

b. We see in the factored form that the number -1 is a root of the numerator and the denominator, but it has a higher multiplicity in the denominator than in the numerator. So, there are no holes in this graph, but the vertical asymptotes are $x = -1$ and $x = 3$. To determine multiplicity, we use the reduced form. So, $x = -1$ has multiplicity one and $x = 3$ also has multiplicity one. There will be a sign change in the y-values as we move across these roots of $d(x)$.

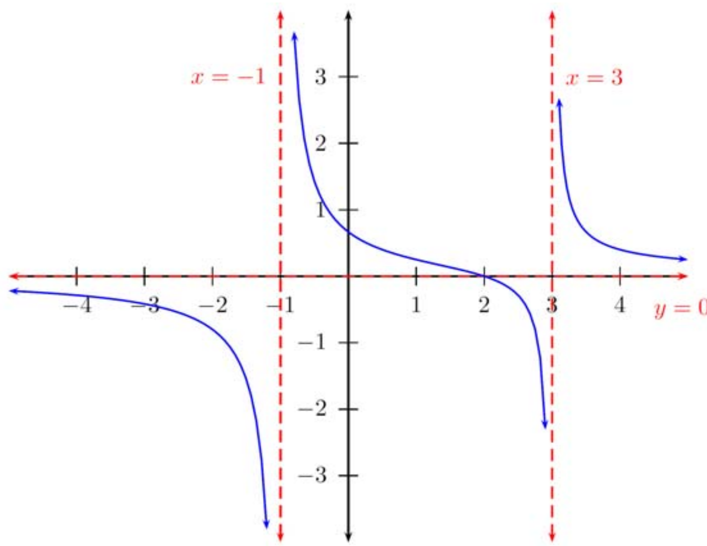
c. The x-intercept is $(2, 0)$; it has multiplicity one. The graph will cross the x-axis at this point. The y-intercept is $\left(0, \frac{2}{3}\right)$.

d. The horizontal asymptote is $y = 0$.

e. Since $x = 0$ is in the interval $(-1, 2)$ and the y-value is positive, we may conclude $R(x)$ will be positive over the intervals $(-1, 2) \cup (3, \infty)$ and it will be negative over the intervals $(-\infty, -1) \cup (2, 3)$. This follows directly from the multiplicities of these roots previously mentioned and the Signs of Rational Functions Theorem.

f. Yes, it crosses the horizontal asymptote at $x = 2$. In accordance with the Intersection with Asymptotes Theorem, the remainder of this rational function is $y = x - 2$ and this has a root of multiplicity one at $x = 2$. Thus, the graph not only intersects the asymptote, but also the graph will cross this asymptote.

Examining the graph of $R(x)$, we can see that our analysis is supported:



Our next example is rather complicated. This is done to demonstrate the power of our results over the traditional use of test numbers.

Example 4. Determine the following and then sketch the graph of

$$R(x) = \frac{x^5 + 4x^4 - 2x^2 + 7x - 1}{x^3 - 2x^2 - 5x + 6}:$$

a. The domain of $R(x)$.

b. The vertical asymptotes and holes of $R(x)$, if any.

c. The oblique asymptote of $R(x)$.

d. The intervals where $R(x)$ is positive and negative.

e. Does $R(x)$ intersects the oblique asymptote and if so, give the locations.

Answer. Let us factor this rational function and then we can answer each of the questions

$$R(x) = \frac{x^5 - 7x^4 - 3x^3 + 79x^2 - 46x - 120}{x^3 - 2x^2 - 5x + 6}$$
$$= \frac{(x+1)(x-2)(x+3)(x-4)(x-5)}{(x-1)(x+2)(x-3)}:$$

a. We can see from the factored form, the domain is $\{x : x \in \mathbb{R}\backslash\{-2, 1, 3\}\}$.

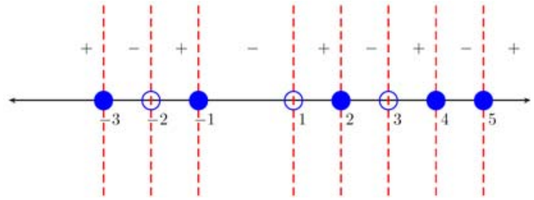
b. None of the factors of the denominator are repeated in the numerator, so each of these values corresponds to a vertical asymptote. Thus, $x = -2$, $x = 1$ and $x = 3$ are all vertical asymptotes to this rational function. There are no holes.

c. We shall use generalized synthetic division [2] to obtain the quotient and remainder

	1	–7	–3	79	–46	–120
–6				–6	30	48
5			5	–25	–40	
2		2	–10	–16		
	1	–5	–8	32	–56	–72

So, the asymptote is $y = x^2 - 5x - 8$.

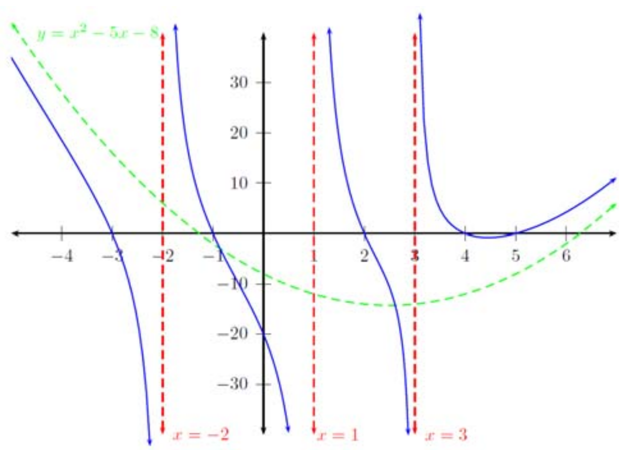
d. Below we show a number line of all critical numbers and the associated signs of $R(x)$:



We note the y-intercept of this rational function was $(0, -20)$, so in the interval $(-1, 1)$ the rational function will be negative. From the Second Signs of Rational Functions Lemma, the sign diagram can be filled in without additional test numbers. We have the rational function is positive in the intervals $(-\infty, -3) \cup (-2, -1) \cup (1, 2) \cup (3, 4) \cup (5, \infty)$. The rational function is negative over the intervals $(-3, -2) \cup (-1, 1) \cup (4, 5)$.

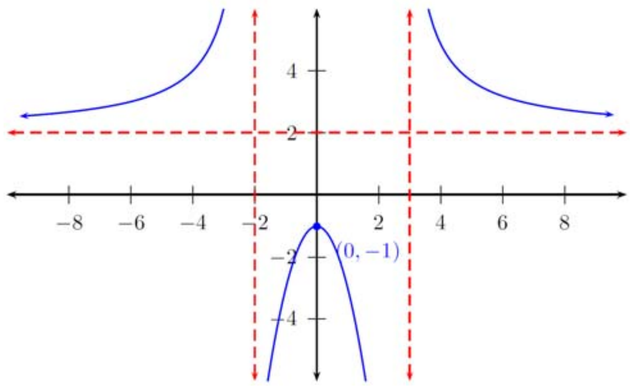
e. Since the remainder is $r(x) = 32x^2 - 56x - 72$, it will have two distinct real solutions. So, the rational function will intersect the asymptote in two distinct locations. These locations are $\left(\frac{7+\sqrt{193}}{8}\right)$ and $\left(\frac{7-\sqrt{193}}{8}\right)$. The Intersection with Asymptotes Theorem also tells us the graph of the rational function will cross this asymptote in these two places because both the mutliplicity of these roots are odd.

Now that we have answered these questions, we sketch the graph of $R(x)$ and see the graph and our answers concur:



In the next example, we reverse the situation. We provide a graph and ask for a possible construction of an algebraic representation for the rational function. Here is where our results can prove even more effective.

Example 5. Suppose you are provided the following graph of a rational function:



Construct a rational function with the lowest possible multiplicity of the roots in the numerator and denominator.

Answer. We can see the graph has the following information:

- There are vertical asymptotes at $x = 3$ and $x = -2$ and there are no holes.
- There is a horizontal asymptote at $y = 2$.
- The graph does not intersect the horizontal asymptote, so the remainder must be a constant.
- The graph changes sign across the vertical asymptotes so the roots of the denominator polynomial must be of odd multiplicity.
- The point $(0, -1)$ is on the graph.

Combining these results yields the following function:

$$R(x) = 2 + \frac{c}{(x-3)(x+2)}.$$

Using the y-intercept, we may determine the value of the constant c

$$R(0) = -1 = 2 + \frac{c}{(0-3)(0+2)}$$

$$-3 = \frac{c}{-6}$$

$$18 = c.$$

So, our rational function can have the form: $R(x) = 2 + \dfrac{18}{(x-3)(x+2)}$.

In our final example, we again reverse the situation, but this time instead of providing a graph of the rational function we provide information in a textual format. Again, our results can prove effective in constructing an appropriate rational function.

Example 6. Suppose you are provided the following information about a rational function:

a. It has vertical asymptotes at $x = 1$.

b. The graph has an horizontal asymptote $y = 1$.

c. The graph will cross the horizontal asymptote at $x = -2$ and bounce off the horizontal asymptote at $x = -1$.

d. The graph changes sign across the vertical asymptote.

Construct a rational function with the lowest possible multiplicity of the roots in the numerator and denominator.

Answer. Since it has a vertical asymptote at $x = 1$, $x - 1$ must be a factor of the denominator. Also, because it has a horizontal asymptote at $y = 1$ we know the following:

$$R(x) = 1 + \frac{r(x)}{x-1}.$$

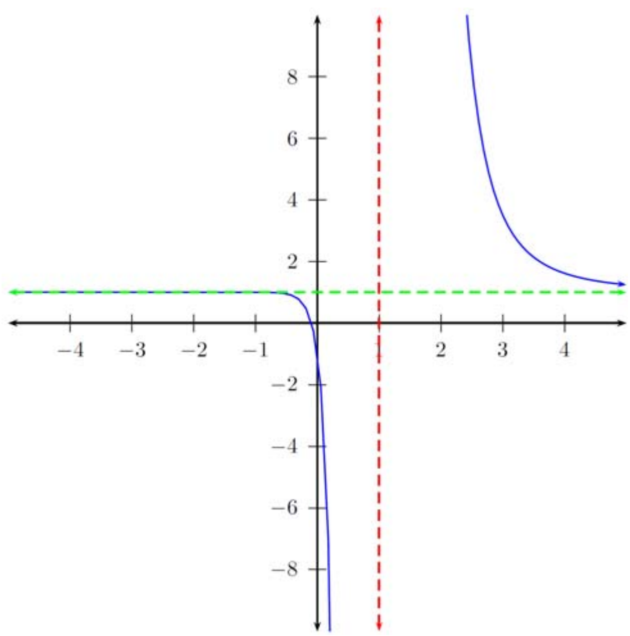
The information the graph intersecting the horizontal asymptote tells us about the remainder $r(x)$. Because we are told the graph of the rational function will cross the horizontal asymptote at $x = -2$, we know that $x + 2$ is a factor of $r(x)$. We are also told the graph bounces off the horizontal asymptote at $x = -1$ which implies that $(x+1)^2$ is a factor of $r(x)$ (using lowest powers for construction). So, now, we have the following:

$$R(x) = 1 + \frac{(x+1)^2(x+2)}{x-1}.$$

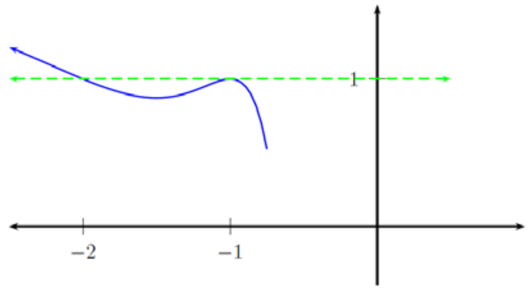
This cannot be correct at this time because the remainder is a polynomial of degree 3 while the denominator in the rational function is a degree one. Our denominator must be of a higher degree and it must also be odd since we are told the graph changes sign across the vertical asymptote. Hence our rational function could appear as

$$R(x) = 1 + \frac{(x+1)^2(x+2)}{(x-1)^5}.$$

A graph of this function is given below:



Examining the graph around $x = -1$ and $x = -2$, we have the following:



4. Concluding Remarks

The main objective of this work is to apply the concept of mutliplicity with the difficult concept of analyzing rational functions. The goal of such application is to minimize the use of test values in each interval between roots of the numerator and denominator. Other applications of multiplicity also arose in the possibility of intersection points between the rational function and a horizontal or oblique asymptote. These concepts can be used to aid in sketching the graphs of rational functions or to create algebraic expressions for graphs that are provided. While current technology can certainly perform the former, it is not capable of providing the latter. It is here that the students can be tested as to the understanding of ideas from polynomial functions and shown how vital these concepts can be.

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