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STABILITY AND UNIQUENESS OF THE SOLUTION OF AN ILL-POSED ELLIPTIC PROBLEM

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Abstract

In this work, we consider an ill-posed Cauchy problem for an elliptic equation with variable coefficients. We assume the existence of a solution $u(x, \cdot)$ in $H^2(R)$ and we use a wavelet regularization method. Furthermore, we get the numerical estimates for the convergence of the method and prove the uniqueness of the solution for the problem.

1. Introduction

In paper [3], we studied the following ill-posed parabolic problem with variable coefficients:

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$$K(x)u_{xx}(x, t) = u_t(x, t), \quad t \geq 0, \quad 0 < x < 1,$$

$$u(0, \cdot) = g, \quad u_x(0, \cdot) = 0,$$

$$0 < \alpha \leq K(x) < +\infty, \ K \textit{ continuous}.$$

By assuming the existence of a solution for this problem, we had regularized the ill-posedness of the problem approximating it by well-posed problems in the scaling spaces of the Meyer multiresolution analysis with an estimate error.

Furthermore, in our work [2], under the hypothesis that $\frac{1}{K(x)}$ is Lipschitz, we proved that the above mentioned problem has at most one solution $u(x, \cdot)$ in the Sobolev space $H^1(R)$.

In another work [4], we had extended the results in [2] and [3] to the hyperbolic problem with $u(x, \cdot) \in H^2(R)$:

$$K(x)u_{xx}(x, t) = u_{tt}(x, t), \quad t \geq 0, \quad 0 < x < 1,$$

$$u(0, \cdot) = g, \quad u_x(0, \cdot) = 0,$$

$$0 < \alpha \leq K(x) < +\infty, \quad K \textit{ continuous}.$$

Now, we will consider the following elliptic problem:

$$K(x)u_{xx}(x, y) + u_{yy}(x, y) = 0, \quad 0 < x < 1, \quad -\infty < y < +\infty,$$

$$u(0, y) = g(y), \quad u_x(0, y) = 0, \quad -\infty < y < +\infty,$$

$$0 < \alpha \leq K(x) < +\infty, \ K \textit{ continuous}. \tag{1.1}$$

In [6] and [7], the authors considered the Cauchy problem for the Laplace equation (1.1) in the case $K(x) \equiv 1$, and in [8], the authors considered the three dimensional problem.

We assume $g \in L^2(R)$ and the problem (1.1) to have a solution $u(x, \cdot) \in H^2(R)$.

Our approach follows quite closely to the one used in [2-4].

The problem (1.1) is ill-posed in the following sense: if the problem has a solution, then a small disturbance on the boundary specification g of the problem can produce a large alteration in the solution (see Note (1)).

We consider a multiresolution analysis with the Meyer's wavelet because its Fourier transform has compact support. So, the orthogonal projections onto scaling spaces cutoff the high frequencies.

From the variational formulation of the approximating problem on the scaling space V_j, we get an infinite-dimensional system of second order ordinary differential equations with variable coefficients. An estimate obtained for the solution of this evolution problem is used to regularize the ill-posed problem approaching it by well-posed problems. Using an estimate obtained for the difference between the exact solution of problem (1.1) and its orthogonal projection onto V_j, we get an estimate for the difference between the exact solution of problem (1.1) and the orthogonal projection, onto V_j, of the solution of the approximating problem defined on the scaling space V_{j-1}. Further we consider that $1/K(x)$ is Lipschitz and we prove that the existence of a solution $u(x, \cdot) \in H^2(R)$ implies its uniqueness.

It is very important to point out that our result is weaker than the overall uniqueness of a solution $u(\cdot, \cdot)$ of problem (1.1), which cannot be discussed without further conditions on this problem. Our uniqueness result supposes that $x \in (0, 1)$ is fixed and it is the solution $u(x, \cdot) \in H^2(R)$, as function of the second variable, which is proved to be unique. More precisely, a solution $u(x, \cdot)$ can only be modified in a subset of $(-\infty, +\infty)$ of measure zero.

In Section 2, we construct the Meyer multiresolution analysis. In Section 3, we get the estimates of the numerical stability and the convergence of the wavelet Galerkin method. In Section 4, we prove the uniqueness of the solution.

For a function $h \in L^1(R) \cap L^2(R)$, its Fourier transform is given by $\hat{h}(\xi) := \int_{\mathbb{R}} h(x)e^{-ix\xi}dx$. We use the notation e^x and $\exp x$ indistinctly.

2. Meyer Multiresolution Analysis

Definition 2.1. A *multiresolution analysis*, as defined in [1], is a sequence of closed subspaces V_j in $L^2(R)$, called *scaling spaces*, satisfying:

(M1) $V_j \subseteq V_{j-1}$ for all $j \in Z$.

(M2) $\bigcup_{j \in Z} V_j$ is dense in $L^2(R)$.

(M3) $\bigcap_{j \in Z} V_j = \{0\}$.

(M4) $f \in V_j$ if and only if $f(2^j \cdot) \in V_0$.

(M5) $f \in V_0$ if and only if $f(\cdot - k) \in V_0$ for all $k \in Z$.

(M6) There exists $\phi \in V_0$ such that $\{\phi_{0,k} : k \in Z\}$ is an orthonormal basis in V_0, where $\phi_{j,k}(x) = 2^{-j/2}\phi(2^{-j}x - k)$ for all $j, k \in Z$. The function ϕ is called the *scaling function* of the multiresolution analysis.

The scaling function of the Meyer multiresolution analysis is the function φ defined by its Fourier transform:

$$\hat{\varphi}(\xi) := \begin{cases} 1, & |\xi| \le \dfrac{2\pi}{3}, \\ \cos\left[\dfrac{\pi}{2}\nu\left(\dfrac{3}{2\pi}|\xi| - 1\right)\right], & \dfrac{2\pi}{3} \le |\xi| \le \dfrac{4\pi}{3}, \\ 0, & |\xi| > \dfrac{4\pi}{3}, \end{cases}$$

where ν is a differentiable function satisfying

$$\nu(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x \geq 1 \end{cases}$$

and

$$\nu(x) + \nu(1 - x) = 1.$$

The associated mother wavelet ψ, called *Meyer's wavelet*, is given by (see [1])

$$\hat{\psi}(\xi) := \begin{cases} e^{i\xi/2} \sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\xi| - 1\right)\right], & \frac{2\pi}{3} \leq |\xi| \leq \frac{4\pi}{3}, \\ e^{i\xi/2} \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\xi| - 1\right)\right], & \frac{4\pi}{3} \leq |\xi| \leq \frac{8\pi}{3}, \\ 0, & |\xi| > \frac{8\pi}{3}. \end{cases}$$

We will consider the Meyer multiresolution analysis with scaling function φ. We have

$$\widehat{\psi_{jk}}(\xi) = 2^{j/2} e^{-i2^{j}k\xi} \hat{\psi}(2^{j}\xi).$$

So, since $\operatorname{supp}(\hat{\psi}) = \left\{\xi : \frac{2}{3}\pi \leq |\xi| \leq \frac{8}{3}\pi\right\}$, we have that

$$\operatorname{supp}(\widehat{\psi_{jk}}) = \left\{\xi;\ \frac{2}{3}\pi 2^{-j} \leq |\xi| \leq \frac{8}{3}\pi 2^{-j}\right\}, \quad \forall k \in \mathbb{Z}. \tag{2.1}$$

Furthermore,

$$\operatorname{supp}(\widehat{\varphi_{jk}}) = \left\{\xi;\ |\xi| \leq \frac{4}{3}\pi 2^{-j}\right\}, \quad \forall k \in \mathbb{Z}. \tag{2.2}$$

The orthogonal projection onto V_j, $P_j : L^2(R) \to V_j$, is given by

$$P_j f(t) = \sum_{k \in \mathbb{Z}} \langle f, \varphi_{jk} \rangle \varphi_{jk}(t).$$

From (2.2), we see that P_j filters away the frequencies higher than $\frac{4}{3}\pi 2^{-j}$.

We have, for all $f \in L^2(R)$,

$$\begin{aligned} f &= P_j f - P_j f + f \\ &= P_j f + (I - P_j) f \\ &= \sum_{k \in \mathbb{Z}} \langle f, \varphi_{jk} \rangle \varphi_{jk} + \sum_{l \le j} \sum_{k \in \mathbb{Z}} \langle f, \psi_{lk} \rangle \psi_{lk}. \end{aligned}$$

This implies

$$\widehat{P_j f}(\xi) = \hat{f}(\xi) \text{ for } |\xi| \le \frac{2}{3}\pi 2^{-j} \tag{2.3}$$

since, by (2.1), $\hat{\psi}_{lk}(\xi) = 0$ for all $l \le j$ and $|\xi| \le \frac{2}{3}\pi 2^{-j}$.

Considering the corresponding orthogonal projections in the frequency space, $\widehat{P_j} : L^2(R) \to \widehat{V_j} = \overline{\text{span}\{\widehat{\varphi_{jk}}\}}_{k \in \mathbb{Z}}$,

$$\widehat{P_j} f = \sum_{k \in \mathbb{Z}} \frac{1}{2\pi} \langle f, \widehat{\varphi_{jk}} \rangle \widehat{\varphi_{jk}},$$

we have

$$\widehat{P_j} \hat{f} = \sum_{k \in \mathbb{Z}} \frac{1}{2\pi} \langle \hat{f}, \widehat{\varphi_{jk}} \rangle \widehat{\varphi_{jk}} = \sum_{k \in \mathbb{Z}} \langle f, \varphi_{jk} \rangle \widehat{\varphi_{jk}} = \widehat{P_j f}.$$

Then (2.3) implies that

$$\begin{aligned} \| (I - P_j) f \| &= \frac{1}{\sqrt{2\pi}} \| \widehat{f - P_j f} \| \\ &= \frac{1}{\sqrt{2\pi}} \| (I - \widehat{P_j}) \hat{f} \| \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \| (I - \widehat{P_j}) \chi_j \hat{f} \|$$

$$\leq \| \chi_j \hat{f} \|, \tag{2.4}$$

where χ_j is the characteristic function in $\left(-\infty, -\frac{2}{3}\pi 2^{-j}\right] \cup \left[\frac{2}{3}\pi 2^{-j}, +\infty\right)$.

3. Stability and Convergence of the Method

In this section, we approach the ill-posed problem (1.1) by well-posed problems, and we show, with an estimate error, the convergence of the wavelet method used. The next lemma is given in [3].

Lemma 3.1. *Let u and v be positive continuous functions, $x \geq a$ and $c > 0$. If $u(x) \leq c + \int_a^x \int_a^s v(\tau) d\tau ds$, then*

$$u(x) \leq c \exp\left(\int_a^x \int_a^s v(\tau) d\tau ds \right).$$

Proof. See [3, p. 215]. □

Applying the Fourier transform with respect to variable y in problem (1.1), we obtain the following problem in the frequency space:

$$\hat{u}_{xx}(x, \xi) = \frac{\xi^2}{K(x)} \hat{u}(x, \xi), \quad 0 < x < 1, \quad \xi \in R,$$

$$\hat{u}(0, \xi) = \hat{g}(\xi), \quad \hat{u}_x(0, \cdot) = 0$$

whose solution satisfies

$$| \hat{u}(x, \xi) | \leq | \hat{g}(\xi) | + \int_0^x \int_0^s \frac{\xi^2}{K(\tau)} | \hat{u}(\tau, \xi) | d\tau ds.$$

Then, by Lemma 3.1, for $\hat{g}(\xi) \neq 0$, we have

$$| \hat{u}(x, \xi) | \leq | \hat{g}(\xi) | \exp\left[\xi^2 \int_0^x \int_0^s \frac{1}{K(\tau)} d\tau ds \right]. \tag{3.1}$$

Lemma 3.2. *The operator* $D_j(x)$ *defined by*

$$[(D_j)_{lk}(x)]_{l \in \mathbb{Z},\, k \in \mathbb{Z}} = \left[\frac{1}{K(x)} \langle \varphi''_{jl}, \varphi_{jk} \rangle\right]_{l \in \mathbb{Z},\, k \in \mathbb{Z}}$$

satisfies the following three conditions:

(1) $(D_j)_{lk}(x) = (D_j)_{kl}(x)$.

(2) $(D_j)_{lk}(x) = (D_j)_{(l-k)0}(x)$. *Hence*, $(D_j)_{lk}(x)$ *is a Toeplitz matrix*.

(3) $\| D_j(x) \| \leq \dfrac{\pi^2 4^{-j+1}}{K(x)}$.

Proof. The proof follows as in [2], regarding that in (1), we can integrate twice by parts, since φ and φ' are reals and $\varphi_{jk}(x) \to 0$, $\varphi'_{jk}(x) \to 0$, when $x \to \pm\infty$, and in (3), Γ_j is defined by:

$$\Gamma_j(t) = -2^{-j}[(t - 2^{-j+1}\pi)^2 |\widehat{\varphi_{j0}}(t - 2^{-j+1}\pi)|^2 + t^2 |\widehat{\varphi_{j0}}(t)|^2$$
$$+ (t + 2^{-j+1}\pi)^2 |\widehat{\varphi_{j0}}(t + 2^{-j+1}\pi)|^2]. \qquad \square$$

Let us now consider the following approximating problem in V_j:

$$\begin{cases} K(x)u_{xx}(x, y) + P_j u_{yy}(x, y) = 0, \ 0 < x < 1, -\infty < y < +\infty, \\ u(0, \cdot) = P_j g, \\ u_x(0, \cdot) = 0, \\ u(x, y) \in V_j, \end{cases} \tag{3.2}$$

where the projection in the first equation of (3.2) is needed because we can have $\varphi \in V_j$ with $\varphi'' \notin V_j$ (see Note (2) below).

Its variational formulation is

$$\begin{cases} \langle K(x)u_{xx} + u_{yy}, \varphi_{jk} \rangle = 0, \\ \langle u(0, \cdot), \varphi_{jk} \rangle = \langle P_j g, \varphi_{jk} \rangle, \quad \langle u_x(0, \cdot), \varphi_{jk} \rangle = \langle 0, \varphi_{jk} \rangle, \quad k \in Z, \end{cases}$$

where φ_{jk} is the orthonormal basis of V_j given by the scaling function φ. Consider u_j is a solution of the approximating problem (3.2), given by $u_j(x, y) = \sum_{l\in\mathbb{Z}} w_l(x)\varphi_{jl}(y)$. Then we have

$$(u_j)_{yy}(x, y) = \sum_{l\in\mathbb{Z}} w_l(x)\varphi''_{jl}(y)$$

and $(u_j)_{xx}(x, y) = \sum_{l\in\mathbb{Z}} w''_l(x)\varphi_{jl}(y)$. Therefore,

$$K(x)(u_j)_{xx}(x, y) + (u_j)_{yy}(x, y)$$

$$= K(x)\sum_{l\in\mathbb{Z}} w''_l(x)\varphi_{jl}(y) + \sum_{l\in\mathbb{Z}} w_l(x)\varphi''_{jl}(y).$$

Hence,

$$\langle K(x)(u_j)_{xx} + (u_j)_{yy}, \varphi_{jk}\rangle = 0$$

$$\Leftrightarrow \left\langle \sum_{l\in\mathbb{Z}} K(x)w''_l\varphi_{jl} + \sum_{l\in\mathbb{Z}} w_l\varphi''_{jl}, \varphi_{jk} \right\rangle = 0$$

$$\Leftrightarrow \sum_{l\in\mathbb{Z}} K(x)w''_l\langle\varphi_{jl}, \varphi_{jk}\rangle = -\sum_{l\in\mathbb{Z}} w_l\langle\varphi''_{jl}, \varphi_{jk}\rangle$$

$$\Leftrightarrow K(x)w''_k = -\sum_{l\in\mathbb{Z}} w_l\langle\varphi''_{jl}, \varphi_{jk}\rangle, \quad k \in \mathbb{Z}$$

$$\Leftrightarrow \frac{d^2}{dx^2}w_k = -\sum_{l\in\mathbb{Z}} w_l \frac{1}{K(x)}\langle\varphi''_{jl}, \varphi_{jk}\rangle$$

$$\Leftrightarrow \frac{d^2}{dx^2}w_k = -\sum_{l\in\mathbb{Z}} w_l(D_j)_{lk}(x),$$

where, as defined before, $(D_j)_{lk}(x) = \frac{1}{K(x)}\langle\varphi''_{jl}, \varphi_{jk}\rangle$. Thus, we get an infinite-dimensional system of ordinary differential equations:

$$\begin{cases} \dfrac{d^2}{dx^2} w = -D_j(x) w, \\ w(0) = \gamma, \\ w'(0) = 0, \end{cases} \tag{3.3}$$

where γ is given by

$$P_j g = \sum_{z \in \mathbb{Z}} \gamma_z \varphi_{jz} = \sum_{z \in \mathbb{Z}} \langle g, \varphi_{jz} \rangle \varphi_{jz}.$$

Lemma 3.3. *If w is a solution of the evolution problem of second order* (3.3), *then*

$$\| w(x) \| \leq \| \gamma \| \exp\left(4^{-j+1} \pi^2 \int_0^x \int_0^s \frac{1}{K(\tau)} d\tau ds \right).$$

Proof. Follows by Lemma 3.1 and Lemma 3.2. See details in [3]. □

Theorem 3.4 (Stability)**.** *Let* u_j *and* v_j *be solutions in* V_j *of the approximating problems* (3.2) *for the boundary specifications g and* $\tilde{g}$, *respectively. If* $\| g - \tilde{g} \| \leq \varepsilon,$ *then*

$$\| u_j(x, \cdot) - v_j(x, \cdot) \| \leq \varepsilon \exp\left(\frac{4^{-j+1} \pi^2}{2\alpha} x^2 \right),$$

where α *satisfies* $0 < \alpha \leq K(x) < +\infty$ *as in the definition of the problem* (1.1). *For j such that* $4^{-j} \leq \dfrac{\alpha}{2\pi^2} \log \varepsilon^{-1},$ *we have*

$$\| u_j(x, \cdot) - v_j(x, \cdot) \| \leq \varepsilon^{1 - x^2}.$$

Proof. By Lemma 3.3 and linearity of (3.3), the proof follows quite closely to the one of Theorem 3.4 in [3, p. 221]. □

We will consider the problem (1.1), for the functions $g \in L^2(R)$ such that $\hat{g}(\xi)\exp(\xi^2/(2\alpha)) \in L^2(R)$, where $\hat{g}$ is the Fourier transform of g. The inverse Fourier transform of $\exp\left(-\frac{\xi^2 + |\xi|}{2\alpha}\right)$, for instance, satisfies this condition. Define

$$f := \hat{g}(\xi)\exp\left(\frac{\xi^2}{2\alpha}\right) \in L^2(R). \tag{3.4}$$

Proposition 3.5. *If* $u(x, t)$ *is a solution of problem* (1.1), *then*

$$\| u(x, \cdot) - P_j u(x, \cdot) \| \leq \| f \|_{L^2(\mathbb{R})} \exp\left(-\frac{2}{9}\frac{\pi^2}{\alpha} 4^{-j}(1 - x^2)\right),$$

where f is given by (3.4).

Proof. Follows by (2.4) and (3.1) as Proposition 3.5 in [3]. □

Proposition 3.6. *If u is a solution of problem* (1.1) *and* u_{j-1} *is a solution of the approximating problem in* V_{j-1}, *then*

$$\hat{u}(x, \xi) = \hat{u}_{j-1}(x, \xi) \text{ for } |\xi| \leq \frac{4}{3}\pi 2^{-j}. \tag{3.5}$$

Consequently,

$$P_j u(x, \cdot) = P_j u_{j-1}(x, \cdot). \tag{3.6}$$

Proof. See Proposition 3.6 in [3]. □

Theorem 3.7. *Let u be a solution of* (1.1) *with the condition* $u(0, \cdot) = g$, *and let f be given by* (3.4). *Let* v_{j-1} *be a solution of* (3.2) *in* V_{j-1} *for the boundary specification* $\tilde{g}$ *such that* $\| g - \tilde{g} \| \leq \varepsilon$. *If* $j = j(\varepsilon)$ *is such that* $4^{-j} = \frac{\alpha}{8\pi^2}\log\varepsilon^{-1}$, *then*

$$\| P_j v_{j-1}(x, \cdot) - u(x, \cdot) \| \leq \varepsilon^{1-x^2} + \| f \|_{L^2(R)} \cdot \varepsilon^{\frac{1}{36}(1-x^2)}.$$

Proof.

$$\begin{aligned} &\| P_j v_{j-1}(x, \cdot) - u(x, \cdot) \| \\ &\leq \| P_j v_{j-1}(x, \cdot) - P_j u(x, \cdot) + P_j u(x, \cdot) - u(x, \cdot) \| \\ &\leq \| P_j v_{j-1}(x, \cdot) - P_j u(x, \cdot) \| + \| P_j u(x, \cdot) - u(x, \cdot) \|. \end{aligned}$$

Let u_{j-1} be a solution of (3.2) in V_{j-1} for the boundary specification g. By (3.6), $P_j u(x, \cdot) = P_j u_{j-1}(x, \cdot)$. Thus, by Theorem 3.4, we have

$$\begin{aligned} \| P_j v_{j-1}(x, \cdot) - P_j u(x, \cdot) \| &= \| P_j v_{j-1}(x, \cdot) - P_j u_{j-1}(x, \cdot) \| \\ &\leq \| v_{j-1}(x, \cdot) - u_{j-1}(x, \cdot) \| \\ &\leq \varepsilon^{1-x^2}. \end{aligned}$$

Now, by Proposition 3.5,

$$\begin{aligned} \| P_j u(x, \cdot) - u(x, \cdot) \| &\leq \| f \|_{L^2(\mathbb{R})} \exp\left(-\frac{2}{9} \frac{\pi^2}{\alpha} 4^{-j} (1 - x^2) \right) \\ &\leq \| f \|_{L^2(\mathbb{R})} \cdot \varepsilon^{\frac{1}{36}(1-x^2)}. \end{aligned}$$

Then $\| P_j v_{j-1}(x, \cdot) - u(x, \cdot) \| \leq \varepsilon^{1-x^2} + \| f \|_{L^2(R)} \varepsilon^{\frac{1}{36}(1-x^2)}$. □

4. Uniqueness of the Solution

The infinite-dimensional system of ordinary differential equations (3.3) can be written in the following way:

$$\begin{cases} \dfrac{dv}{dx} = -D_j(x) w + 0v, \\ \dfrac{dw}{dx} = 0w + v, \\ w(0) = \gamma \text{ and } v(0) = 0, \end{cases} \qquad \begin{cases} \dfrac{dV}{dx} = A_j(x) V, \\ V(0) = (0, \gamma)^T, \end{cases}$$

where $V = (v, w) \in X := l^2(R) \times l^2(R),\ x \in [0, 1)$ and

$$A_j(x) = \begin{bmatrix} 0 & -D_j(x) \\ 1 & 0 \end{bmatrix}$$

with $\| A_j(x)V \|_X = \| (D_j(x)w, v) \|_X = \sqrt{\| D_j(x)w \|_{l^2}^2 + \| v \|_{l^2}^2}$.

Lemma 4.1. *For all* $j \in Z$, $A_j(x): X \to X$ *is a uniformly bounded linear operator on* $x \in [0, 1)$.

Proof. By Lemma 3.2 and the hypothesis $0 < \alpha \le K(x) < +\infty$, we have

$$\| D_j(x) \| \le \frac{\pi^2 4^{-j+1}}{K(x)} \le \frac{\pi^2 4^{-j+1}}{\alpha} := K_j.$$

If $\| V \|_X = 1$, then $\| w \|_{l^2} \le 1$ and $\| v \|_{l^2} \le 1$. So,

$$\| A_j(x)V \|_X = \sqrt{\| D_j(x)w \|_{l^2}^2 + \| v \|_{l^2}^2} \le \sqrt{K_j^2 + 1}.$$

Hence, the operator $A_j(x)$ is uniformly bounded on $x \in [0, 1)$. □

Lemma 4.2. *If* $\frac{1}{K(x)}$ *is Lipschitz on* $[0, 1)$, *then* $x \mapsto D_j(x)$ *is Lipschitz on* $[0, 1)$, $\forall j \in Z$. *Consequently*, $x \mapsto A_j(x)$ *is Lipschitz on* $[0, 1)$.

Proof. $D_j(x) = \frac{1}{K(x)} B_j$, where $(B_j)_{lk} = \langle \varphi''_{jl}, \varphi_{jk} \rangle$ and $\| B_j \| \le \pi^2 4^{-j+1}$. Then

$$\| D_j(x) - D_j(\tilde{x}) \| \le \left| \frac{1}{K(x)} - \frac{1}{K(\tilde{x})} \right| \pi^2 4^{-j+1} \le L_j | x - \tilde{x} |$$

with $L_j = L \cdot \pi^2 4^{-j+1}$, where L is the Lipschitz constant of $\frac{1}{K(x)}$.

Now,

$$\begin{aligned}\| A_j(x) - A_j(\tilde{x}) \| &= \sup_{V \in X, \| V \| = 1} \| (A_j(x) - A_j(\tilde{x}))V \|_X \\ &= \sup_{w \in l^2, \| w \| = 1} \| (D_j(x) - D_j(\tilde{x}))w \|_{l^2} \\ &= \| D_j(x) - D_j(\tilde{x}) \| \\ &\leq L_j | x - \tilde{x} |. \qquad \square\end{aligned}$$

Lemma 4.3. *For each* $j \in Z$, *the operator* $[0, 1) \ni x \mapsto A_j(x)$ *is continuous in the uniform operator topology.*

Proof. Let $x \in [0, 1)$ and $\varepsilon > 0$. By Lemma 4.2, $A_j(x)$ is Lipschitz with Lipschitz constant L_j. Let $\delta_\varepsilon := \varepsilon / L_j$. We have, for $\tilde{x} \in [0, 1)$:

$$| x - \tilde{x} | < \delta_\varepsilon \Rightarrow \| A_j(x) - A_j(\tilde{x}) \| \leq L_j | x - \tilde{x} | < L_j \cdot \delta_\varepsilon = \varepsilon. \qquad \square$$

By previous lemmas, we have:

Theorem 4.4. *The infinite-dimensional system of ordinary differential equations* (3.3) *has a unique solution.*

Proof. The result follows by Lemma 4.1, Lemma 4.2, Lemma 4.3 above and Theorem 5.1 in [5, p. 127]. $\square$

Theorem 4.5. *Let u be a solution of problem* (1.1) *with condition* $u(0, \cdot) = g$, *where g satisfies* (3.4). *Then, for any sequence* j_n *such that* $j_n \to -\infty$ *as* $n \to +\infty$, *there exists a unique sequence* u_{j_n} *of solutions of the approximating problems* (3.2) *in* V_{j_n} *with conditions* $u_{j_n}(0, \cdot) = P_{j_n} g$ *and* $\forall x \in [0, 1)$ *such that*

$$P_{j_n + 1} u_{j_n}(x, \cdot) \to u(x, \cdot) \text{ in } L^2(R).$$

Proof. By Theorem 4.4, each approximating problem has a unique solution. Then the result follows by Theorem 3.7, with $\widetilde{g} = g$, since that j and ε are functionally related by $4^{-j} = \frac{\alpha}{8\pi^2} \log \varepsilon^{-1}$ independently of u. □

Corollary 4.6. *Problem* (1.1) *has at most one solution*, *for each* $x \in [0, 1)$, *where g satisfies* (3.4).

5. Conclusions

We have considered a solution $u(x, \cdot) \in H^2(R)$ for the ill-posed elliptic problem $K(x)u_{xx} + u_{yy} = 0$, $0 < x < 1$, $-\infty < y < +\infty$, $u(0, \cdot) = g \in L^2(R)$ and $u_x(0, \cdot) = 0$, where $K(x)$ is continuous, $0 < \alpha \le K(x) < +\infty$, $\frac{1}{K(x)}$ is Lipschitz and $\hat{g}(\xi)\exp(\xi^2/(2\alpha)) \in L^2(R)$. Utilizing a wavelet Galerkin method with the Meyer multiresolution analysis, we regularize the ill-posedness of the problem, approaching it by well-posed problems in the scaling spaces and we have shown the convergence of the wavelet Galerkin method applied to our problem, with an estimate error. We have shown that the solution $u(x, \cdot)$ is unique, for each $x \in [0, 1)$, fixed.

Notes. (1) Consider the Laplace equation with Cauchy conditions on x:

$$u_{xx}(x, y) + u_{yy}(x, y) = 0, \quad 0 < x < 1, \quad -\infty < y < +\infty,$$

$$u(0, \cdot) = g_n,\ u_x(0, \cdot) = 0,$$

where

$$g_n(y) = \begin{cases} n^{-2} \cos \sqrt{2}ny, & \text{if } 0 \le y \le y_0, \\ 0, & \text{otherwise.} \end{cases}$$

The solution of this problem is

$$u_n(x, y) = \begin{cases} \sum_{j=0}^{\infty} n^{-2} \cos(\sqrt{2}ny) \frac{(\sqrt{2}nx)^{2j}}{(2j)!}, & \text{if } 0 \le y \le y_0, \\ 0, & \text{otherwise.} \end{cases}$$

We have that $g_n(y)$ converges uniformly to zero as n tends to infinity, while for $x > 0$, the solution $u_n(x, y)$ does not tend to zero.

(2) Note that $(\varphi_{jl})'' \notin V_j$. In fact, if $(\varphi_{jl})'' \in V_j$, then $(\varphi_{jl})'' = \sum_{k \in Z} \alpha_k \varphi_{jk}$. Hence,

$$\widehat{(\varphi_{jl})''} = \sum_{k \in Z} \alpha_k \widehat{\varphi_{jk}}.$$

So, we would have

$$-2^{j/2} e^{-i2^j l\xi} \hat{\varphi}(2^j \xi) = \sum_{k \in Z} \alpha_k 2^{j/2} e^{-i2^{j/2}\xi} \hat{\varphi}(2^j \xi).$$

This equality implies $\xi^2 = \sum_{k \in Z} -\alpha_k e^{-i[2^j(k-l)\xi]}$.

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