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THE GENERALIZED HYERS-ULAM STABILITY OF ADDITIVE FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN SPACE

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Abstract

In this paper, we investigate the following additive functional inequality

$$\| f(x) + f(y) + f(2z) \| \leq \| f(x+y) - f(-2z) \|$$

and prove the generalized Hyers-Ulam stability of it in non-Archimedean spaces.

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1. Introduction and Preliminaries

The stability problems concerning group homomorphisms were raised by Ulam [11] in 1940 and affirmatively answered for Banach spaces by Hyers [7] in the next year. Hyers theorem was generalized by Aoki [1] for additive mappings and by Rassias [10] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [3] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach.

The functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

is called the *Cauchy additive functional equation*. In particular, every solution of the Cauchy additive functional equation is said to be an *additive mapping*.

In [4], Gilányi showed that if a mapping $f : X \to Y$ satisfies the following functional inequality

$$\| 2f(x) + 2f(y) - f(xy^{-1}) \| \le \| f(xy) \|, \tag{1.2}$$

then f satisfies the Jordan-Von Neumann functional equation

$$2f(x) + 2f(y) - f(xy^{-1}) = f(xy).$$

Gilányi [5] and Fechner [2] proved the Hyers-Ulam stability of (1.2). Park et al. [9] proved the Hyers-Ulam stability of the following functional inequalities:

$$\| f(x) + f(y) + f(z) \| \le \| f(x+y+z) \|. \tag{1.3}$$

Hensel [6] has introduced a normed space which does not have the Archimedean property. During the last three decades, the theory of non-Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings and superstrings [8].

A *valuation* is a function $|\cdot|$ from a field K to $[0, \infty)$ such that for all $r, s \in K,$ the following conditions hold: (i) $|r| = 0$ if and only if $r = 0,$ (ii) $|rs| = |r||s|,$ (iii) $|r + s| \le |r| + |s|.$

A field K is called a *valuation field* if K carries a valuation. A valuation $|\cdot|$ is called a *non-Archimedean valuation* if $|x + y| \le \max\{|x|, |y|\}$ for all $x, y \in K.$ A field K with a non-Archimedean valuation is called a *non-Archimedean field*. Clearly, $|-1| = |1| = 1$ and $|n| \le 1$ for all $n \in N.$

Let $\{x_n\}$ be a sequence in a non-Archimedean space X. Then $\{x_n\}$ is called a *Cauchy sequence* if, for any $\varepsilon > 0,$ there is a positive integer k such that $\| x_n - x_m \| \le \varepsilon$ for all $m, n \ge k$ and all $z \in X$ and $\{x_n\}$ is called *convergent* to x, denoted by $\lim_{n\to\infty} x_n = x,$ if for any $\varepsilon > 0,$ there is a positive integer k such that $\| x_n - x \| \le \varepsilon$ for all $n \ge k.$ Hence by the definition of non-Archimedean normed space (see [6] and [8]), we have

$$\| x_n - x_m \| \le \max\{\| x_{j+1} - x_j \| | m \le j \le n-1\} \quad (n > m),$$

a sequence $\{x_n\}$ is Cauchy in a non-Archimedean space $(X, \|\cdot\|)$ if and only if $\{x_{n+1} - x_n\}$ converges to zero in $(X, \|\cdot\|).$ A non-Archimedean space in which every Cauchy sequence is a convergent sequence is called a *non-Archimedean Banach space*.

In this paper, we investigate the following functional inequality

$$\| f(x) + f(y) + f(2z) \| \le \| f(x+y) - f(-2z) \| \tag{1.4}$$

and prove the generalized Hyers-Ulam stability of it in non-Archimedean spaces.

2. Solutions and Stability of (1.4)

In this section, let X be a non-Archimedean space and Y be a non-Archimedean Banach space. We start with the following theorem:

Theorem 2.1. *A mapping* $f : X \to Y$ *satisfies* (1.4) *if and only if f is additive*.

Proof. Suppose that f satisfies (1.4). Setting $x = y = z = 0$ in (1.4), we have $\| 3f(0) \| \leq \| 0 \|$ for all $w \in X$ and so we have

$$f(0) = 0. \tag{2.1}$$

Putting $y = -x$ and $z = 0$ in (1.4), we have $\| f(x) + f(-x) \| \leq \| 0 \|$ for all $w \in X$ and so $\| f(x) + f(-x) \| \leq 0$ for all $w \in X$. Hence we have

$$f(-x) = -f(x) \tag{2.2}$$

for all $x \in X$. Replacing x, y, z by $-2x -2y,\ 2x,\ 2y$, respectively, in (1.4), by (2.2), we have

$$\| f(-2x - 2y) + f(2x) + f(2y) \| \leq \| f(-2y) - f(-2y) \| = 0 \tag{2.3}$$

for all $x, y, w \in X$. Hence $\| f(-2x - 2y) + f(2x) + f(2y) \| = 0$ for all $w \in X$, and by (2.2), we have

$$f(2x + 2y) = f(2x) + f(2y) \tag{2.4}$$

for all $x, y \in X$. Replacing $2x$ and $2y$ by x and y, respectively, in (2.4), we can show that f is additive. The converse is trivial. □

Now, we will prove the generalized Hyers-Ulam stability of (1.4) in non-Archimedean spaces.

Theorem 2.2. *Assume that* $\phi : X^3 \to [0, \infty)$ *is a function such that*

$$\lim_{n\to\infty} \frac{\phi((-2)^n x, (-2)^n x, (-2)^n x)}{|2|^n} = 0 \tag{2.5}$$

for all $x \in X$ *and for any* $x \in X$, *the limit*

$$\lim_{n\to\infty} \max\left\{ \frac{\phi((-2)^k x, (-2)^k x, -(-2)^k x)}{|2|^{k-1}} \mid 0 \leq k \leq n-1 \right\} \tag{2.6}$$

exists. Let $f : X \to Y$ *be a mapping such that*

$$\| f(x) + f(y) + f(2z) \| \leq \| f(x + y) - f(-2z) \| + \phi(x, y, z) \tag{2.7}$$

for all $x, y, z \in X$. *Then there exists an additive mapping* $A : X \to Y$ *such that A satisfies* (1.4) *and*

$$\| f(x) - A(x) \|$$

$$\leq \lim_{n\to\infty} \max\left\{ \frac{\phi((-2)^k x, (-2)^k x, -(-2)^k x)}{|\, 2\,|^{k-1}} \mid 0 \leq k \leq n-1 \right\} \tag{2.8}$$

for all $x \in X$. *Moreover, if* $\phi : X^4 \to [0, \infty)$ *satisfies*

$$\lim_{k\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\phi((-2)^i x, (-2)^i x, (-2)^i x)}{|\, 2\,|^{i}} \mid k \leq i \leq n-1 \right\} = 0, \tag{2.9}$$

then A is a unique additive mapping satisfying (2.8).

Proof. Replacing x, y, z by $(-2)^n x$, $(-2)^n x$, $-(-2)^n x$ in (2.7), respectively, and dividing (2.7) by $|\, 2\,|^{n+1}$, since $f(0) = 0$, we have

$$\left\| \frac{f((-2)^n x)}{(-2)^n} - \frac{f((-2)^{n+1} x)}{(-2)^{n+1}} \right\| \leq \frac{\phi((-2)^n x, (-2)^n x, -(-2)^n x)}{|\, 2\,|^{n+1}}$$

for all $x \in X$ and all $n \in \mathbb{N}$. By (2.5), $\left\{ \frac{f((-2)^n x)}{(-2)^n} \right\}$ is a Cauchy sequence in *Y* for all $x \in X$ and since *Y* is a non-Archimedean Banach space, there is a function $A : X \to Y$ such that $A(x) = \lim_{n\to\infty} \frac{f((-2)^n x)}{(-2)^n}$ for all $x \in X$. Moreover, for $0 \leq m < n$, we have

$$\left\| \frac{f((-2)^n x)}{(-2)^n} - \frac{f((-2)^m x)}{(-2)^m} \right\|$$

$$\leq \max\left\{ \frac{\phi((-2)^k x, (-2)^k y, (-2)^k z)}{|\, 2\,|^{k+1}} \mid m \leq k \leq n-1 \right\} \tag{2.10}$$

for all $x \in X$. Replacing x, y, z by $(-2)^n x$, $(-2)^n y$, $(-2)^n z$ in (1.4), respectively, and dividing (1.4) by $|2|^n$, we have

$$\left\| \frac{f((-2)^n x)}{(-2)^n} + \frac{f((-2)^n y)}{(-2)^n} + \frac{f((-2)^n 2z)}{(-2)^n} \right\|$$

$$\leq \frac{\| f((-2)^n (x-y)) - f((-2)^{n+1} 2z) \|}{|2|^n} + \frac{\phi((-2)^n x, (-2)^n y, (-2)^n z)}{|2|^n}$$

$$\leq \left\| \frac{f((-2)^n (x-y))}{(-2)^n} - \frac{f((-2)^{n+1} 2z)}{(-2)^n} \right\| + \frac{\phi((-2)^n x, (-2)^n y, (-2)^n z)}{|2|^n} \tag{2.11}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.11), by (2.5), we have

$$\| A(x) + A(y) + A(2z) \| \leq \| A(x-y) - A(2z) \|$$

for all $x, y, z \in X$. By Theorem 2.1, A is additive and by (2.6) and (2.10), we have (2.8).

Now, we show the uniqueness of A. Suppose that (2.5) holds and A_0 is an additive mapping with (2.8). Then for any positive integer n, $2^n A(x) = A(2^n x)$ and $2^n A_0(x) = A_0(2^n x)$ for all $x \in X$. Hence by (2.8), we have

$$\| A(x) - A_0(x) \|$$

$$\leq \frac{\| A((-2)^k x) - f((-2)^k x) \|}{|2|^k} + \frac{\| A_0((-2)^k x) - f((-2)^k x) \|}{|2|^k}$$

$$\leq \lim_{n \to \infty} \max \left\{ \frac{\phi((-2)^{i+k} x, (-2)^{i+k} x, -(-2)^{i+k} x)}{|2|^{i+k}} \mid 0 \leq i \leq n-1 \right\}$$

$$\leq \lim_{n \to \infty} \max \left\{ \frac{\phi((-2)^i x, (-2)^i x, -(-2)^i x)}{|2|^i} \mid k \leq i \leq n-1 \right\}$$

for all $x \in X$ and all $k \in \mathbb{N}$. Hence, letting $k \to \infty$ in the above inequality, by (2.5), we have $A(x) = A_0(x)$ for all $x \in X$. $\square$

Related with Theorem 2.2, we can also have the following theorem. And the proof is similar to that of Theorem 2.2.

Theorem 2.3. *Assume that* $\phi : X^3 \to [0, \infty)$ *is a function such that*

$$\lim_{n\to\infty} |2|^n \phi\left(\frac{x}{(-2)^n}, \frac{x}{(-2)^n}, \frac{x}{(-2)^n}\right) = 0 \tag{2.12}$$

for all $x \in X$ *and for any* $x \in X$, *the limit*

$$\lim_{n\to\infty} \max\left\{|2|^{k-1}\phi\left(\frac{x}{(-2)^k}, \frac{x}{(-2)^k}, \frac{-x}{(-2)^{k+2}}\right) \mid 0 \le k \le n-1\right\} \tag{2.13}$$

exists. Let $f : X \to Y$ *be a mapping satisfying* (2.7). *Then there exists an additive mapping* $A : X \to Y$ *such that A satisfies* (1.4) *and*

$$\|f(x) - A(x)\|$$

$$\le \lim_{n\to\infty} \max\left\{|2|^{k-1}\phi\left(\frac{x}{(-2)^k}, \frac{x}{(-2)^k}, \frac{-x}{(-2)^k}\right) \mid 0 \le k \le n-1\right\} \tag{2.14}$$

for all $x \in X$. *Moreover, if* $\phi : X^3 \to [0, \infty)$ *satisfies*

$$\lim_{k\to\infty} \lim_{n\to\infty} \max\left\{|2|^{i}\phi\left(\frac{x}{(-2)^i}, \frac{x}{(-2)^i}, \frac{x}{(-2)^i}\right) \mid k \le i \le n-1\right\} = 0, \tag{2.15}$$

then A is a unique additive mapping satisfying (2.8).

As an example of $\phi(x, y)$ in Theorem 2.2 and Theorem 2.3, we can take $\phi(x, y, z) = \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$. Then we can formulate the following corollary:

Corollary 2.4. *Let* $f : X \to Y$ *be a mapping such that*

$$\|f(x) + f(y) + f(2z)\| \le \|f(x+y) - f(-2z)\| + \phi(x, y, z) \tag{2.16}$$

for all $x, y, z \in X$. *Suppose that* $|2| < 1$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that A satisfies* (1.4) *and*

$$\| A(x) - f(x) \| \leq \begin{cases} |\,2\,|^{1-p}(2|\,2\,|^{p} + 1)\| x \|^{p}, & \text{if } 1 < p, \\ |\,2\,|^{-1-p}(2|\,2\,|^{p} + 1)\| x \|^{p}, & \text{if } 0 < p < 1 \end{cases}$$

for all $x \in X$.

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