JP Journal of Algebra, Number Theory and Applications
© 2014 Pushpa Publishing House, Allahabad, India
Published Online: June 2014
Available online at http://pphmj.com/journals/jpanta.htm
Volume 32, Number 2, 2014, Pages 79-86

A NOTE ON MORE RAPID CONVERGENCE TO A DENSITY

József Bukor, János T. Tóth and László Zsilinszky*

Department of Mathematics and Informatics
J. Selye University
P. O. Box 54
945 01 Komárno, Slovakia
e-mail: bukor.jozsef@selyeuni.sk
toth.janos@selyeuni.sk

*Department of Mathematics and Computer Science
University of North Carolina at Pembroke
Pembroke, NC 28304, U. S. A.
e-mail: laszlo@uncp.edu

Abstract

The aim of this paper is to answer an open problem proposed by Giuliano et al. [3]. A set A of positive integers is constructed with the following properties:

(a) A has asymptotic density d and therefore logarithmic density d,

(b) the sequence $\left| d - \left(\sum_{a \le n,\, a \in A} 1/a \right) \Big/ \ln n \right|$, $n = 2, 3, \ldots,$ does not tend to 0 on the mean more rapidly than the sequence $|\, d - A(n)/n \,|$.

Received: January 18, 2014; Revised: February 18, 2014; Accepted: March 4, 2014
2010 Mathematics Subject Classification: 11B05.
Keywords and phrases: asymptotic density, logarithmic density, weighted density.
Supported by VEGA Grant no. 1/1022/12.

1. Introduction

Denote by $\mathbb{N}$ the set of all positive integers. For $A \subseteq \mathbb{N}$, and a real number x let $A(x)$ denote the counting function of the set A. Let $f : \mathbb{N} \to (0, \infty)$ be a weight function. For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ denote

$$S_f(A, n) = \sum_{a \le n,\, a \in A} f(a), \qquad S_f(n) = \sum_{a \le n} f(a)$$

and define

$$\underline{d}_f(A) = \liminf_{n\to\infty} \frac{S_f(A, n)}{S_f(n)}, \text{ and } \overline{d}_f(A) = \limsup_{n\to\infty} \frac{S_f(A, n)}{S_f(n)}$$

the lower, and upper f-density of A, respectively. In the case when $\underline{d}_f(A) = \overline{d}_f(A)$ we say that A has *f-density* $d_f(A)$.

Note that the so-called asymptotic density corresponds to $f(n) = 1$, and the logarithmic density to $f(n) = \frac{1}{n}$. It is well-known that each set which has an asymptotic density also has a logarithmic one (and the values are equal), but a set may have a logarithmic density without having an asymptotic one (cf. [5], Chap. III.1, Section 1.2, Theorem 2). The concept of weighted densities was introduced in [4] and [1].

Write $a\mathbb{N} + b = \{an + b : n \in \mathbb{N}\}$. We call a weight function f, and the corresponding weighted density, *regular*, provided for arbitrary positive integers a, b,

$$d_f(a\mathbb{N} + b) = \frac{1}{a}$$

(the f-density of the terms of an arbitrary infinite arithmetical progression with the same difference is equal). It is easy to check that if a monotone weight function f satisfies the conditions

$$\sum_{n=1}^{\infty} f(n) = \infty, \qquad \lim_{n\to\infty} \frac{f(n)}{S_f(n)} = 0,$$

then the corresponding weighted density is regular (cf. [2, Example 2.1]).

Denote by $A \in \mathcal{FUAP}$ the fact that $A \subseteq \mathbb{N}$ is a finite union of arithmetic progressions. It is worth mentioning that $\mathcal{FUAP}$ is closed under complement, finite union and finite intersection, so, in fact, $\mathcal{FUAP}$ is the field of sets generated by arithmetic progressions. Clearly, if $A \in \mathcal{FUAP}$ and d is a regular density, then $d(A)$ is rational; furthermore, $d_1(A) = d_2(A)$ for any $A \in \mathcal{FUAP}$, and regular densities d_1, d_2.

The paper of Giuliano et al. [3] is a collection of open problems on densities, it is the purpose of this note to solve Open Problem 8.2 which reads as follows:

Suppose that the set $A \subseteq \mathbb{N}$ has asymptotic density d. Let $F(n) = |\, d - A(n)n^{-1} \,|$ tends to zero as n tends to ∞. The set A has logarithmic density also equal to d. Let

$$G(n) = \left| d - \left(\sum_{a \le n,\, a \in A} \frac{1}{a} \right) \left(\sum_{a \le n} \frac{1}{a} \right)^{-1} \right|.$$

Is it true that $G(n)$ tends to zero *on the mean* more rapidly than $F(n)$? For instance, does

$$\frac{\frac{1}{n}\sum_{k \le n} G(k)}{\frac{1}{n}\sum_{k \le n} F(k)} = \frac{\sum_{k \le n} G(k)}{\sum_{k \le n} F(k)}$$

tend to 0 as n tends to ∞?

We will give an answer for the above mentioned open problem in a more general case, considering arbitrary two regular densities instead of the asymptotic and logarithmic ones. Our result shows that the speed of convergence to the weighted density, on the mean, does not depend on the growth of the regular weight function.

Theorem 1. *Let f, g be arbitrary regular weight functions, and* d_f, d_g *be the corresponding weighted densities. Then there exists a set* $A \subsetneq \mathbb{N}$ *such that* $d_f(A) = d_g(A) = 1$ *and*

$$\lim_{n\to\infty} \frac{\sum_{k\le n} F(k)}{\sum_{k\le n} G(k)} = 1, \tag{1}$$

where

$$F(n) = \left| d_f(A) - \frac{S_f(A,\, n)}{S_f(n)} \right| \quad \textit{and} \quad G(n) = \left| d_g(A) - \frac{S_g(A,\, n)}{S_g(n)} \right|.$$

2. Preliminaries

We will start with the following lemma. The proof is obvious.

Lemma 1. *Let the series* $\sum a_n$ *with positive terms be divergent. Suppose*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

Then

$$\lim_{n\to\infty} \frac{\sum_{k\le n} a_k}{\sum_{k\le n} b_k} = 1.$$

Lemma 2. *Let d be a regular density. Let* $B = \{b_1 < b_2 < \cdots\} \in \mathcal{FUAP}$, *and* $d(B) = \delta$. *Then for an arbitrary rational number* $\alpha = \frac{a}{b} > 1$ *there exists* $C \in \mathcal{FUAP}$ *such that* $C \subsetneq B$ *and* $d(C) = \frac{\delta}{\alpha}$.

Proof. Observe, that

$$\lfloor \alpha n \rfloor \equiv \lfloor \alpha(n+b) \rfloor (\operatorname{mod} a),$$

therefore the set $\{\lfloor \alpha n \rfloor : n \in \mathbb{N}\}$ is the union of b congruence classes modulo a. The congruence classes can be represented by the numbers

$$\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots, \lfloor b\alpha \rfloor.$$

Let

$$C = \{b_{\lfloor \alpha n \rfloor} : n \in \mathbb{N}\}.$$

The density d is regular and $C \in \mathcal{FUAP}$ so $d(C) = \delta \frac{b}{a} = \frac{\delta}{\alpha}$. □

The following consequence of the previous lemma will be used:

Lemma 3. *Let d be a regular density. Let $E \in \mathcal{FUAP}$, and $d(E) \in (0, 1)$. Then for an arbitrary rational number β, $1 < \beta < \frac{1}{d(E)}$ there exists $H \in \mathcal{FUAP}$ such that $E \subsetneq H$ and $d(H) = \beta d(E)$.*

Proof. The assertion follows immediately from Lemma 2 for the sets $B = \mathbb{N} \backslash E$, $C = \mathbb{N} \backslash H$ and the rational number

$$\alpha = \frac{1 - d(E)}{1 - \beta d(E)}.$$ □

3. Proof of Theorem 1

Proof. Consider the sequence (r_i):

$$r_0 = 0,\ r_1 = \frac{1}{2},\ r_2 = \frac{2}{3},\ \ldots,\ r_i = 1 - \frac{1}{i+1},\ \ldots.$$

Let B_1 be the set of all positive even numbers. Using Lemma 3, define the sets $B_i \in \mathcal{FUAP}$, $i = 2, 3, 4, \ldots$ for which

$$d_f(B_i) = d_g(B_i) = r_i, \quad \text{and} \quad B_1 \subsetneq B_2 \subsetneq B_3 \subsetneq \cdots$$

Let $0 = n_0 < n_1 < n_2 < \cdots$ denote integers (to be determined). The required set A will be of the form

$$A = \bigcup_{i=1}^{\infty} (A_i \cap (0, n_i]).$$

We will construct the sets A_i by induction. Put $A_1 = B_1$.

By the definition of $B_1, B_2,$ there exists an n_1 such that for any $n > n_1$ we have

$$r_0 < \frac{S_f(A_2, n)}{S_f(n)} < r_3, \quad \text{and} \quad r_0 < \frac{S_g(A_2, n)}{S_g(n)} < r_3,$$

where

$$A_2 = (B_1 \cap (0, n_1]) \cup (B_2 \cap (n_1, \infty)).$$

Indeed, observe that $A_1 \subsetneq A_2 \subsetneq B_2$ and

$$r_1 = d_f(B_1) = d_g(B_1) < d_f(B_2) = d_g(B_2) = r_2.$$

Suppose now, that the sets $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_k$ and positive integers $n_1, n_2,$ ..., n_{k-1} have been defined for some $k \geq 2$ so that

- $n_i > i + n_{i-1}$ $(i = 1, 2, ..., k-1)$,
- $A_i = \bigcup_{j=1}^{i-1} (B_j \cap (n_{j-1}, n_j]) \cup (B_i \cap (n_{i-1}, \infty))$ $(i = 1, 2, ..., k)$,
- $r_{i-2} < \dfrac{S_f(A_i, n)}{S_f(n)} < r_{i+1}$ and $r_{i-2} < \dfrac{S_g(A_i, n)}{S_g(n)} < r_{i+1}$ for all $n > n_{i-1}$ $(i = 2, 3, ..., k)$.

As $d_f(A_k) = d_g(A_k) = r_k$ (since A_k differs from B_k in, at most, finite many places), and $d_f(B_{k+1}) = d_g(B_{k+1}) = r_{k+1},$ we can choose a sufficiently large integer $n_k > k + n_{k-1}$ such that if we define

$$\begin{aligned} A_{k+1} &= (A_k \cap (0, n_k]) \cup (B_{k+1} \cap (n_k, \infty)) \\ &= \bigcup_{j=1}^{k} (B_j \cap (n_{j-1}, n_j]) \cup (B_{k+1} \cap (n_k, \infty)), \end{aligned} \tag{2}$$

the following inequalities hold for any integer $n > n_k$:

$$r_{k-1} < \frac{S_f(A_{k+1}, n)}{S_f(n)} < r_{k+2}, \text{ and } r_{k-1} < \frac{S_g(A_{k+1}, n)}{S_g(n)} < r_{k+2},$$

since, in view of (2), we have that $A_k \subsetneq A_{k+1} \subsetneq B_{k+1}$.

The definition of A implies that $d_f(A) = d_g(A) = 1$. Now, for an arbitrary positive integer n, let k be the smallest integer such that $A \cap (0, n] = A_k \cap (0, n]$. This gives $n_{k-1} < n \leq n_k$. Consequently,

$$r_{k-2} < \frac{S_f(A, n)}{S_f(n)} < r_{k+1}, \text{ and } r_{k-2} < \frac{S_g(A, n)}{S_g(n)} < r_{k+1},$$

therefore,

$$\frac{1}{k+2} < F(n) < \frac{1}{k-1}, \text{ and } \frac{1}{k+2} < G(n) < \frac{1}{k-1}.$$

Then, by the conditions $n_k > k + n_{k-1}$, $k = 1, 2, \ldots,$ and the inequalities above, we get that the series $\sum F(n)$ and $\sum G(n)$ diverge, thus, (1) follows by Lemma 1. □

In conclusion, we include an application of our Theorem 1 exhibiting a pair of regular weight functions with diametrically opposing growth behaviors, such that on a subset of the positive integers the rates of convergence to the weighted densities on the mean are equal. Indeed, the weight functions

$$f(n) = \frac{e^{\sqrt{n}}}{\sqrt{n}}, \text{ and } g(n) = \frac{1}{n \ln n}$$

are regular, $f(n)$ has a rapid growth to infinity, $g(n)$ tends to 0, and, by Theorem 1, there exists an $A \subseteq \mathbb{N}$ such that $\sum_{k \leq n} F(k)$ is asymptotically equal to the sum $\sum_{k \leq n} G(k)$.

Acknowledgement

The authors are grateful to the anonymous referee for helpful comments and suggestions.

References

[1] R. Alexander, Density and multiplicative structure of sets of integers, Acta Arith. 12 (1967), 321-332.

[2] J. Bukor, L. Mišík and J. T. Tóth, Dependence of densities on a parameter, Inf. Sci. 179 (2009), 2903-2911.

[3] R. Giuliano, G. Grekos and L. Mišík, Open problems on densities II, Diophantine Analysis and Related Fields 2010, 114-128, AIP Conf. Proc. 1264, Amer. Inst. Phys., Melville, NY, 2010.

[4] H. Rohrbach and B. Volkmann, Verallgemeinerte asymptotische Dichten, J. Reine Angew. Math. 194 (1955), 195-209.

[5] G. Tenenbaumm, Introduction to analytic and probabilistic number theory, Cambridge Studies in Advanced Mathematics, Vol. 46, Cambridge University Press, Cambridge, 1995.