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ON SUMS OF FIGURATE NUMBERS BY USING ALGORITHMS OF DIFFERENTIATION OF POSETS

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Abstract

We use techniques of the theory of algorithms of differentiation of posets and $\mathcal{P}$-partitions to describe identities of some one-dimensional compositions involving polygonal and cubic numbers. We also describe with these techniques numbers which can be written as a sum of three square of numbers of a given shape or sequences of numbers which can be written as sums of three, four or five cubic numbers.

1. Introduction

The theory of algorithms of differentiation of posets was introduced by Nazarova and Roiter in 1972. Actually, they introduced the theory of

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representations of posets and used it to give a proof of the second conjecture of Brauer-Thrall. Researches in this theory are oriented to give a complete description of the indecomposable objects of the additive category rep $\mathcal{P}$ of k-linear representations of a given poset $\mathcal{P}$ [2, 7-9] and [20].

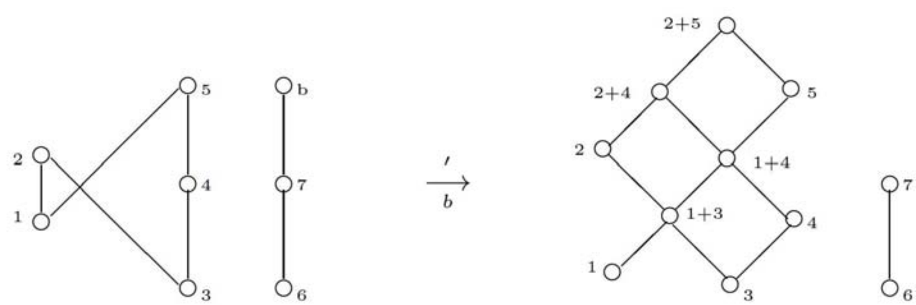
Algorithms of differentiation of posets are the main tool in the theory of representations of posets, such algorithms are functors $D : \text{rep } \mathcal{P} \to \text{rep } \mathcal{P}'$ from a category of representations of a poset $\mathcal{P}$ to the category of representations of a poset $\mathcal{P}'$, in this case D reduces the dimension of the category rep $\mathcal{P}$ and induces a categorical equivalence between some quotient categories. For example, the following is the definition of the algorithm of differentiation with respect to a maximal point given by Nazarova and Roiter:

Let $(\mathcal{P}, \leq)$ be an ordinary poset. Then a maximal point $x \in \max \mathcal{P}$ is suitable for this differentiation if the subset $N \subset \mathcal{P}$ of all points $n \in \mathcal{P}$ incomparable with x has width $w(N) \leq 2$.

The algorithm of differentiation with respect to a suitable point $b \in \max \mathcal{P}$ is defined in such a way that if $N = \mathcal{P} \backslash b_{\Delta}$, then

$$\mathcal{P}'_b = (\mathcal{P} \backslash b) \cup \hat{N}$$

with a partial order induced by $\mathcal{P} \backslash b$ and $\hat{N} = N \cup \{x + y \mid x, y \in N, x \not\leq y, y \not\leq x\}$. Figure 1 below shows the Hasse diagram of this differentiation.



$\mathcal{P} = N + \{6 < 7 < b\}$ $\qquad$ $\mathcal{P}'_b = \hat{N} + \{6 < 7\}$

Figure 1

Gabriel proved that $|\operatorname{Ind}\mathcal{P}| = |\operatorname{Ind}\mathcal{P}'_b| + |\hat{N}| + 1,$ where $|\operatorname{Ind}\mathcal{P}|$ denotes the number of classes of indecomposable objects in rep $\mathcal{P}$ and $|\hat{N}|$ is the size of $\hat{N}$ [13, 17, 20].

A representation $U' = (U'_0; U'_x \mid x \in \mathcal{P}'_b)$ of $\mathcal{P}'_b$ is defined by the following formulae from a representation $U = (U_0; U_x \mid x \in \mathcal{P})$ of $\mathcal{P}$, where U_0 is a finite-dimensional k-vector space and $U_x \subseteq U_0$ is a subspace of U_0 for each $x \in \mathcal{P}$.

$$U'_0 = U_b,$$

$$U'_x = U_{xb} \text{ for } x \in \mathcal{P}\backslash b,$$

$$U'_{x+y} = U_{(x+y)b} \text{ for each dyad } \{x, y\} \subset N,$$

$$\varphi' = \varphi|U_b \text{ for any linear map-morphism } \varphi : U_0 \to V_0 \in \operatorname{rep}\mathcal{P}. \tag{1}$$

Gabriel also proved that there exists a categorical equivalence $\operatorname{rep}\mathcal{P}/\langle\operatorname{Ind} N\rangle \xrightarrow{\sim} \operatorname{rep}\mathcal{P}'_b,$ where $\langle\operatorname{Ind} N\rangle$ denotes the ideal of all morphisms passing through sums of indecomposable objects defined by the subset N, in this case $\operatorname{rep}\mathcal{P}/\langle\operatorname{Ind} N\rangle$ is a quotient category. In this paper, we use these ideas in order to describe advances to the following open problems mentioned by Guy in [14-16] and Cañadas and Irlande in [4]:

(1) What theorems are there, stating that all numbers of a suitable shape are expressible as the sum of three (say) squares of numbers of a given shape? For instance, can all sufficiently large numbers be expressed as the sum of three pentagonal (hexagonal, heptagonal) numbers of nonnegative rank? Equivalently, is every sufficiently large number of shape $24n + 3$ $(8n + 3,\ 40n + 27)$ expressible as the sum of three squares of numbers of shape $6r - 1\ (4r - 1,\ (10r \pm 3))$?

(2) There are theorems giving the number of representations of a number n, as the sum of triangular or square numbers. Can we find corresponding results for any of the other polygonal numbers?

(3) Is every number of the form $9n \pm 4$ the sum of four cubes? Deshouillers et al. believe that 7373170279850 is the largest integer which cannot be expressed as the sum of four nonnegative integral cubes [12]. Actually more demanding is to ask if every number is the sum of four cubes with two of them equal.

Regarding partitions and compositions, we recall that a *partition* of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the *parts* of the partition [1]. A *composition* is a partition in which the order of the summands is considered.

Partitions of positive numbers may be treated as a linear array whose sum is prescribed

$$n = n_1 + n_2 + \cdots + n_s = \sum_{i=1}^{s} n_i, \quad n_i \geq n_{i+1},$$

higher-dimensional partitions are arrays whose sum is n. In this case;

$$n = \sum_{i_1, \ldots, i_r \geq 0} n_{i_1 i_2 \ldots i_r}, \quad \text{where } n_{i_1 i_2 \ldots i_r} \geq n_{j_1 j_2 \ldots j_r} \tag{2}$$

whenever $i_1 \leq j_1, i_2 \leq j_2, \ldots, i_r \leq j_r$ (all $n_{i_1 i_2 \ldots i_r}$ are nonnegative integers) [1]. In particular, the plane partitions of n are two-dimensional arrays of nonnegative integers in the first quadrant subject to a nonincreasing condition along rows and columns. For example there are six plane partitions of 3:

$$\begin{matrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 3 & 0 & 0 & \cdots \end{matrix} \quad \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{matrix} \quad \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{matrix} \quad \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{matrix} \quad \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{matrix}, \text{ and } \begin{matrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}.$$

According to Andrews [1], there is much of interest when the dimension is 1 or 2, and very little when the dimension exceeds 2. In this paper, we use ideas from the theory of algorithms of differentiation to obtain identities for some one-dimensional compositions involving polygonal and cubic numbers.

Concerning higher-dimensional partitions, we recall that, Stanley has shown that numerous partition and permutation problems can be treated through the use of $\mathcal{P}$-partitions, i.e., order-preserving maps from a partially ordered set $\mathcal{P}$ to a chain with special rules specifying where equal values may occur [1, 21, 22]. For instance, if $\mathcal{P}$ is a *p*-element chain, then a $\mathcal{P}$-partition of a positive integer *n* is equivalent to an ordinary partition of n into at most *p* parts. Some relationships between $\mathcal{P}$-partitions and the counting of chains in the set of order ideals of $\mathcal{P}$ ordered by inclusion are well described by Stanley in [21] and [22]. Furthermore, Stanley's work on $\mathcal{P}$-partitions allows him to deduce results (regarding *r*-dimensional partitions), easily from a general reciprocity theorem [1, 21]. We also recall that the first author et al. in [6] describe some compositions of dimension three by using $\mathcal{P}$-partitions.

This paper is organized as follows: Some of the basic definitions and notations concerning posets and $\mathcal{P}$-partitions are included in Section 2. In Section 3, we describe numbers which can be written as a sum of three square of numbers of a given shape, in Section 4, we solve Diophantine equations involving cubic numbers, in particular, we describe some sequences whose terms can be written as a sum of four cubes with two of them equal. Finally, in Section 5, we give examples of compositions defined in Section 3 with the help of some algorithms of differentiation and $\mathcal{P}$-partitions.

Remark 1. We will use the customary symbols $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ for the set of natural numbers, integers and real numbers, respectively.

2. Preliminaries

This section introduces some other basic definitions, and notations to be used throughout the paper [5, 6, 10, 11, 19, 21, 22].

2.1. Posets

An *ordered set* (or *partially ordered set* or *poset*) is an ordered pair of the form $(\mathcal{P}, \leq)$ of a set $\mathcal{P}$ and a binary relation $\leq$ contained in $\mathcal{P} \times \mathcal{P}$, called the *order* (or the *partial order*) on $\mathcal{P}$ such that $\leq$ is reflexive, antisymmetric and transitive [11]. The elements of $\mathcal{P}$ are called the *points* of the ordered set. We will write $x < y$ for $x \leq y$ and $x \neq y$, in this case we will say x is *strictly less than* y. An ordered set will be called *finite* (*infinite*) if and only if the underlying set is finite (infinite). Usually we shall be a little slovenly and say simply $\mathcal{P}$ is an *ordered set* where it is necessary to specify the order relation overtly we write $(\mathcal{P}, \leq)$.

Let $\mathcal{P}$ be an ordered set and let $x, y \in \mathcal{P}$ we say x is *covered* by y if $x < y$ and $x \leq z < y$ implies $z = x$.

An ordered set C is called a *chain* (or a *totally ordered set* or a *linearly ordered set*) if and only if for all $p, q \in C$ we have $p \leq q$ or $q \leq p$ (i.e., p and q are comparable). On the other hand, an ordered set $\mathcal{P}$ is called an *antichain* if $x \leq y$ in $\mathcal{P}$ only if $x = y$ [11].

Let $\mathcal{P}$ be an ordered set. A chain C in $\mathcal{P}$ will be called a *maximal chain* if and only if for all chains $K \subseteq \mathcal{P}$ with $C \subseteq K$ we have $C = K$.

If n is a positive integer we let $\mathbf{n}$ denote the n-element poset with the special property that any two elements are comparable [22]. We also define a subposet Q of a poset $\mathcal{P}$ to be *convex* if $y \in Q$ whenever $x < y < z$ in $\mathcal{P}$ and $x, z \in Q$.

Let $\mathcal{P}$ be a finite ordered set. We can represent $\mathcal{P}$ by a configuration of circles (representing the elements of $\mathcal{P}$) and interconnecting lines (indicating the covering relation). The construction goes as follows.

(1) To each point $x \in \mathcal{P}$, associate a point $p(x)$ of the Euclidean plane $\mathbb{R}^2$, depicted by a small circle with center at $p(x)$.

(2) For each covering pair $x < y$ in $\mathcal{P}$, take a line segment $l(x, y)$ joining the circle at $p(x)$ to the circle at $p(y)$.

(3) Carry out (1) and (2) in such a way that

(a) if $x < y$, then $p(x)$ is lower than $p(y)$,

(b) the circle at $p(z)$ does not intersect the line segment $l(x, y)$ if $z \neq x$ and $z \neq y$.

A configuration satisfying (1)-(3) is called a *Hasse diagram* or *diagram* of P. In the other direction, a diagram may be used to define a finite ordered set; an example is given below, for a poset $(\mathcal{M}_3, \preceq) = \{(i, j) | 0 \leq i \leq 3, 0 \leq j \leq 3\} \subset \mathbb{N}^2$ whose points satisfy the following condition:

$$(i, j) \preceq (i', j') \text{ if and only if } i \leq i' \text{ and } j \leq j', \text{ for all } (i, j), (i', j') \in \mathcal{M}_3. \quad (3)$$

In this case, $\mathbb{N}$ has been equipped with its natural ordering.

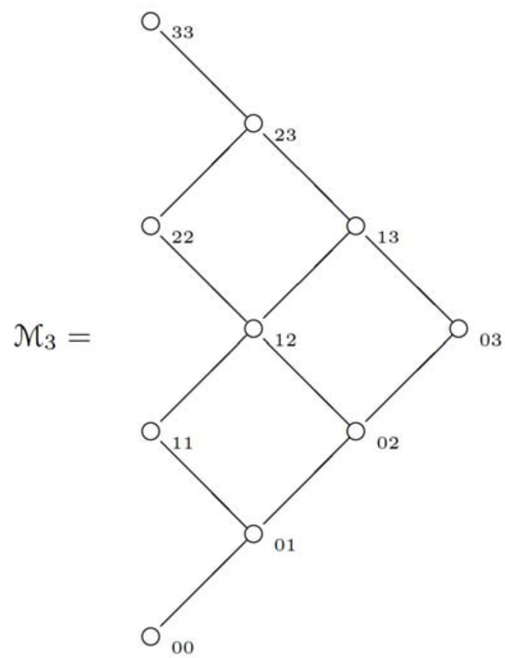


Figure 2

Let $(\mathcal{P}, \preceq)$ and $(\mathcal{Q}, \trianglelefteq)$ be ordered sets and let $f : \mathcal{P} \to \mathcal{Q}$ be a map. Then *f* is called an *order-preserving function* if and only if for all $x, y \in \mathcal{P}$ we have:

$$x \preceq y \Rightarrow f(x) \trianglelefteq f(y).$$

We shall say that two posets P and Q are *isomorphic* if there exists an order-preserving bijection $f : P \to Q$, whose inverse is order-preserving. In such a case, we shall write $P \cong Q$.

Let $(\mathcal{P}, \preceq)$ and $(\mathcal{Q}, \trianglelefteq)$ be ordered sets. Then $f : \mathcal{P} \to \mathcal{Q}$ is called an (*order*) *embedding* if and only if f is injective, and for all $x, y \in \mathcal{P}$, we have:

$$x \preceq y \Leftrightarrow f(x) \trianglelefteq f(y).$$

If $(P, \leq)$ and $(Q, \trianglelefteq)$ are posets, then the *direct* (or cartesian) *product* of P and Q is the poset $(P \times Q, \preceq)$ on the set $\{(x, y) : x \in P \text{ and } y \in Q\}$ such that $(x, y) \preceq (x', y')$ in $P \times Q$ if $x \leq x'$ in P and $y \trianglelefteq y'$ in Q. To draw the Hasse diagram of $P \times Q$ (when P and Q are finite), draw the Hasse diagram of P, replace each element x of P by a copy Q_x of Q and connect corresponding elements of Q_x and Q_y (with respect to some isomorphism $Q_x \cong Q_y$) if x and y are connected in the Hasse diagram of P.

A further operation that we wish to consider is the *dual* of a poset P. This is the poset P^* on the same set as P, but such that $x \leq y$ in P^* if and only if $y \leq x$ in P. If P and P^* are isomorphic, then P is called *self-dual*.

An *order ideal* of a poset $(\mathcal{P}, \leq)$ is a subset I of $\mathcal{P}$ such that if $x \in I$ and $y \leq x$, then $y \in I$. We let $J(\mathcal{P})$ denote the set of all order ideals of $\mathcal{P}$, ordered by inclusion. In particular, we define the order ideal or *down-set* of $a \in \mathcal{P}$ to be $a_\Delta = \{q \in \mathcal{P} : q \leq a\}$. Dually, $a^\nabla = \{q \in \mathcal{P} : a \leq q\}$ is the *filter* or *up-set* of a [19].

Note that, k-element antichains in $\mathcal{P}$ correspond to elements of $J(\mathcal{P})$ that cover exactly k-elements.

If x, y belong to a poset $\mathcal{P}$, then an *upper bound* of x and y is an element $z \in \mathcal{P}$, satisfying $x \leq z$ and $y \leq z$. A *least upper bound* of x and y is an upper bound z of x and y such that every upper bound w of x and y satisfies

$z \leq w.$ If a least upper bound of x and y exists, then it is clearly unique and is denoted $x \vee y.$ Dually one can define the greatest lower bound $x \wedge y,$ when it exists. A *lattice* is a poset L for which every pair of elements has a least upper bound and greatest lower bound. We say that a poset $\mathcal{P}$ *has a* $\hat{0}$ if there exists an element $\hat{0} \in \mathcal{P}$ such that $\hat{0} \leq x$ for all $x \in \mathcal{P}.$ Similarly, $\mathcal{P}$ *has a* $\hat{1}$ if there exists $\hat{1} \in \mathcal{P}$ such that $x \leq \hat{1}$ for all $x \in \mathcal{P}.$ Clearly all finite lattices have $\hat{0}$ and $\hat{1}.$ Since the union and intersection of order ideals is again an order ideal, it follows from the well-known distributivity of set union and intersection over one another that $J(\mathcal{P})$ is indeed a distributive lattice [22].

A finite nonnegative *lattice path* in the plane (with unit steps to the right and down) is a sequence $L = (v_1, v_2, ..., v_k),$ where $v_i \in \mathbb{N}^2$ and $v_{i+1} - v_i = (1, 0)$ or $(0, -1)$ [22].

As an example, let $\mathcal{S}$ be the set of all sequences of nonnegative integers $\{f_i\}_{i=1}^{\infty},$ where only a finite number of terms are not null. To each sequence $\{f_i\} \in \mathcal{S}$ there is associated a partition $\lambda = (1^{f_1} 2^{f_2} 3^{f_3} \ldots),$ where f_i denotes the number of times that the part i occurs in $\lambda.$ $(\mathcal{S}, \preceq)$ is a poset if $\preceq$ is defined in such a way that

$$\{f_i\} \preceq \{g_i\}, \text{ if } f_i \leq g_i, \text{ for all } i.$$

In fact $\mathcal{S}$ is a lattice with operations $\wedge$ and $\vee$ given by

$$\{f_i\} \wedge \{g_i\} = \{\min(f_i, g_i)\}, \qquad \{f_i\} \vee \{g_i\} = \{\max(f_i, g_i)\}.$$

A subset $C \subset \mathcal{S}$ such that $\{f_i\} \in C$ and $\{g_i\} \preceq \{f_i\}$ imply $\{g_i\} \in C$ is called an *ideal partition*. Moreover, two ideal partitions C_1 and C_2 are equivalent if $P(C_1, n) = P(C_2, n)$ for all integer n, in such a case we write, $C_1 \sim C_2.$ We recall here that the fundamental problem for ideal partitions consists of giving a complete characterization of its equivalence classes [1].

Given a finite poset $\mathcal{P}$ with $|\mathcal{P}| = n$ in [22] it is defined an *extension of* $\mathcal{P}$ *to a total order or linear extension of* $\mathcal{P}$ as an order-preserving bijection $\sigma : \mathcal{P} \to \mathbf{n}$. The number of extensions of $\mathcal{P}$ to a total order is denoted $e(\mathcal{P})$. Actually, $e(\mathcal{P})$ is also equal to the number of maximal chains of $J(\mathcal{P})$.

We may identify a maximal chain of $J(\mathcal{P})$ with a certain type of lattice path in Euclidean space as follows [22]. Let $C_1, \dots, C_k$ be a partition of $\mathcal{P}$ into chains. Define a map $\delta : J(\mathcal{P}) \to \mathbb{N}^k$ by:

$$\delta(I) = (|I \cap C_1|, |I \cap C_2|, \dots, |I \cap C_k|).$$

If we give $\mathbb{N}^k$ the obvious product order, then δ is an injective lattice homomorphism that is cover-preserving (and therefore rank-preserving). Thus in particular $J(\mathcal{P})$ is isomorphic to a sublattice of $\mathbb{N}^k$. Given $\delta : J(\mathcal{P}) \to \mathbb{N}^k$, as above, define $\Gamma_\delta = \bigcup_T cx(\delta(T))$, where *cx* denotes convex hull in $\mathbb{R}^k$, T ranges over all intervals of $J(\mathcal{P})$ that are isomorphic to Boolean algebras. Thus Γ_δ is a compact polyhedral subset of $\mathbb{R}^k$. It is then clear that the number of maximal chains in $J(\mathcal{P})$ is equal to the number of lattice paths from the origin $(0, 0, \dots, 0) = \delta(\hat{0})$ to $\delta(\hat{1})$, with unit steps in the direction of the coordinate axes. In other words, $e(\mathcal{P})$ is equal to the number of ways of writing

$$\delta(\hat{1}) = v_1 + v_2 + \cdots + v_n, \tag{4}$$

where each v_i is a unit coordinate vector in $\mathbb{R}^k$ and where $v_1 + v_2 + \cdots + v_i \in \Gamma_\delta$, for all i.

For example, let $\mathcal{M} = \mathbf{2} \times \mathbf{n}$, and take $C_1 = \{(2, j) \mid j \in \mathbf{n}\}$, $C_2 = \{(1, j) \mid j \in \mathbf{n}\}$. Then $\delta(J(\mathcal{M})) = \{(i, j) \in \mathbb{N}^2 \mid 0 \le i \le j \le n\}$. For example when $n = 3$ we obtain Figure 2. Hence $e(\mathcal{M})$ is equal to the number of lattice paths from $(0, 0)$ to (n, n) with steps $(1, 0)$ and $(0, 1)$, which never rise above the main diagonal $x = y$ of the plane (x, y)-plane. It can be shown that $e(\mathbf{2} \times \mathbf{n}) = \frac{1}{n+1}\binom{2n}{n} = C_n$. These numbers are called *Catalan numbers* [22].

We let $\mathcal{M}_n$ denote the poset $(\delta(J(\mathbf{2} \times \mathbf{n})), \preceq)$, where $\preceq$ is the relation defined in (3).

2.2. $\mathcal{P}$-partitions

The theory of $\mathcal{P}$-partitions which is a common generalization of the theory of partitions and the theory of compositions was introduced by Stanley in 1972 [1, 6, 21]. In order to give a definition of $\mathcal{P}$-partition we must define labeled ordered sets. In this case if $(\mathbb{N}, \leq)$ is the set of natural numbers equipped with its natural ordering and $(\mathcal{P}, \trianglelefteq)$ is a poset with $|\mathcal{P}| = p$, then a *labeling* w of $\mathcal{P}$ is a bijection $w : \mathcal{P} \to \{1, 2, ..., p\} \subset \mathbb{N}$. A labeling w is called a *natural labeling* if it satisfies

$$x \trianglelefteq y \text{ implies } w(x) \leq w(y).$$

w is called a *strict labeling* if

$$x \trianglelefteq y \text{ implies } w(x) \geq w(y).$$

An ordered set together with a labeling w is called a *labeled ordered* set.

If w is a labeling of $(\mathcal{P}, \trianglelefteq)$, then a $(\mathcal{P}, w)$*-partition of n* or *poset partition* is a map $\sigma : \mathcal{P} \to \mathbb{N}$ satisfying the conditions:

1. $x \trianglelefteq y$ in $\mathcal{P}$ implies $\sigma(x) \geq \sigma(y)$, i.e., σ is order-reversing,

2. $x \triangleleft y$ in $\mathcal{P}$ and $w(x) > w(y)$ implies $\sigma(x) > \sigma(y)$,

3. $\sum_{x \in \mathcal{P}} \sigma(x) = n$.

If w is a natural labeling, then σ is called a $\mathcal{P}$*-partition*. If w is a strict labeling, then σ is called a *strict* $\mathcal{P}$*-partition*. If σ is a $(\mathcal{P}, w)$-partition, then the values $\sigma(x)$, $x \in \mathcal{P}$, are called the *parts* of σ.

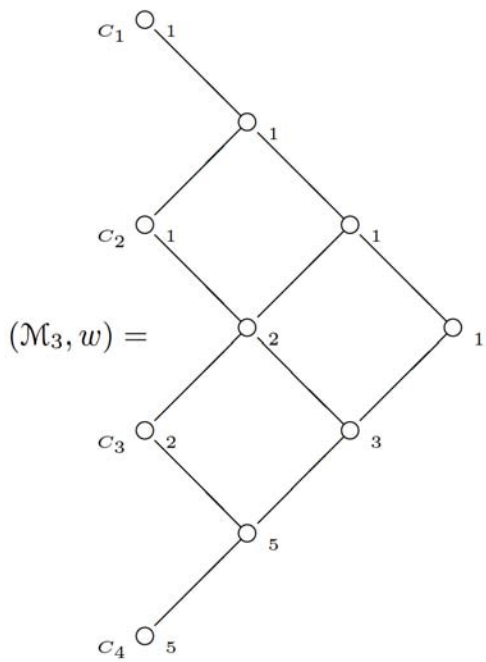


Figure 3

Figure 3 above, shows a $(\mathcal{M}_3, w)$-partition of $22 = C_1 + C_2 + C_3 + C_4$, where C_i denotes the ith Catalan number. In this case, we have labeled $\mathcal{M}_3$ with a map $w : \mathcal{M}_3 \to \mathbb{P}$ such that $\mathbb{P} = \{1, \dots, 10\}$, $w(i, 3) = 4 - i$, if $0 \leq i \leq 3$, $w(j, 2) = 7 - j$, if $0 \leq j \leq 2$, $w(k, 1) = 9 - k$, if $0 \leq k \leq 1$, and $w(0, 0) = 10$.

Some relationships between $\mathcal{P}$-partitions and the counting of chains in the set of order ideals of $\mathcal{P}$ ordered by inclusion are well described by Stanley in [21] and [22]. Actually, he describes in [21] the following relation between the number of some $\mathcal{P}$-partitions of a positive integer n, denoted a_n, and the number $e(\mathcal{P})$ of extensions of $\mathcal{P}$ to a total order. In this case, we have considered that $|\mathcal{P}| = p$ [2, 7, 10]:

$$a_n = \frac{e(\mathcal{P})n^{p-1}\left(1 + o\left(\frac{1}{n}\right)\right)}{p!(1-p)!} \text{ as } n \to \infty.$$

We let $\mathcal{A}(\mathcal{P}, w)$ denote the class of all $(\mathcal{P}, w)$ partitions. Define two labelings w, w' to be equivalent (denoted $w \sim w'$) if $\mathcal{A}(\mathcal{P}, w) = \mathcal{A}(\mathcal{P}, w')$.

In [21, 22], Stanley mentioned a number of interesting combinatorial problems concerning labeling of ordered sets, for example: Given a labeled ordered set $(\mathcal{P}, w)$, how many labelings are equivalent to w?

Most of the preceding concepts can be extended to infinite posets. For example the notion of a $\mathcal{P}$-partition can be extended in such a way that the following finiteness conditions hold:

1. For every element $x \in \mathcal{P}$ there is some $\mathcal{P}$-partition σ such that $\sigma(x) > 0$.

2. There exist only finitely many $\mathcal{P}$-partitions of any given integer n.

Therefore, if $\mathcal{P}$ is a poset, then an order-reversing map w from $\mathcal{P}$ to the set of nonnegative integers is a labeling of $\mathcal{P}$ if additionally only finitely many x have $w(x) > 0$. In this case a labeling w of $\mathcal{P}$ is a $\mathcal{P}$-partition of n if $\sum_{x \in \mathcal{P}} w(x) = n$ [1].

Remark 2. We have considered only the cases for which a $\mathcal{P}$-partition is an order-reversing map. The order-preserving case can be obtained simply by dualizing the poset $\mathcal{P}$ [21].

3. The Main Results

In this section, we follow ideas of the first author who uses in [3] theory of algorithms of differentiation in order to give advances to the solution of the problems (1)-(3) mentioned in pages 101 and 102. To do that, it is associated to a given $\mathcal{P}$-partition a suitable set of partitions.

3.1. Representation of posets over a set of positive integers

Let $(\mathbb{N}^*, \leq)$ be the set of positive integers endowed with the usual order and $(\mathcal{P}, \leq')$ an ordinary poset $\mathcal{P} \neq \varnothing$. A *representation* of the poset $\mathcal{P}$ over

$\mathbb{N}^*$ is a system of positive integers with the form:

$$\Lambda = (\Lambda_0; (n_x, \lambda_x) | x \in \mathcal{P}), \tag{5}$$

where $\Lambda_0 \subset \mathbb{N}^*$, $(\Lambda_0 \neq \varnothing)$ λ_x is a partition of the positive number n_x with parts in Λ_0 and $|\lambda_x|$ is the size or cardinal of λ_x. Moreover,

$$x \leq' y \Rightarrow n_x \leq n_y, \quad |\lambda_x| \leq |\lambda_y| \text{ and } \max\{\lambda_x\} \leq \max\{\lambda_y\}. \tag{6}$$

Two representations (over $\mathbb{N}^*$), $\Lambda^1 = (\Lambda_0^1; (n_x^1, \lambda_x^1) | x \in \mathcal{P})$ and $\Lambda^2 = (\Lambda_0^2; (n_x^2, \lambda_x^2) | x \in \mathcal{P})$ of a given poset $\mathcal{P}$ are equivalent if and only if $\Lambda_0^1 = \Lambda_0^2$ and $n_x^1 = n_x^2$ for each $x \in \mathcal{P}$. In this case the fundamental problem consists of characterizing the corresponding equivalence classes by calculating $P(\Lambda_0, n_x)$ for each $x \in \mathcal{P}$. Note that, this problem is similar to the problem of classification of ideal partitions.

If $\lambda = (1^{f_1} \dots i^{f_i} \dots m^{f_m})$ is a partition with parts in a set H_0 the uth *substitution derivative* of λ with respect to the part i is a partition with parts in H_0 obtained from λ by substituting one or several occurrences of $i \in \lambda$ by the number $u \in H_0$. We let $\lambda_i(u) = \frac{\partial \lambda}{\partial i}(u)$ denote this substitution and write:

$$\lambda_i(u) = \frac{\partial \lambda}{\partial i}(u) = (1^{f_1} \dots u i^{f_i - 1} \dots m^{f_m}),$$

$$\frac{\partial^n \lambda}{\partial i^n}(u) = (1^{f_1} \dots (i-1)^{f_{(i-1)}} u i^{f_i - n} (i+1)^{f_{(i+1)}} \dots m^{f_m}), \text{ if } n \leq f_i.$$

Furthermore, different substitutions can be applied to the same partition λ. We let

$$\frac{\partial^k \lambda}{\partial i_1 \partial i_2 \dots \partial i_k}(u_1, u_2, \dots, u_k) = (1^{f_1} \dots u_t i_t^{f_{i_t} - 1} \dots m^{f_m}), \quad 1 \leq t \leq k$$

denote such substitutions.

If $(\mathcal{P}, \leq')$ is a poset, then we say that a pair $(a, b) \in \mathcal{P}$ is *L-suitable* or *suitable* for differentiation L if a and b are incomparable and there exists a chain $C = \{c_1 <' c_2 \cdots <' c_n\}$ such that $a <' c_1$, $b <' c_1$ and

$$\mathcal{P} = a + b_{\Delta} + C.$$

The differentiation of a poset $\mathcal{P}$ with a pair (a, b), L-suitable is a poset $\mathcal{P}'_{(a,b)}$ such that

$$\mathcal{P}'_{(a,b)} = a^{-} + \mathcal{P}/(a^{\nabla}) + C^{-} + C^{+},$$

where $C^{-} = \{c_1^{-} < \cdots < c_n^{-}\}$, $C^{+} = \{c_1^{+} < \cdots < c_n^{+}\}$ are chains with $a^{-} < c_1^{-} = a$ and for each i, $1 \leq i \leq n$, $c_i^{-} < c_i^{+}$, a^{-}, c_i^{-} and c_i^{+} inherit all relations that points a and c_i had with the other points in $\mathcal{P}$. In $\mathcal{P}/a^{\nabla}$ points and relations have not changes. In fact, these new relations and the original ones defined in $\mathcal{P}/a^{\nabla}$ induce all relations in $\mathcal{P}'_{(a,b)}$. Figure 4 shows the Hasse diagram of this reduction.

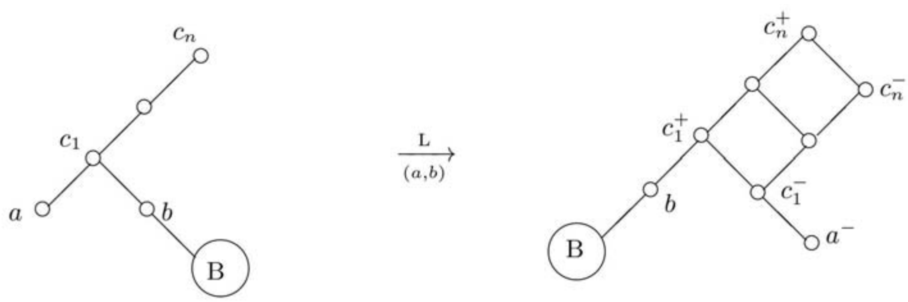


Figure 4

Henceforth, we shall assume that if Λ is a representation of a poset $\mathcal{P}$ with a pair of points (a, b), L-suitable with t fixed, $k_x \in \mathbb{N}$ for each $x \in \mathcal{P}$, then

$$\lambda_x = ((k_x)^t), \text{ for all } x \in b^{\nabla} + B,$$

$$\frac{\partial^{|\lambda_a|} \lambda_x}{\partial k_x^{|\lambda_a|}}(\lambda_a) = ((I_x)(J_x)), \text{ if } x \in C,$$

$$I_x = \lambda_a,$$

$$|\lambda_a| < |\lambda_{c_1}|,$$

$$0 < n_{c_1} - n_a - |J_{c_1}|k_{c_1} < n_a,$$

$$\alpha = n_{c_1} - n_a - |J_{c_1}|k_{c_1} \leq \max\{\lambda_a\}. \tag{7}$$

Representation Λ' of the derived poset $\mathcal{P}'_{(a,b)}$ is given by the following formulae if $2 \leq j \leq i$;

$$\Lambda'_0 = \Lambda_0,$$

$$n'_{a^-} = \alpha, \quad \lambda'_{a^-} = ((\alpha)^1),$$

$$n'_{c_1^-} = n_a, \quad \lambda'_{c_1^-} = \lambda_a,$$

$$n'_{c_i^-} = |I_{c_i}|k_{c_i}, \quad \lambda'_{c_i^-} = ((k_{c_i})^{|I_{c_i}|}),$$

$$n'_{c_i^+} = n_{c_i}, \quad \lambda'_{c_i^+} = ((\alpha)^1\lambda'_{c_1^-}(|I_{c_j}|(k_{c_j} - k_{c_{j-1}}))^{\delta_i} k_{c_i}^{|J_{c_i}|}),$$

$$(n'_x, \lambda'_x) = (h_x, (h_x^1)), \text{ for all } x \in b_\Delta, \text{ if } B^\nabla \cap \{a^-\} \neq \varnothing,$$

$$h_x = \min\{\alpha, \min\{n_a - n_x \mid x \in B\}\},$$

$$(n'_x, \lambda'_x) = (n_x, \lambda_x), \text{ for all } x \in b_\Delta, \text{ if } B^\nabla \cap \{a^-\} = \varnothing,$$

$$\delta_i = \begin{cases} 1, & i \geq 2, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Remark 3. If $i = 1$ in formulas (8), then we assume that the formula for $\lambda'_{c_1^+}$ is obtained by applying the same formula given for $\lambda'_{c_i^+}$ and deleting the term $|I_{c_1}|(k_{c_j} - k_{c_{j-1}})$. Furthermore, $\lambda'_{c_1^+} = n_{c_1}$.

3.2. Compositions of polygonal numbers

In order to describe compositions of multiples of polygonal numbers of positive rank, we consider representations of a poset $(\mathcal{P}_{c_1}, \leq')$, with Hasse diagram showed in Figure 5, in this case, $\mathcal{P}_{c_1} = a^{\nabla} + b_{\Delta}$, the pair (a, b) is L-suitable, $a = a_1$, $a_1 <' c_1 <' c_2 <' \cdots <' c_n$ is a chain, $b_{\blacktriangle} = b_1$ and $a_{\Delta} \backslash a = \varnothing$.

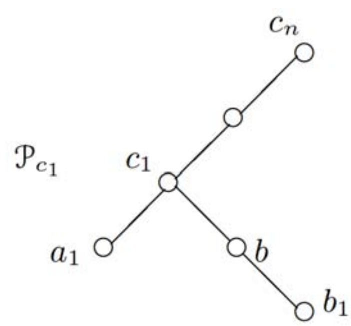


Figure 5

In this section, we describe some compositions of numbers of type $n'_{c_j^+} = n_0 p_{j+2}^{n_0}$, $j \geq 1$, via differentiations of posets $\mathcal{P}_{c_i} = \mathcal{P}_{c_1} \backslash a_1 + a_i$ with $c_n = c_j$ in the chain C. Such posets are defined in such a way that, relations between points of $\mathcal{P}_{c_1} \backslash a_1$ have not changes. Furthermore, $a_i <' c_i$, (a_i, c_{i-1}) is a pair of points L-suitable and a_i inherits all relations that $a_{(i-1)}$ had with points of $c_{(i-2)_{\Delta}}$ (see Figure 5 with $c_0 = b$).

Let Λ_{c_1} be a representation of poset $\mathcal{P}_{c_1}$ such that $\Lambda_{c_1 0} \supset \{t \in \mathbb{N} \mid t = p_r^{n_0}\}$, where $p_r^{n_0}$ denotes the n_0-gonal number of rank r, $n_0 \geq 5$, fixed. Furthermore,

$$(n_{b_1}, \lambda_{b_1}) = (n_0 p_1^{n_0}, ((p_1^{n_0})^{n_0})),$$

$$(n_b, \lambda_b) = (n_0 p_2^{n_0}, ((p_2^{n_0})^{n_0})),$$

$$(n_{a_1}, \lambda_{a_1}) = (p_2^{n_0} + p_3^{n_0}, ((p_2^{n_0})^1 (p_3^{n_0})^1)),$$

$$(n_{c_i}, \lambda_{c_i}) = (n_0 p_{i+2}^{n_0}, ((p_{i+2}^{n_0})^{n_0})), \quad i \geq 1. \tag{9}$$

Each representation Λ_{c_i} of $\mathcal{P}_{c_i}$ is defined in such a way that $\Lambda_{c_1 0} = \Lambda_{c_i 0}$ for each $x \neq a_i$, n_x, λ_x are those defined in the representation Λ_{c_1} of $\mathcal{P}_{c_1}$, in this case, $n_{a_i} = p_{i+2}^{n_0} + p_{\iota}^{n_0}$ and $\lambda_{a_i} = ((p_{i+2}^{n_0})^1(p_{\iota}^{n_0})^1)$, $2 \leq \iota < i+2$ and $1 \leq i \leq j$.

In a poset $\mathcal{P}'_{(a_i, c_{i-1})}$, $1 \leq \iota \leq j$, we denote, $d_\iota = |\, c^+_{\iota_\Delta} \backslash c^-_{\iota_\Delta} \,|$, $n'_{c^+_\iota} = n_0 p^{n_0}_{d_\iota}$, where $p^{n_0}_{d_\iota}$ is the d_ι th n_0 -gonal number, in this case, $p_1^{n_0} = 1$.

If $\Delta_{c^+_\iota} = n'_{c^+_\iota} - n'_{c^+_{\iota-1}}$, $\Delta_{c^+_1} = n'_{c^+_1} - n_b$, $\Delta_b = n'_b - n'_{b_1}$, $\Delta_{b_1} = n'_{b_1}$, then a partition λ of $n'_{c^+_j}$ in the poset $\mathcal{P}'_{(a_1, c_0)}$ is of type I, if it has at least d_j parts with the form:

$$\lambda = ((n'_x)^{f_x}(\Delta_{c^+_\iota})^{f_{c^+_\iota}}(\Delta_b)^{f_b}(\Delta_{b_1})^{f_{b_1}}), \quad x \in c^+_{j_\Delta} \backslash c^-_{j_\Delta},$$

where for each $z \in \mathcal{P}$, $f_z \in \{0, 1\}$ if $z \in c^+_{j_\Delta} \backslash c^-_{j_\Delta}$, $f_z = 0$ otherwise. Furthermore, if $n'_x \in \lambda$, then $n'_y \notin \lambda$ if $y \neq x$ and $y \in c^+_{j_\Delta} \backslash c^-_{j_\Delta}$, $1 \leq \iota \leq j$.

A partition λ is of type II for a positive number $n'_{c^+_j}$ in a poset $\mathcal{P}'_{(a_i, c_{i-1})}$ if it is not of type I. Moreover, either $\lambda = \lambda'_{c^+_j}$ or its parts have the form:

$$\lambda = (s^{l_1} t^{l_1} u^{l_2} v^{f_h} x^{f_5} y^{f_6}), \quad l_1, l_2, f_h, f_5, f_6 \in \{0, 1\},$$

exponents 1, 0, indicate the existence of the corresponding term in the partition, in this case:

$$s = (n'_{a_i^-}),\ t = \lambda_{a_i} = ((p_{i+2}^{n_0})^1(p_\iota^{n_0})^1), \quad 2 \leq \iota < i+2 \leq j+2,$$

$$u = (n'_{a_i^-} + p_{i+2}^{n_0} + p_\iota^{n_0}),\ v = \mu_h,\ x = \Delta_{c^+_g},$$

$$y = ((p^{n_0}_{d_\varpi})^{(n_0 - 2)}) = ((p^{n_0}_{d_\varpi})^{(J_{c_\varpi})}), \quad 1 \leq g \leq j,\ 2 \leq h \leq j,$$

$$1 \leq i, \varpi \leq j,\ l_1 = 1 \Rightarrow l_2 = 0,\ l_2 = 1 \Rightarrow l_1 = 0$$

and if $p_0^3 = 0,$ then in the poset $\mathcal{P}'_{(a_1, c_0)}$ we have that

$$\mu_h = [20 + 6(n_0 - 5) + (2n_0 - 4)(h - 2)]^m ((p_{d_h}^{n_0})^2)^{1-m}, \; m \in \{0, 1\},$$

$$\Delta_{c_g^+} = 54 + 19(n_0 - 6) + 4p_{(n_0-6)}^3 + n_0(n_0 - 2)(g - 1), \; g \geq 1, \; n_0 \geq 6,$$

$$\Delta_{c_g^+} = 35 + 15(g - 1), \text{ if } n_0 = 5,$$

$$\Delta_{c_2^-} = n'_{c_2^-} - n'_{c_1^-} - n'_{a_1^-},$$

$$\Delta_{c_h^-} = n'_{c_h^-} - n'_{c_{h-1}^-}, \; 3 \leq h \leq j,$$

$$\Delta_{c_h^-} = [20 + 6(n_0 - 5) + (2n_0 - 4)(h - 2)], \quad 2 \leq h \leq j.$$

Values $\Delta_{c_h^\pm} = n'_{c_h^\pm} - n'_{c_{h-1}^\pm}$ in $\mathcal{P}'_{(a_i, c_{i-1})}$ are those used in $\mathcal{P}'_{(a_1, c_0)}$ by considering the corresponding translations. Furthermore, if we let H denote the set of parts of type II and assume $H = \{s, t, u, v, x, y\},$ then such parts are assigned to a partition of type II of $n_{c_j} = n_0 p_{j+2}^{n_0}$ as follows:

Let $\Gamma_{(a_i, c_{i-1})}$ be the graph whose vertices are points of the chains $a_i^- + C^- + C^+$ of $\mathcal{P}'_{(a_i, c_{i-1})}$ with edges defined according to the following diagram for $1 \leq i \leq j$:

$$\begin{array}{ccccccccccc}
 & & & & & c_i^+ & & & & & \\
 & & & & \nearrow & & \searrow & & & & \\
\Gamma_{(a_i, c_{i-1})} = & a_i^- & \rightarrow & c_i^- & & & & c_{i+1}^+ & \rightarrow & \cdots \rightarrow c_{j-1}^+ & \rightarrow & c_j^+ \\
 & & & & \searrow & & \nearrow & & & \nearrow & & \uparrow \\
 & & & & & c_{i+1}^- & \rightarrow & \cdots & \rightarrow & c_{j-1}^- & \rightarrow & c_j^- .
\end{array}$$

A partition of type II for n_{c_j} is obtained from $\Gamma_{(a_i, c_{i-1})}$ by assigning a unique element of H (keeping the order induced by $\mathcal{P}'_{(a, c_i)}$) to each vertex of a path $P \subset \Gamma_{(a_i, c_{i-1})}$ which has as initial vertex a point in $a_i^- + C^-$ with final vertex c_j^+. Therefore, numbers $\Delta_{c_{i+\iota}}$ or $(p_{d_{i+\iota}}^{n_0})^2$ are associated to the vertex $c_{i+\iota}^-$, $\iota \geq 1$ whereas, vertices $c_i^+, c_{i+\iota}^+$, have associated numbers of the form $(p_{d_i}^{n_0})^{n_0-2}$, $(p_{d_{i+\iota}}^{n_0})^{n_0-2}$ or $\Delta_{c_{i+\iota}^+}$, respectively. Furthermore, if $c_\iota^- \in P$ is the maximum of the chain C^- in *P*, then to each vertex of type $c_k^+ \in P$ with $\iota < k$ it is assigned the part $\Delta_{c_k^+}$ and the number $(p_{d_\iota}^{n_0})^{n_0-2}$ it is associated to the minimum c_ι^+ of the chain C^+ in *P*.

For $\iota \geq 1$, a term $(p_{d_\iota}^{n_0})^{n_0-2}$, occurs only once in a partition and a term of the shape $(p_{d_{i+\iota}}^{n_0})^2$ occurs if and only if vertex $c_{i+\iota}^-$ is the minimum in the chain C^- in a path *P*. Furthermore, note that $a_i^- \in P\,(c_i^- \in P)$ if and only if $l_1 = 1$ ($l_2 = 1$ in such a case c_i^- is the minimum of the path *P*).

For example the d_3 partitions of type I of $n'_{c_1^+} = n_0 p_3^{n_0}$ are

$$n'_{b_1} + \Delta_b + \Delta_{c_1^+},$$
$$n'_b + \Delta_{c_1^+},$$
$$n'_{c_1^+}.$$

The $d_3 - 1$ partitions of type II of $n_0 p_3^{n_0}$ are

$$(n'_{a_1^-})^1 + \lambda_{c_1^-} + (p_{d_1}^{n_0})^{(n_0-2)} = ((n'_{a_1^-})^1(p_2^{n_0})^1(p_3^{n_0})^1(p_{d_1}^{n_0})^{(n_0-2)}) \text{ and}$$

$$(((n'_{a_1^-})^1 + p_2^{n_0} + p_3^{n_0})^1(p_{d_1}^{n_0})^{(n_0-2)}). \quad (10)$$

In particular, if $n_0 = 5$, then partitions of type I of 60 are 60 = 5 + 20 + 35 = 25 + 35 whereas partitions of type II are 7 + 12 + 5 + 12 + 12 + 12 = 24 + 12 + 12 + 12.

Partitions of type I of 110 are 110 = 5 + 20 + 35 + 50 = 25 + 35 + 50 = 60 + 50. Partitions of type II are 7 + 12 + 5 + 12 + 12 + 12 + 50 = 24 + 12 + 12 + 12 + 50 = 7 + 12 + 5 + 20 + 22 + 22 + 22 = 24 + 20 + 22 + 22 + 22 = 22 + 22 + 22 + 22 + 22 = 17 + 22 + 5 + 22 + 22 + 22 = 44 + 22 + 22 + 22 = 10 + 12 + 22 + 22 + 22 + 22. The last one obtained from poset $\mathcal{P}'_{(a_2, c_1)}$ (in fact, $\lambda'_{c_2^+} = 10 + 12 + 22 + 22 + 22 + 22$).

Note that, in partition, 5 + 20 + 35 + 50, according to the derived poset of the poset showed in Figure 5, we have $n_{b_1} = 5$, $\Delta_b = 20$, $\Delta_{c_1^+} = 35$, $\Delta_{c_2^+} = 50$.

Partition $\lambda'_{c_2^+} = 7 + 5 + 12 + 20 + 22 + 22 + 22$ is defined in such a way that numbers $s = 7$, $t = 5 + 12$, $\Delta_{c_2^-} = \mu_2 = 20$, $(p^5_{d_2})^3 = (p^5_4)^3 = 22 + 22 + 22$ are assigned to the path $P \subset \Gamma_{(a_1, c_0)}$:

$$P = a_1^- \to c_1^- \to c_2^- \to c_2^+.$$

Note that, in the representation of $\mathcal{P}'_{(a_l, c_{l-1})}$ a term $n'_{a_l^-}$ can be given by the identities $n'_{a_l^-} = p^{n_0}_{i+j+1} - p^{n_0}_{j+1} = a_{ij}$, $l \geq 1$, where a_{ij} is the ijth entry of the matrix:

$$a_{ij} = \begin{cases} 2n_0 - 3 + (n_0 - 2)(j - 1), & i = 1,\ j \geq 1, \\ -3n_0 + 6 + (4n_0 - 7)i + (n_0 - 2)i(j - 1), & i > 1,\ j \geq 1. \end{cases}$$

If in the representation over $\mathbb{N}^*$ of a poset $\mathcal{P}_{c_i}$ we have $\lambda_a = ((p^{n_0}_j)(p^{n_0}_k))$ then graphs $\Gamma_{(a_i, c_{i-1})}$ define corresponding partitions of type II (taking into account all possible combinations of l, f_h, f_5 and f_6) of $n'_{c_{r-2}} = n_0 p^{n_0}_r$ fixed. We let $N(j, k, r)$ denote the number of partitions generated in this way.

The following result is a consequence of facts described above.

Theorem 4. *If N denotes the number of partitions of type* I *and* II *of* $n_0 p_r^{n_0}$, $r \geq 4$, $n_0 \geq 5$, *then*

$$N = \frac{(r-3)(r-2)(r-1)}{6} + 2p_{r-3}^3 + p_{r-1}^3 + r - 1.$$

Proof. We apply differentiation L to the poset $\mathcal{P}_{c_1}$ (see Figure 5) with respect to the pair (a_1, b) with $n = r - 2$, then partitions $\lambda'_{c_i^+}$ define partitions of type II of $\lambda'_{c_{r-2}^+} = n_0 p_r^{n_0}$, note that terms $|I|(k_{c_j} - k_{c_{j-1}}) = \Delta_{c_j^-}$, are in bijective correspondence with numbers

$$\mu_h = (20 + 6(n_0 - 5) + (2n_0 - 4)(h - 2)).$$

Since by definition the number of partitions of type I of $n'_{c_{r-2}^+} = n_0 p_r^{n_0}$ is $d_{r-2} = r$ and $n_{a_1} = p_2^{n_0} + p_3^{n_0}$, we have $N(2, 3, 3) = 2$. In the corresponding derived poset it is easy to see that

$$N(2, 3, r) = 2|(c_1^+)^{\nabla}| + \sum_{i=2}^{r-2} |(c_i^+)^{\nabla}| = p_{r-1}^3 - 1.$$

If we consider that for $1 < k \leq r - 2$ fixed, in poset $\mathcal{P}_{c_k}$, relations and points in $B^{\nabla} \subset \mathcal{P}$ are invariants and that the pair of points (a_k, c_{k-1}) is L-suitable, then we apply differentiation L to the posets $\mathcal{P}_{c_k}$ with respect to these points for each $1 < k \leq r - 2$ with a representation of the form $\lambda_{a_k} = ((p_2^{n_0})^1 (p_{k+2}^{n_0})^1)$, $n_{a_k} = p_2^{n_0} + p_{k+2}^{n_0}$, $n_{c_i} = n_0 p_{i+2}^{n_0}$, $\lambda_{c_i} = ((p_{i+2}^{n_0})^{n_0})$, $1 < k \leq r - 2$, $k \leq i \leq r - 2$ keeping without changes the other identities in formulas (9). We can see that the total number of new parts furnished for these differentiations to the partitions of type II of the new $n'_{c_i^+}$ are

$2|(c_k^+)^{\nabla}| = 2(r-k-1)$ for each k. Then for $k_0 \le r$ fixed we have:

$$N(2,\, k_0,\, r) = 2\,(r - k_0 - 1), \quad 4 \le k_0 \le r,$$

thus

$$N(2,\, k > 3,\, r) = \sum_{k_0 = 4}^{r} N(2,\, k_0,\, r) = 2\,p_{r-3}^3.$$

If we use same arguments for each $j_0,\ 3 \le j_0 \le r-1,$ we conclude that

$$N(j_0,\, k_0,\, r) = p_{r-j_0}^3, \quad 3 \le j_0 \le r-1, \quad j_0 \le k_0 \le r,$$

thus

$$N(j \ge 3,\, k,\, r) = \sum_{i=1}^{r-3} p_i^3 \,\frac{(r-3)(r-2)(r-1)}{6}.$$

Therefore, if $2 \le j < k \le r,$ then

$$N = \frac{(r-3)(r-2)(r-1)}{6} + 2\,p_{r-3}^3 + p_{r-1}^3 + r - 1. \qquad \square$$

A partition γ is said to be of *type* III if it is obtained from a partition λ of type II by applying the substitution $\dfrac{\partial^{(n_0-2)}\lambda}{\partial(p_{d_i}^{n_0})^{n_0-2}}((n_0-2)\,p_{d_i}^{n_0})$. If $\lambda'_{a_k} = ((p_j^{n_0})^1(p_{d_k}^{n_0})^1)$ in the derived poset $\mathcal{P}'_{(a_k,\, c_{k-1})}$ and the partition λ of $n'_{c_i^+}$ is of type III, then the partition $\dfrac{\partial^2\lambda}{\partial p_j^{n_0}\partial p_{d_k}^{n_0}}(n'_{a_k})$ is of type IV whereas, if λ is of type II for the same number then, $\dfrac{\partial^2\lambda}{\partial p_j^{n_0}\partial p_{d_k}^{n_0}}(n'_{a_k})$ is a partition of type V. We let Γ denote the set of partitions of these types and $P_r(\Gamma,\, n_0 p_r^{n_0})$ denotes the number of partitions of type Γ of the number $n_0 p_r^{n_0}$.

Remark 5. By definition, it is easy to see that partitions of type II of $n_0 p_r^{n_0}$ equal partitions of type III. And partitions of type IV equal partitions of type V, $r \geq 4$, $n_0 \geq 5$.

Lemma 6. *If* $P_r(\mathrm{IV}, n_0 p_r^{n_0})$ *is the number of partitions of type* IV *of* $n_0 p_r^{n_0}$, *then* $P_r(\mathrm{IV}, n_0 p_r^{n_0}) = \frac{(r-2)(r-1)r}{6}$.

Proof. Note that to each pair of numbers $(p_{i+2}^{n_0}, p_\iota^{n_0})$ with $2 \leq \iota < i+2$, $1 \leq i \leq r-2$, it is possible to assign a unique poset $\mathcal{P}_{c_i}$ with a representation satisfying $n_{a_i} = p_{i+2}^{n_0} + p_\iota^{n_0}$ in which case $|(c_i^+)^\nabla| = r - (i+1)$, in the corresponding derived poset $\mathcal{P}'_{(a_i, c_{i-1})}$. Thus, if we denote $\mathcal{P}'_i$ this family of posets, then the number of partitions of type IV can be obtained by calculating $\sum_{c_i^+ \in \mathcal{P}'_i} |(c_i^+)^\nabla| = \sum_{j=1}^{r-2} p_j^3 = \frac{(r-2)(r-1)r}{6}$. □

Theorem 4, Remark 5 and Lemma 6 allow us to obtain the following result.

Theorem 7. *The number of partitions of types* I, II, IV, V *of* $n_0 p_r^{n_0}$ *with* $r \geq 4$ *and* $n_0 \geq 5$ *is given by the formula*

$$M = (r-1)p_{r-2}^3 + \frac{3r^2 - 9r + 10}{2}.$$

Corollary 8. $P_r(\Gamma, n_0 p_r^{n_0}) \equiv r \pmod 2$.

Proof. $P_r(\Gamma, n_0 p_r^{n_0}) = M + N - r$. □

We say that a partition is of *type* VI, if it is obtained as a partition of type III by defining the substitution $\frac{\partial^{n_0-2}\lambda}{\partial(p_{d_i}^{n_0})^{n_0-2}}((n_0-3)p_{d_i}^{n_0}, (p_{d_i}^{n_0})^1)$. These partitions allow us to obtain the following results:

Lemma 9. *The number of partitions of type* VI *of* $n_0 p_r^{n_0}$, $n_0 \geq 5$, $r \geq 4$, *equals the number of partitions of type* III.

Theorem 10. *The number of partitions of type* Ξ *of* $n_0 p_r^{n_0}$, $n_0 \geq 5$, $r \geq 4$ *is*

$$P_r(\Xi, n_0 p_r^{n_0}) = M + 2N - 2r,$$

where a partition λ *is said to be of type* Ξ *if and only if* λ *is a partition of one of the types* I *to* VI. *As an example, the following are the list of partitions of types* III, IV, V *and* VI *of* 110:

Partitions of type III *are* 7 + 12 + 5 + 36 + 50 = 24 + 36 + 50 = 7 + 12 + 5 + 20 + 66 = 24 + 20 + 66 = 22 + 22 + 66 = 17 + 22 + 5 + 66 = 44 + 66 = 10 + 12 + 22 + 66,

Partitions of type IV *are* 7 +17 +36 + 50 = 7 +17 + 20 + 66 = 17 + 27 + 66 = 10 + 34 + 66,

Partitions of type V *are* 7 + 17 + 12 + 12 + 12 + 50 = 7 + 17 + 20 + 22 + 22 + 22 = 17 + 27 + 22 + 22 + 22 = 10 + 34 + 22 + 22 + 22,

Partitions of type VI *are* 7 + 12 + 5 + 24 + 12 + 50 = 24 + 24 + 12 + 50 = 7 + 12 + 5 + 20 + 44 + 22 = 24 + 20 + 44 + 22 = 22 + 22 + 44 + 22 = 17 + 22 + 5 + 44 + 22 = 44 + 44 + 22 = 10 + 12 + 22 + 44 + 22.

4. The Guy's Problems

In this section, we give some advances to the problems (1) and (3) mentioned in pages 101 and 102.

4.1. Sums of three squares of a given shape

In this section, we give advances to the solutions of problem (1) mentioned in pages 101 and 102.

The following result is a consequence of the definition of substitution derivative:

Theorem 11. *If* $p_0^3 = p_{-1}^3 = 0$, $p_1^3 = 1$, *then each number of the form* $24n + 3$ *can be written as a sum of three square of numbers of the form* $(6r - 1)$, *where* $n = s_{(j,i)} - 2p_{i+1}^5$ *and*

$$s_{(j,i)} = 97 + 57(j-1) + 15p_{j-2}^3 + 9p_{i-2}^3 + 21(i-1), \quad i, j \geq 1. \tag{11}$$

Proof. Note that, each term of $s_{(j,i)}$ can be obtained by applying the following derivatives:

$$\frac{\partial^2 \lambda_k}{\partial p_k^5 \partial p_k^5}(p_j^5, p_{2j+1}^5), \quad j \geq 3, \qquad \left(\frac{\partial^2 \lambda_k}{\partial p_k^5 \partial p_k^5}\right)(p_2^5, p_i^5), \quad i > 2,$$

to the partition $\lambda_k = ((p_k^5)^5)$, of elements of the sequence $a_k = 5(3p_{k-1}^3 + 7k + 5)$, $k > 1$. Therefore, $s_{(j,i)}$ is a sum of five pentagonal numbers of the form $s_{(j,i)} = p_{i+1}^5 + p_{j+2}^5 + p_{2j+5}^5 + 2p_{i+1}^5$ thus $s_{(j,i)} - 2p_{i+1}^5 = p_{i+1}^5 + p_{j+2}^5 + p_{2j+5}^5$. Therefore for i, j fixed, $24(s_{(j,i)} - 2p_{i+1}^5) + 3 = (6(i+1)-1)^2 + (6(j+2)-1)^2 + (6(2j+5)-1)^2 = (6i+5)^2 + (6j+11)^2 + (12j+29)^2$. □

The arguments used in proof of Theorem 11 allow to obtain the following result.

Corollary 12. *If* $p_0^3 = p_{-1}^3 = 0$, $p_1^3 = 1$, *then each number of the form* $24n + 3$ *can be written as a sum of three square of numbers of the form* $(6r - 1)$, *where* $n = t_{(j,i)} - 2p_{i+1}^5$ *and*

$$t_{(j,i)} = 32 + 10(j-1) + 3p_{j-2}^3 + 9p_{i-2}^3 + 21(i-1), \quad i, j \geq 1. \tag{12}$$

If in Theorem 11, we set p_j^6 or p_j^7 instead of p_j^5 and define in the corresponding proof sequence a_k as $a_k = 6(-4 + 5k + 4p_{k-2}^3)$ or $a_k = 7(-5 + 6k + 5p_{k-2}^3)$, $k > 1$, respectively, then we obtain the following results which are other advances to the first problem mentioned in pages 101 and 102, regarding hexagonal and heptagonal numbers.

Corollary 13. *If* $p_0^3 = p_{-1}^3 = 0,\ \ p_1^3 = 1,$ *then any number of the form* $40n + 27$ *is a sum of three square of numbers of the form* $10r - 3,$ *where* $n = n_{(j,i)} - 4p_{i+1}^7$ *and*

$$n_{(j,i)} = 165 + 93(j-1) + 25p_{j-2}^3 + 25p_{i-2}^3 + 55(i-1), \quad i,\ j \geq 1. \quad (13)$$

Corollary 14. *If* $p_0^3 = p_{-1}^3 = 0,\ \ p_1^3 = 1,$ *then any number of the form* $8n + 3$ *is a sum of three square of numbers of the form* $4r - 1,$ *where* $n = m_{(j,i)} - 3p_{i+1}^6$ *and*

$$m_{(j,i)} = 130 + 75(j-1) + 20p_{j-2}^3 + 16p_{i-2}^3 + 36(i-1), \quad i,\ j \geq 1. \quad (14)$$

4.2. The equation $x^3 + y^3 + kz^3 = n$

In this section, we use arguments used in the proof of Theorem 11 to give sequences of positive integers which are solutions of a Diophantine equation of the form

$$x^3 + y^3 + kz^3 = n, \quad (15)$$

where x, y, z, n are positive integers and $1 \leq k \leq 5.$ Cases $k = 1, 2$ are indexed in [15] as *problem* D5. In [18], Koyama describes efficient algorithms to solve equations of type (15) if $k = 2.$

We let $\rho_t^3 = \dfrac{t(t+1)(t+2)}{6}$ the *t*th tetrahedral number.

The following result describes sequences of positive integers which can be written as a sum of five cubic numbers.

Theorem 15. *If* $p_0^3 = 0,\ p_1^3 = 1$ *and* $i,\ j \geq 1,$ *then all positive numbers of the form* $s_{(j,i)} - q_{2j+i+8}$ *is a sum of four cubes with two of them equal,* *where* $s_{(j,i)} = 508 + (j+1)891 + 690p_j^3 + (507 + 306j + 72p_{j-1}^3)(i+1) + (144 + 36j)p_i^3 + 18\rho_{i-1}^3 + 198\rho_{j-1}^3.$

Proof. If q_k is the kth cubic number, then the sequence $s_{(j,i)}$ can be obtained by deriving each partition $\lambda_k = ((q_{k+2})^5)$, of the sequence $a_{k+2} = 40 + 95k + 90p_{k-1}^3 + 30\sum_{t=1}^{k} p_{t-2}^3$, in such a way that

$$\frac{\partial^2 \lambda_k}{\partial q_{k+2} \partial q_{k+2}}(q_s, q_{2s+1}), \quad s \geq 4, \quad 2s - 1 \leq k.$$

Therefore $s_{(j,i)} = q_{j+3} + q_{2j+7} + q_{2j+i+8} + 2q_{2j+i+8}$ is a sum of five cubes with two of them equal. □

As an example, the following is the matrix $S = (s_{(j,i)})$ of size 4×3:

$$S = \begin{bmatrix} 4786 & 5977 & 7384 \\ 8047 & 9688 & 11581 \\ 12538 & 14701 & 17152 \\ 18457 & 21214 & 24295 \end{bmatrix}.$$

Theorem 15 allows to obtain formulas for numbers which can be written as sums of two or three cubic numbers. To do that it suffices to evaluate the following differences:

$$s_{(j,i)} - 3q_{2j+i+8} = q_{j+3} + q_{2j+7},$$

$$s_{(j,i)} - 2q_{2j+i+8} = q_{j+3} + q_{2j+7} + q_{2j+i+8}. \tag{16}$$

Formulas (16) and Theorem 15 allow to obtain the following result.

Corollary 16. *For all* $i, j \geq 1$, *the triplet* (x, y, z) *with* $x = j+3$, $y = 2j+7$ *and* $z = 2j+i+8$ *is a solution of the Diophantine equation*

$$x^3 + y^3 + 2z^3 = s_{(j,i)} - q_{2j+i+8}. \tag{17}$$

Corollary 17. *If* $i,\ j \geq 1,$ *then* $x = -(4j + 8),\ y = -(2j + 4)$ *and* $z = 2j + i + 8$ *is a solution of the Diophantine equation*

$$x^3 + y^3 + 2z^3 = s_{(j,i)} - n_{(j,i)},$$

where

$$m_{(j,i)} = 508 + (j + 1)891 + 690p_j^3 + (507 + 306j + 72p_{j-1}^3)(i + 1)$$

$$+ (144 + 36j)\, p_i^3 + 18\rho_{i-1}^3 + 198\rho_{j-1}^3,$$

and

$$n_{(j,i)} = 4068 + 4193(j - 1) + 1167(j - 2)(j - 1) + 89(j - 3)(j - 2)(j - 1)$$

$$+ (397 + 150(j - 1) + 12(j - 2)(j - 1))(i - 1)$$

$$+ 6(5 + j)(i - 2)(i - 1) + (i - 3)(i - 2)(i - 1).$$

That is,

$$S = \begin{bmatrix} 4786 & 5977 & 7384 & \cdots \\ 8047 & 9688 & 11581 & \cdots \\ 12538 & 14701 & 17152 & \cdots \\ 18457 & 21214 & 24295 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \quad N = \begin{bmatrix} 4068 & 4465 & 4934 & \cdots \\ 8261 & 8808 & 9439 & \cdots \\ 14788 & 15509 & 16326 & \cdots \\ 24183 & 25102 & 26129 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Proof. If q_k is the kth positive cube, each term $m_{(j,i)},\ n_{(j,i)}$ can be obtained via the substitutions

$$\frac{\partial^2 \lambda_k}{\partial q_{k+2} \partial q_{k+2}}(q_s,\ q_{2s+1}), \quad \frac{\partial^2 \lambda_{ji}}{\partial q_{2j+i+8} \partial q_{2j+i+8}}(q_{2j+4},\ q_{4j+8}),$$

$$s \geq 4, \quad 2s - 1 \leq k,$$

where λ_k, λ_{ji} are partitions such that

$$\lambda_k = ((q_{k+2})^5),$$

$$\lambda_{ji} = ((q_{2j+7})^1 (q_{j+3})^1 (q_{2j+i+8})^3). \tag{18}$$

□

4.3. Representations of posets and compositions into cubic numbers

Let q_k be the kth positive cubic number. Then the poset $\mathcal{P}_{c_1}$ (see Figure 5) is represented over $\mathbb{N}^*$ in the following form:

$$(n_{b_1}, \lambda_{b_1}) = (q_1, ((q_1)^5)),$$

$$(n_b, \lambda_b) = (q_2, ((q_2)^5)),$$

$$(n_{a_1}, \lambda_{a_1}) = (q_2 + q_3, ((q_2)^1 (q_3)^1)),$$

$$(n_{c_i}, \lambda_{c_i}) = (5q_{i+2}, ((q_{i+2})^5)), \quad i \geq 1. \tag{19}$$

We let Λ_q denote this representation. Posets $\mathcal{P}_{c_i}$ of Subsection 3.2 are represented in the same way, taking into account the following changes:

$n'_{a_l^-} = q_{i+j+1} - q_{j+1} = a_{ij}, \; i, j \geq 1, \; l \geq 1,$ where

$$a_{ij} = \begin{cases} 6p_{j+1}^3 + 1, & i = 1, \; j \geq 1, \\ 6(\rho_{i+j}^3 - \rho_j^3) + i, & i > 1, \; j \geq 1. \end{cases}$$

Furthermore, for $j \geq 2, \; k, \; h \geq 1,$

$$n'_{c_k^-} = 2q_{k+2}, \quad \Delta_{c_j^-} = 2 + 12p_{j+1}^3, \quad \Delta_{c_h^+} = 5 + 30p_{h+1}^3. \tag{20}$$

Therefore all results regarding partitions of type Γ and Ξ for $n_0 p_r^{n_0}$ obtained in Section 3 are valid for numbers $5q_r$, where q_r denotes the rth cubic number. In particular, we have the following results by using the notation Ξ_q

for partitions of type Ξ (induced by representation Γ_q and its differentiations) of numbers of the form $5q_r$:

Lemma 18. *If N is the number of partitions of type* I_q *and* II_q *of* $5q_r$, $r \geq 4$, *then*

$$N = \frac{(r-3)(r-2)(r-1)}{6} + 2p_{r-3}^{3} + p_{r-1}^{3} + r - 1.$$

Theorem 19. *If* $r \geq 4$, *then*

$$P_r(\Gamma_q, 5q_r) - r \equiv 0 \pmod{2}.$$

Corollary 20. *The number of partitions of type* Ξ_q *of* $5q_r$, $r \geq 4$, *is*

$$P_r(\Xi_q, 5q_r) = P_r(\Xi, n_0 p_r^{n_0}) = M + 2N - 2r.$$

As an example, the following are the partitions of type Ξ_q of $320 = 5q_4$.

Partitions of type I are 320 = 5 + 35 + 95 + 185 = 40 + 95 + 185 = 135 + 185,

Partitions of type II are 19 + 8 + 27 + 27 + 27 + 27 + 185 = 54 + 27 + 27 + 27 + 185 = 19 + 8 + 27 + 74 + 64 + 64 + 64 = 54 + 74 + 64 + 64 + 64 = 64 + 64 + 64 + 64 + 64 = 56 + 8 + 64 + 64 + 64 + 64 = 128 + 64 + 64 + 64 = 37 + 27 + 64 + 64 + 64 + 64,

Partitions of type III are 19 + 8 + 27 + 81 + 185 = 54 + 81 + 185 = 19 + 8 + 27 + 74 + 192 = 54 + 74 + 192 = 64 + 64 + 192 = 56 + 8 + 64 + 192 = 128 + 192 = 37 + 27 + 64 + 192,

Partitions of type IV are 19 + 35 + 81 + 185 = 19 + 35 + 74 + 192 = 56 + 72 + 192 = 37 + 91 + 192,

Partitions of type V are 19 + 35 + 27 + 27 + 27 + 185 = 19 + 35 + 74 + 64 + 64 + 64 = 56 + 72 + 64 + 64 + 64 = 37 + 91 + 64 + 64 + 64, Partitions of type VI are 19 + 8 + 27 + 27 + 54 + 185 = 54 + 54 + 27 + 185 = 19 + 8 + 27 + 74 + 128 + 64 = 54 + 74 + 128 + 64 = 64 + 64 + 128 + 64 = 56 + 8 + 64 + 128 + 64 = 128 + 128 + 64 = 37 + 27 + 64 + 128 + 64.

We also have the following result.

Theorem 21. *If*

$$\begin{aligned} t_{(j,i)} = {} & 4068 + 4193(j-1) + 1167(j-2)(j-1) + 89(j-3)(j-2)(j-1) \\ & + (397 + 150(j-1) + 12(j-2)(j-1))(i-1) \\ & + 6(5+j)(i-2)(i-1) + (i-3)(i-2)(i-1),\ i,\ j \geq 1, \end{aligned}$$

then the quintuplet $v = 2j + i + 8$, $w = j + 3$, $x = 2j + 7$, $y = 2j + 4$ and $z = 4j + 8$ *is a solution of the Diophantine equation*

$$v^3 + w^3 + x^3 + y^3 + z^3 = t_{(j,i)}. \tag{21}$$

Proof. Since for each $i, j, \lambda_{ji} = ((q_{2j+7})^1 (q_{j+3})^1 (q_{2j+i+8})^3)$ is a partition of $s_{(j,i)}$, $t_{(j,i)}$ can be obtained by calculating the derivatives

$$\frac{\partial^2 \lambda_{ji}}{\partial q_{2j+i+8} \partial q_{2j+i+8}} (q_{2j+4}, q_{4j+8}). \qquad \square$$

The following is a matrix $T = (t_{(j,i)})$ of size 4×3:

$$T = \begin{bmatrix} 4068 & 4465 & 4934 \\ 8261 & 8808 & 9439 \\ 14788 & 15509 & 16326 \\ 24183 & 25102 & 26129 \end{bmatrix}.$$

The following result is a consequence of formulas (16).

Corollary 22. *If* $i,\ j \geq 1$, *then* $w = j + 3$, $x = 2j + 7$, $y = 2j + i + 8$, $z = 2j + 4$ *is a solution of the Diophantine equation*

$$w^3 + x^3 + y^3 + z^3 = s_{(j,i)} - 2q_{2j+i+8} + q_{2j+4}. \tag{22}$$

In particular, there exist integers $a, b, c \in \mathbb{Z}$ *such that* $s_{(j,i)} - t_{(j,i)} = a^3 + b^3 + 2c^3$.

Proof. For the difference $s_{(j,i)} - t_{(j,i)}$ it suffices to write $c = 2j + i + 8$, $b = -(2j + 4)$, $a = -(4j + 8)$. □

Remark 23. Note that, it is easy to see that if $\xi(n)$ denotes the sum of the digits of n mod 9, then $\xi(t_{(j,i)} - q_{4j+8}) = 5$ and $\xi(s_{(j,i)} - q_{2j+i+8}) = 5$.

If $i, j \equiv 2 \pmod 3$, then $s_{(j,i)} - q_{2j+i+8} \equiv -4 \pmod 9$ and $t_{(j,i)} - q_{4j+8} \equiv -4 \pmod 9$.

Furthermore, since $s_{(j,i)} - 3q_{2j+i+8} + 2q_{2j+i+k} \equiv -4 \pmod 9$ if $k = 2 + 9l$ and $s_{(j,i)} - 3q_{2j+i+8} + 2q_{2j+i+k} \equiv -4 \pmod 9$.

If $k = 6 + 45l$, $l \geq 1$, $j \equiv 2 \pmod 3$, $i \equiv 1 \pmod 3$.

We have the following corollary:

Corollary 24. *If* $i, j \equiv 2 \pmod 3$, $h \equiv 1 \pmod 3$, $k = 6 + 45l$, $m = 2 + 9l$, $h, i, j, l \geq 1$, *then*

$$s_{(j,i)} - q_{2j+i+8} \equiv -4 \pmod 9,$$

$$t_{(j,i)} - q_{4j+8} \equiv -4 \pmod 9,$$

$$s_{(j,i)} - 2q_{2j+i+8} + q_{2j+4} \equiv -4 \pmod 9,$$

$$s_{(j,h)} - 3q_{2j+h+8} + 2q_{2j+h+k} \equiv -4 \pmod 9,$$

$$s_{(j,i)} - 3q_{2j+i+8} + 2q_{2j+i+m} \equiv -4 \pmod 9, \tag{23}$$

and each number of the forms: $s_{(j,i)} - q_{2j+i+8}$, $t_{(j,i)} - q_{4j+8}$, $s_{(j,i)} -$

$2q_{2j+i+8} + q_{2j+4}$ *and* $s_{(j,i)} - 3q_{2j+i+8} + 2q_{2j+i+n}$, $8 \neq n \geq 1$ *is a sum of four positive cubic numbers*.

Proof. It suffices to note that the triplet $x = j + 3$, $y = 2j + 7$, $z = 2j + i + n$ is a solution of the Diophantine equation

$$x^3 + y^3 + 2z^3 = s_{(j,i)} - 3q_{2j+i+8} + 2q_{2j+i+n}. \tag{24}$$

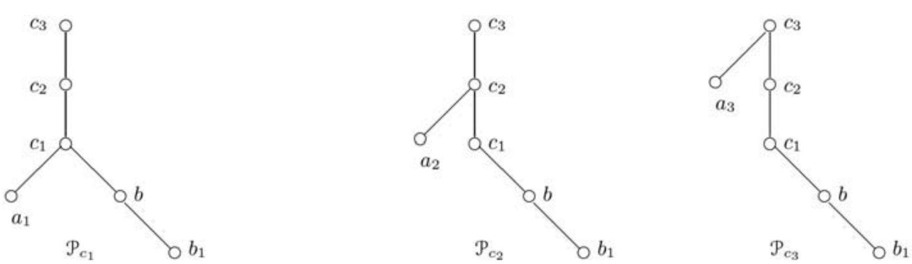
□

As an example an $m \times n$ matrix of type $\xi(s_{(j,i)} - q_{2j+i+8})$ (see Corollary 24) has the form:

$$U = \begin{bmatrix} 8 & 1 & 3 & 8 & 1 & 3 & \cdots \\ 9 & 5 & 7 & 9 & 5 & 7 & \cdots \\ 1 & 3 & 8 & 1 & 3 & 8 & \cdots \\ 8 & 1 & 3 & 8 & 1 & 3 & \cdots \\ 9 & 5 & 7 & 9 & 5 & 7 & \cdots \\ 1 & 3 & 8 & 1 & 3 & 8 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

5. Partitions of Types I and II of $5p_5^5 = 175$

In this section, we illustrate how partitions of types I and II can be obtained via differentiations of posets $\mathcal{P}_{c_i}$, $1 \leq i \leq 3$, with the following Hasse diagrams:



These posets are of the type $\mathcal{P}_{c_1}$ as showed in Figure 5 with $n = 3$.

In this case, we consider representations over $\mathbb{N}^*$ of posets $\mathcal{P}_{c_i}$ and its corresponding derivatives $\mathcal{P}'_{(a_i, c_{i-1})}$, $1 \leq i \leq 3$, where for each $1 \leq i \leq 3$, we have

$$\mathcal{P}_{c_i} = a_i + b + b_1 + C,$$

where $C = \{c_1 <' c_2 <' c_3\}$ is a chain such that $\{a_i <' c_i\}$, $1 \leq i \leq 3$. Furthermore, $b_1 <' b <' c_1$, the pair of points (a_i, c_{i-1}) is L-suitable in each poset $\mathcal{P}_{c_i}$ and $c_0 = b$.

As an example of a representation of poset $\mathcal{P}_{c_1}$ over $\mathbb{N}^*$ with $n_0 = 5$ (see formulas (9)), has the form:

$$\begin{aligned}
(n_{b_1}, \lambda_{b_1}) &= (5, 1 + 1 + 1 + 1 + 1),\\
(n_b, \lambda_b) &= (25, 5 + 5 + 5 + 5 + 5),\\
(n_{a_1}, \lambda_{a_1}) &= (17, 5 + 12),\\
(n_{c_1}, \lambda_{c_1}) &= (60, 12 + 12 + 12 + 12 + 12),\\
(n_{c_2}, \lambda_{c_2}) &= (110, 22 + 22 + 22 + 22 + 22),\\
(n_{c_3}, \lambda_{c_3}) &= (175, 35 + 35 + 35 + 35 + 35).
\end{aligned} \tag{25}$$

The following is the representation of the corresponding derived poset $\mathcal{P}'_{(a_1, c_0)}$:

$$\begin{aligned}
(n'_{b_1}, n'_{b_1}) &= (5, 1 + 1 + 1 + 1 + 1),\\
(n'_b, \lambda'_b) &= (25, 5 + 5 + 5 + 5 + 5),\\
(n'_{a_1^-}, \lambda'_{a_1^-}) &= (7, 7),\\
(n'_{c_1^-}, \lambda'_{c_1^-}) &= (17, 5 + 12),
\end{aligned}$$

$$(n'_{c_2^-}, \lambda'_{c_2^-}) = (44,\ 22 + 22),$$

$$(n'_{c_3^-}, \lambda'_{c_3^-}) = (70,\ 35 + 35),$$

$$(n'_{c_1^+}, \lambda'_{c_1^+}) = (60,\ 7 + 5 + 12 + 12 + 12 + 12),$$

$$(n'_{c_2^+}, \lambda'_{c_2^+}) = (110,\ 7 + 5 + 12 + 20 + 22 + 22 + 22),$$

$$(n'_{c_3^+}, \lambda'_{c_3^+}) = (175,\ 7 + 5 + 12 + 20 + 26 + 35 + 35 + 35). \tag{26}$$

In the sequel, we give representations of posets $\mathcal{P}_{c_2}$ and $\mathcal{P}'_{(a_2, c_1)}$, respectively, with $n_0 = 5$:

$$(n_{b_1}, \lambda_{b_1}) = (5, 1 + 1 + 1 + 1 + 1),$$

$$(n_b, \lambda_b) = (25, 5 + 5 + 5 + 5 + 5),$$

$$(n_{a_2}, \lambda_{a_2}) = (27, 5 + 22),$$

$$(n_{c_1}, \lambda_{c_1}) = (60, 12 + 12 + 12 + 12 + 12),$$

$$(n_{c_2}, \lambda_{c_2}) = (110,\ 22 + 22 + 22 + 22 + 22),$$

$$(n_{c_3}, \lambda_{c_3}) = (175,\ 35 + 35 + 35 + 35 + 35). \tag{27}$$

The representation of the derived poset $\mathcal{P}'_{(a_2, c_1)}$ is given by the following formulas:

$$(n'_{b_1}, \lambda'_{b_1}) = (5, 1 + 1 + 1 + 1 + 1),$$

$$(n'_b, \lambda'_b) = (25,\ 5 + 5 + 5 + 5 + 5),$$

$$(n'_{a_2^-}, \lambda'_{a_2^-}) = (17, 17),$$

$$(n'_{c_2^-}, \lambda'_{c_2^-}) = (27,\ 5 + 12),$$

$$(n'_{c_2^-}, \lambda'_{c_2^-}) = (44,\ 22 + 22),$$

$$(n'_{c_3^-}, \lambda'_{c_3^-}) = (70,\ 35 + 35),$$

$$(n'_{c_1^+}, \lambda'_{c_1^+}) = (60,\ 7 + 5 + 12 + 12 + 12 + 12),$$

$$(n'_{c_2^+}, \lambda'_{c_2^+}) = (110,\ 7 + 5 + 12 + 20 + 22 + 22 + 22),$$

$$(n'_{c_3^+}, \lambda'_{c_3^+}) = (175,\ 7 + 5 + 12 + 20 + 26 + 35 + 35 + 35). \tag{28}$$

Formulas given above can be used for posets $\mathcal{P}_{c_3}$ and $\mathcal{P}'_{(a_3, c_2)}$ taking into account the following changes:

$$(n_{a_3}, \lambda_{a_3}) = (40,\ 5 + 35),\ \text{in } \mathcal{P}_{c_3},$$

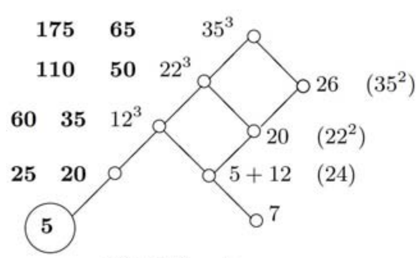
$$(n'_{a_3^-}, \lambda'_{a_3^-}) = (30,\ 30),$$

$$(n'_{c_3^-}, \lambda'_{c_3^-}) = (40,\ 5 + 35),\ \text{in } \mathcal{P}'_{(a_3, c_2)}. \tag{29}$$

The following is the matrix $A = (a_{ij})$ furnishing terms $n_{a_l^-} = p^5_{i+j+1} - p^5_{j+1}$:

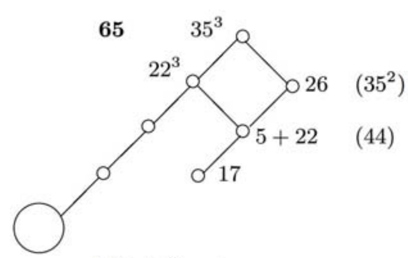
$$A = \begin{array}{|c|c|c|} \hline 7 & 10 & 13 \\ \hline 17 & 23 & \\ \hline 30 & & \\ \hline \end{array}.$$

In these graphics, we show all partitions of types I and II of $5p_r^5$, $2 \le r \le 5$ induced by the derivatives of $\mathcal{P}_{c_1}$.



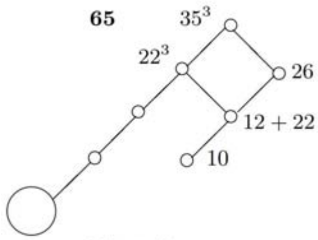
$N(2, 3, 5) = 9$

$\mathcal{P}'_{(a_1, c_0)}$

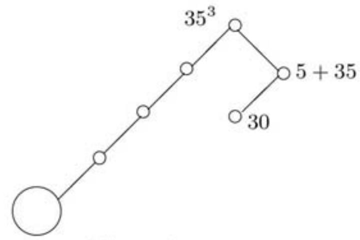


$N(2, 4, 5) = 4$

$\mathcal{P}'_{(a_2, c_1)}$

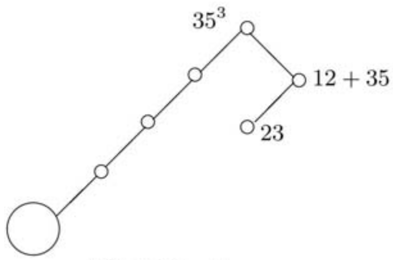


$N(3,4,5)=2$

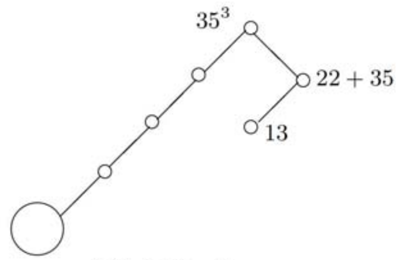


$N(2,5,5)=2$

$\mathcal{P}'_{(a_3,c_2)}$ (70)



$N(3,5,5)=1$



$N(4,5,5)=1$

A partition of type II of 175 can be obtained in this example by choosing a graph $\Gamma_{(a_i, c_{i-1})}$ whose vertices are points in chains C^+, $a_i^- + C^-$ of a poset $\mathcal{P}'_{(a_i, c_{i-1})}$ with edges linking points as shown below (for $\mathcal{P}'_{(a_1, c_0)}$). We consider paths $P \subset \Gamma_{(a_i,\, c_{i-1})}$ with initial vertex in a point of the chain $a_i^- + C^-$ of $\mathcal{P}'_{(a_i, c_{i-1})}$ and final vertex c_3^+,

$$\Gamma_{(a_1,c_0)} = \begin{tikzcd}
& & c_1^+ \arrow[dr] & & \\
a_1^- \arrow[r] & c_1^- \arrow[ur] \arrow[dr] & & c_2^+ \arrow[dr] & \\
& & c_2^- \arrow[ur] \arrow[r] & c_3^- \arrow[r] & c_3^+
\end{tikzcd}$$

$1 \le i \le 3$, $c_0 = b$ and choosing a unique number according to the graphic for each point of P. Numbers in parentheses mean that they occur in the partition and that the corresponding point is an initial vertex of P. The first term of a partition of type II is induced by a vertex of the form a_i^- and it does not contain parts in parentheses. Note that, if $c_2^+ \in P$, then the number associated to c_3^+ is **65** and that an expression of type $((p_i^5)^3)$ occurs once in the partition. Actually, the term $((p_i^5)^3)$ occurs in the partition if and only if c_{i-2}^+ is the minimum of the chain C^+ in the path *P*.

As an example, we can consider the following path $P \subset \Gamma_{(a_1, c_0)}$ such that

$$P = c_1^- \to c_2^- \to c_2^+ \to c_3^+,$$

then the corresponding partition of type II associated to *P* has the form:

$$24 + 20 + 22 + 22 + 22 + 65.$$

A partition of type II of 175 associated to the path $P = c_3^- \to c_3^+ \subset \Gamma_{(a_3, c_2)}$ has the form:

$$70 + 35 + 35 + 35.$$

Partitions of type I are obtained in the same way taking into account that each path is contained in the graph Γ^+ (shown below) with initial point in $b_1^\nabla \subset \mathcal{P}'_{(a_1, c_0)}$. Furthermore, each path must be finished in c_3^+ (there exists only one path for each point in b_1^∇):

$$\Gamma^+ = b_1 \to b \to c_1^+ \to c_2^+ \to c_3^+.$$

In this case, $n_{b_1} = \mathbf{5}$, $n_b = \mathbf{25}$ and so on. We assume that there exist the path $c_3^+ \to c_3^+$ such that $n_{c_3^+} = \mathbf{175}$. There are not two terms of the form n_x, $x \in b_1^\nabla$ associated to a path and no paths associate a term of the form n^3.

Therefore, partitions of type Γ of 175 are:

Partitions of type I, 5 + 20 + 35 + 50 + 65 = 25 + 35 + 50 + 65 = 60 + 50 + 65 = 110 + 65 = 175.

Partitions of type II, 7 + 5+ 12 + 12 + 12 + 12 + 50 + 65 = 24 + 12 + 12 + 12 + 50 + 65 = 7 + 5 + 12 + 20 + 22 + 22 + 22 + 65 = 24 + 20 + 22 + 22 + 22 + 65 = 22 + 22 + 22 + 22 + 22 + 65 = 7 + 5 + 12 + 20 + 26 + 35 + 35 + 35 = 24 + 20 + 26 + 35 + 35 + 35 = 22 + 22 + 26 + 35 + 35 + 35 = 35 + 35 + 35 + 35 + 35.

17 + 5 + 22 + 22 + 22 + 22 + 65 = 44 + 22 + 22 + 22 + 65 = 17 + 5 + 22 + 26 + 35 + 35 + 35 = 44 + 26 + 35 + 35 + 35 = 30 + 5 + 35 + 35 + 35 + 35 = 70 + 35 + 35 + 35 = 10 + 12 + 22 + 22 + 22 + 22 + 65 = 10 + 12 + 22 + 26 + 35 + 35 + 35 = 23 + 12 + 35 + 35 + 35 + 35 = 13 + 22 + 35 + 35 + 35 + 35.

Partitions of type III, 7 + 5 + 12 + 36 + 50 + 65 = 24 + 36 + 50 + 65 = 7 + 5 + 12 + 20 + 66 + 65 = 24 + 20 + 66 + 65 = 22 + 22 + 66 + 65 = 7 + 5 + 12 + 20 + 26 + 105 = 24 + 20 + 26 + 105 = 22 + 22 + 26 + 105 = 35 + 35 + 105.

17 + 5 + 22 + 66 + 65 = 44 + 66 + 65 = 17 + 5 + 22 + 26 + 105 = 44 + 26 + 105 = 30 + 5 + 35 + 105 = 70 + 105 = 10 + 12 + 22 + 66 + 65 = 10 + 12 + 22 + 26 + 105 = 23 + 12 + 35 + 105 = 13 + 22 + 35 + 105.

Partitions of type IV, 7 + 17 + 36 + 50 + 65 = 7 + 17 + 20 + 66 + 65 = 7 + 17 + 20 + 26 + 105 = 17 + 27 + 66 + 65 = 17 + 27 + 26 + 105 = 30 + 40 + 105 = 10 + 34 + 66 + 65 = 10 + 34 + 26 + 105 = 23 + 47 + 105 = 13 + 57 + 105.

Partitions of type V, 7 + 17 + 12 + 12 + 12 + 50 + 65 = 7 + 17 + 20 + 22 + 22 + 22 + 65 = 7 + 17 + 20 + 26 + 35 + 35 + 35 = 17 + 27 + 22 + 22 + 22 + 65 = 17 + 27 + 26 + 35 + 35 + 35 = 30 + 40 + 35 + 35 + 35 = 10 + 34 + 22 + 22 + 22 + 65 = 10 + 34 + 26 + 35 + 35 + 35 = 23 + 47 + 35 + 35 + 35 = 13 + 57 + 35 + 35 + 35.

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