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A NEW CHAOTIC ATTRACTOR FROM MODIFIED CHEN EQUATION

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Abstract

In this work we study the dynamical behavior of a new continuous-time three-dimensional autonomous chaotic system obtained from direct modification of the Chen equation. Equilibrium and their stability are discussed. Basic dynamical behaviors are briefly described. The possibility of circuitry realization is presented. The existence of chaotic attractors is justified with various numerical results which give some new chaotic solutions.

1. Introduction

In 1963, Edward Lorenz [6], described a simple mathematical model of a weather system that was made up of three linked nonlinear differential equations that showed rates of change in temperature and wind speed. Some surprising results showed complex behavior from supposedly simple equations; also, the behavior of the system of equations was sensitively dependent on the initial conditions of the

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model. He spelled out the implications of his discovery, saying it implied that if there were any errors in observing the initial state of the system and this is inevitable in any real system, then prediction as to a future state of the system was impossible.

Lorenz labeled these systems that exhibited sensitive dependence on initial conditions as having the "butterfly effect": this unique name came from the proposition that a butterfly flapping its wings in Hong Kong can affect the course of a tornado in Texas.

A new chaotic attractor of a three-dimensional system is coined by Chen and Ueta in 1999, in the pursuit of anti controlling chaos for Lorenz model [3], [11]. This new chaotic model reassembles the Lorenz and Rôssler systems [8]. The Chen model itself is modified by Aziz Alaoui [1] and it obtains another new "non-symmetric" chaotic attractor.

The Chen model appears to be more complex and sophisticated [11]. It has the same complexity as the Lorenz equation; they are both three-dimensional autonomous with only two quadratic terms, however it is topologically not equivalent to the Lorenz equation.

The Chen model is given by the following closed-form dimensionless equations:

$$\begin{aligned} dx/dt &= a(y - x), \\ dy/dt &= (c - a)x + cy - xz, \\ dz/dt &= xy - bz, \end{aligned} \tag{1}$$

which, for parameters $a = 35.0$, $b = 3.0$, $c = 28.0$, there is a chaotic attractor called *Chen attractor* and it is shown in Fig. 1(b).

In this work we present a new chaotic system obtained from direct modification of the Chen model: the term a is replaced by some real r and x by $\operatorname{sgn}(x)$ in the formula of dx/dt, the terms $(c - a)$ and xz in the dy/dt formula are replaced by $-(c + a)$ and $x\operatorname{sgn}(z)$, respectively, then one has a new continuous-time three-dimensional autonomous with only one nonlinear term which is xy. We have recently discovered that the new model can generate various chaotic attractors.

We study the following modified Chen's system:

$$dx/dt = r(y - \operatorname{sgn}(x)),$$

$$dy/dt = -(c + a)x + cy - x\operatorname{sgn}(z),$$

$$dz/dt = xy - bz, \tag{2}$$

where a, b, c and r are constants parameters, sgn(.) is the standard signum function that gives the sign of its argument.

It is interesting to compare the new attractor with the familiar Lorenz and Chen attractors to see the difference between them.

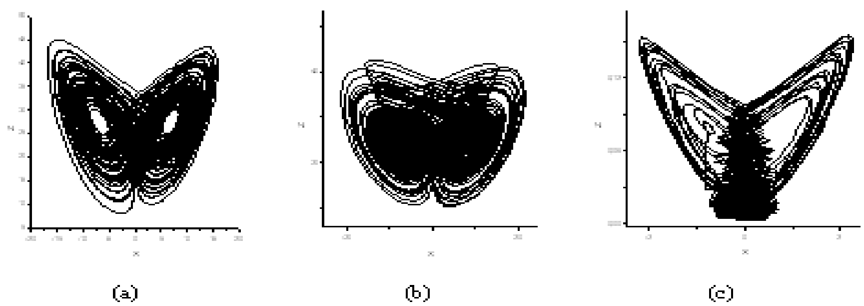

Figure 1. Comparison between the three chaotic attractors: (a) The Lorenz attractor, for $s = 10.$, $r = 28.$ and $b = 8/3.$, (b) The Chen attractor, for $a = 35.$, $b = 3.$, and $c = 28.$, (c) The new attractor, for $a = 2.$, $b = 10.$, $c = 9.$, $r = 40.$

2. Some Basic Properties

The new chaotic attractor has several important properties with both the Lorenz and Chen attractors with some differences between them. It has a natural symmetry under the coordinate transform $(x, y, z) \to (-x, -y, z)$ which persists for all values of the system parameters.

One remarks that all trajectories that start from the z-axis do not remain on it and not tend towards the origin, since for such trajectories, $dx/dt = -r$, $dy/dt = 0$ and $dz/dt = -bz$, thus, the z-axis $(x = y = 0)$ is not invariant except for $r = 0$. Therefore, for system (2), the divergence of the flow is given by

$$\Delta V = \frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial y} + \frac{\partial z'}{\partial z} = c - b.$$

Then one has the following proposition:

Proposition 1. *If* $c < b$, *then the system* (2) *has a bounded globally attracting* ω-*limit set.*

Then the system (2) is dissipative just like the Lorenz and Chen systems. Thus, all trajectories ultimately are confined to a specific subset having zero volume and the asymptotic motion settles onto an attractor, this result has been confirmed by some computer simulations.

3. Equilibrium Point and their Stability

Due to the shape of the vector field the phase space can be divided into four nonlinear regions denoted by E_i, $i = 1, 4$,

$$E_1 = \{(x, y, z) \in \mathbf{R}^3 / x \geq 0, z \geq 0\}$$

$$E_2 = \{(x, y, z) \in \mathbf{R}^3 / x \geq 0, z \leq 0\}$$

$$E_3 = \{(x, y, z) \in \mathbf{R}^3 / x \leq 0, z \geq 0\}$$

$$E_4 = \{(x, y, z) \in \mathbf{R}^3 / x \leq 0, z \leq 0\}.$$

In each of these regions, there exists a "symmetric" point $P^-(-x, -y, z)$ for each equilibrium $P^+(x, y, z)$, due to the symmetry of the vector field.

The equilibrium point of the system (2) can be found by solving the three equations: $dx/dt = dy/dt = dz/dt = 0$, which lead to

$$a(y - \operatorname{sgn}(x)) = 0,$$

$$-(c + a)x + cy - x \operatorname{sgn}(z) = 0,$$

$$xy - bz = 0.$$

Thus, one obtains that all equilibrium points of system (2) are given by

$$X_{\text{eq}(x)} = \left(x,\ \text{sgn}(x),\ \frac{x\,\text{sgn}(x)}{b}\right),$$

where x is the solution of the equation

$$-(c + a + \text{sgn}(z))x + c\,\text{sgn}(x) = 0.$$

One remarks that the origin is not an equilibrium point for the system (2). Thus one has the following propositions:

Proposition 2. *If* $a + c > 1$, *then there exist four equilibrium points for the system* (2),

$$P_1^{\pm}\left(\pm\frac{c}{a+c+1},\ \pm 1,\ \frac{c}{b(a+c+1)}\right)$$

$$P_2^{\pm}\left(\pm\frac{c}{a+c-1},\ \pm 1,\ \frac{c}{b(a+c-1)}\right)$$

else there exist only two equilibrium points

$$P_1^{\pm}\left(\pm\frac{c}{a+c+1},\ \pm 1,\ \frac{c}{b(a+c+1)}\right).$$

One remarks that

$$P_1^{+}\left(\frac{c}{a+c+1},\ 1,\ \frac{c}{b(a+c+1)}\right) \in E_1$$

$$P_1^{-}\left(\frac{c}{a+c-1},\ 1,\ \frac{c}{b(a+c-1)}\right) \in E_2$$

$$P_2^{+}\left(-\frac{c}{a+c+1},\ -1,\ \frac{c}{b(a+c+1)}\right) \in E_3$$

$$P_2^{-}\left(-\frac{c}{a+c-1},\ 1,\ \frac{c}{b(a+c-1)}\right) \in E_4$$

and in each region there exists only one equilibrium point.

Let us now study their stability. For this it must compute the Jacobian matrix evaluated at these equilibria, as follows:

$$J_{\pm}(P) = \begin{pmatrix} 0 & r & 0 \\ -(c + a + t_{\pm}) & c & 0 \\ s_{\pm} & x & -b \end{pmatrix},$$

where $s_{\pm} = \operatorname{sgn}(x)$ and $t_{\pm} = \operatorname{sgn}(z)$ and P is an equilibrium point. Thus one has the following propositions:

Proposition 3. *The equilibrium points* P_1^+ *and* P_2^+ *of system* (2) *have the same type of stability.*

Proposition 4. *The equilibrium points* P_1^- *and* P_2^- *of system* (2) *have the same type of stability.*

Because in each case the two equilibria have the same characteristic equation. The exact value of the eigenvalues is obtained by using the Cardan method for solving a cubic characteristic equation $P(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C$. By setting $\omega = \frac{-A}{3}$, these yield

$$G(X) = X^3 + PX + Q,$$

where

$$P = -\frac{A^2}{3} + B \text{ and } Q = \frac{2A^3}{27} - \frac{AB}{3} + C.$$

We set $\Delta = 4P^3 + 27Q^2$, resulting the following:

(i) If $\Delta > 0$, then there is a unique real eigenvalue $\lambda_R = -\frac{A}{3} + X_R$, where

$$X_R = \left(-\frac{Q}{2} + \sqrt{\frac{Q^2}{2} + \frac{P^3}{27}}\right)^{1/3} + \left(-\frac{Q}{2} - \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}\right)^{1/3}$$

along with two complex conjugate eigenvalues $(\lambda_C)^{\pm} = \omega + (X_C)^{\pm}$, where

$$(X_C)^{\pm} = \frac{-X_R}{2} \pm \frac{i}{2}\sqrt{4P + 3(X_R)^2}.$$

(ii) If $\Delta < 0,$ then the system has three real and distinct eigenvalues:

$$\lambda_1 = -\frac{A}{3} + 2\sqrt{-\frac{P}{3}}\sin\left(\frac{\theta}{3}\right),$$

$$\lambda_2 = -\frac{A}{3} + 2\sqrt{-\frac{P}{3}}\sin\left(\frac{2\pi + \theta}{3}\right),$$

$$\lambda_3 = -\frac{A}{3} + 2\sqrt{-\frac{P}{3}}\sin\left(\frac{4\pi + \theta}{3}\right),$$

where

$$\theta = \arcsin\left(\sqrt{\frac{-27Q^2}{4P^3}}\right) \in [0, \pi].$$

The case $\Delta = 0$ corresponds to a measure-zero set of parameters. So, by a slight perturbation of parameters, without changing the behavior of the system, a system belonging to one of the two cases is obtained. The values of A, B, and C are determined for each equilibrium point as follows:

For P_1^+ and P_2^+, one has

$$A_1 = b - c,$$

$$B_1 = -(bc + a + c + 1),$$

$$C_1 = br(a + c + 1).$$

For P_1^- and P_2^-, one has

$$A_2 = b - c,$$

$$B_2 = -(bc + a + c + 1),$$

$$C_2 = br(a + c - 1).$$

For the parameter values $a = 2.0, b = 10.0, c = 9.0, r = 40.0,$ the system has a chaotic attractor shown in Fig. 2. In this case the equilibria are

$$P_1^{\pm} = (\pm 0.75, \pm 1, 0.075), \quad P_2^{\pm} = (\pm 0.9, \pm 1, 0.09).$$

The eigenvalues corresponding to these equilibrium points are

For P_1^+ and P_2^+, one has

$$\lambda_R = -19.251, \quad (\lambda_C)^{\pm} = 9.1254 \pm 12.887i.$$

For P_1^- and P_2^-, one has

$$\lambda_R = -18.384, \quad (\lambda_C)^{\pm} = 8.6919 \pm 11.918i.$$

Then there exist, for all equilibrium points, two conjugate complex eigenvalues, then in each case locally the equilibrium points are not stable, they are attracting in one direction but repelling in the other two directions.

4. Observation of New Chaotic Attractors

Notably, this new system (2) is not diffeomorphic with any of the two mentioned Lorenz and Chen systems, since their eigenvalues at the corresponding equilibria are not the same, nor in any sense equivalent. In addition, the system (2) is not equivalent to the one given in [1]. Since the former has 2 or 4 equilibria, but the latter has only three. To see some chaotic behavior of the system (2) we present various numerical results to show its chaoticity, including its sensitive dependence on initial conditions.

Thus, we fix

$$b = 10., \; c = 9., \text{ and } r = 40.$$

and assume

$$c < b.$$

Then we vary the parameter a.

For $a = 2.0$, the system (2) has the chaotic attractor shown in Fig. 2. The aperiodicity of these attractors can be seen from the calculation of the power spectrum of the time series (here we have chosen the x-component).

It seems that the attractor is aperiodic; the spectrum is broadband and contains a dominant discrete pick at a low frequency that is due to the presence of unstable limit cycles (see Figs. 5 and 6). This noise-like spectrum is an essential characteristic of chaotic systems.

The attractor in Fig. 4 resembles by its shape a "real butterfly" (projection into the *x*-*z* plane) and it is very sensitive to a change of the initial data but not for the change of parameters, it persists for a big region of values.

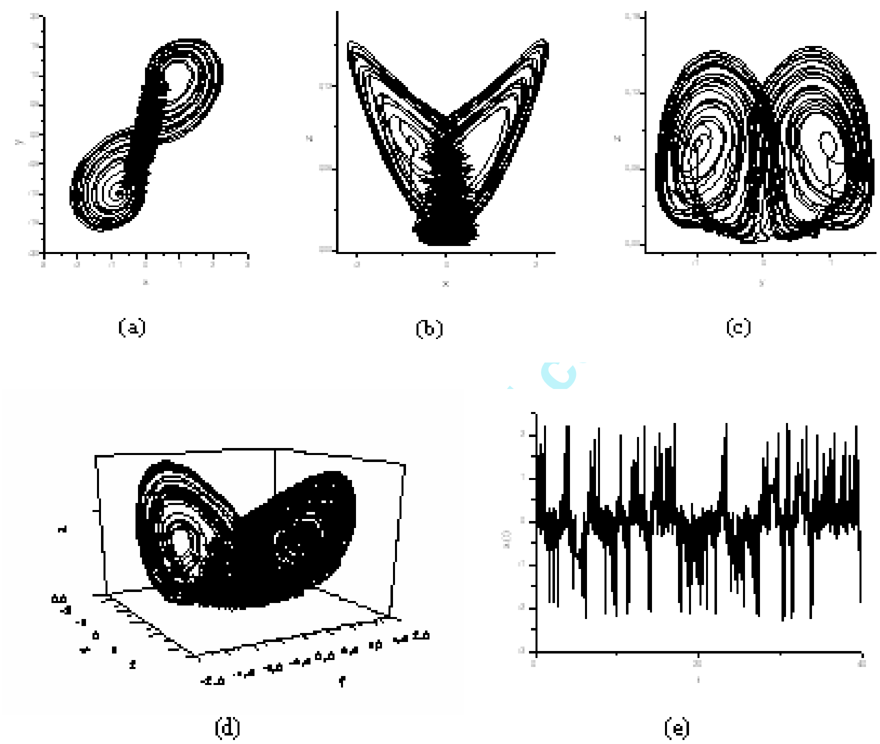

Figure 2. The new chaotic attractor for $a = 2.$, $b = 10.$, $c = 9.$, $r = 40.$ (a) Projection into the *x*-*y* plane, (b) Projection into the *x*-*z* plane, (c) Projection into the *y*-*z* plane, (d) The shape of the attractor in the *x*-*y*-*z* space, (e) The time waveform of the time series $x(t)$

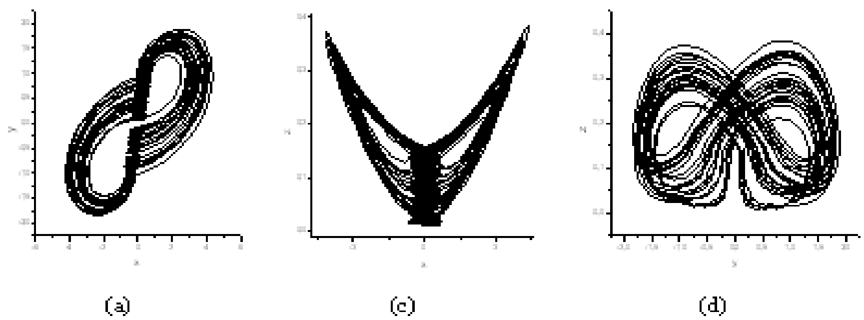

Figure 3. Another chaotic attractor for $a = -3.$, $b = 10.$, $c = 9.$, $r = 40.$ (a) Projection into the x-y plane, (b) Projection into the x-z plane, (c) Projection into the y-z plane

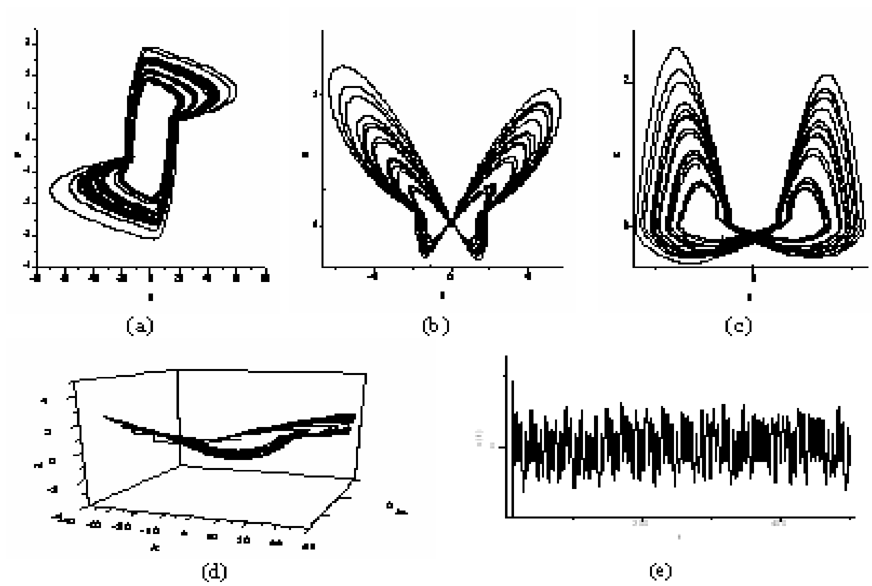

Figure 4. Another chaotic attractor obtained for $a = -1.$, $b = 45.$, $c = 0.$, $r = -40.$ (a) Projection into the x-y plane, (b) Projection into the x-z plane, (c) Projection into the y-z plane, (d) The shape of the attractor in the x-y-z space, (e) The time waveform of the time series $x(t)$

In this work we reported the study of the attractors shown in Fig. 4 and a detailed analysis for these phenomena will be provided in the near future.

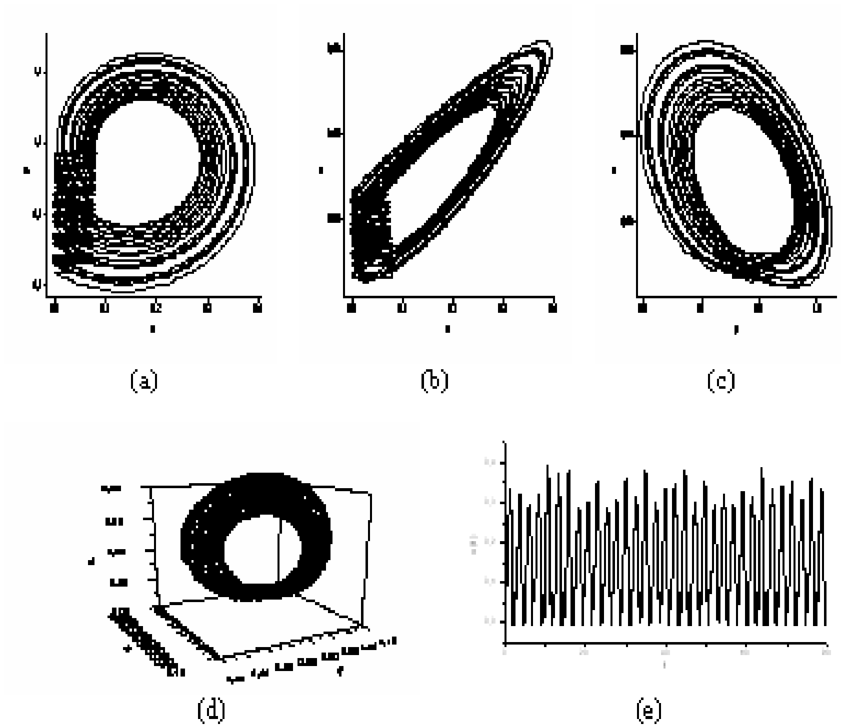

Figure 5. Another chaotic attractor obtained for $a = 1.0$, $b = 4.0$, $c = 0.0$, $r = 4.0$. (a) Projection into the x-y plane, (b) Projection into the x-z plane, (c) Projection into the y-z plane, (d) The shape of the attractor in the x-y-z space, (e) The time waveform of the time series $x(t)$

5. Sensitive Dependence on Initial Conditions

To prove this assertion, we compute two orbits with initial points (x_0, y_0, z_0), (x_1, y_0, z_0), where $x_1 = x_0 + 0.0000001$ is a very small perturbation of x_0. The result is shown in Fig. 6. We remark that after a few number of iterations the difference between them builds up rapidly by an enlargement in Fig. 6.

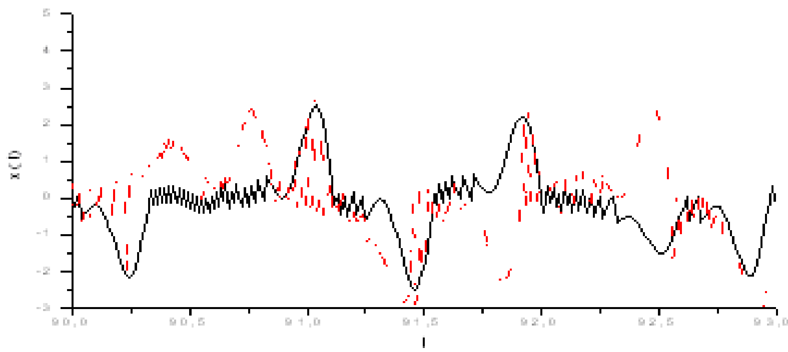

Figure 6. Sensitive dependence on initial condition: x-coordinate of the two orbits, for system with the x-coordinates and the parameters: $a = 2.$, $b = 10.$, $c = 9.$, $r = 40.$ and the initial conditions differs by 0.000001, the other coordinates kept equal

6. Possibility of the Circuitry Realization of the New System

An electronic circuit is designed to realize a new continuous-time three-dimensional autonomous system with three nonlinearities [5], it consists of three channels, conducting the integration of the three state variables, operational amplifiers, analog multipliers, and linear resistors and capacitors are employed to perform the required addition, subtraction and multiplication operations. In our case there is only single nonlinearity, this affects simplify the circuitry realization of the new system (2).

7. A New Chaotic Attractor from a PWL Version

A piecewise-linear version of the modified model (2) (replacing xy by $x + y$) to be studied in the near future. This statement is very important because it is possible to build a simple electronic circuit with piecewise linear (PWL) functions, to realize chaos in differential systems [4], [10] (see Fig. 7).

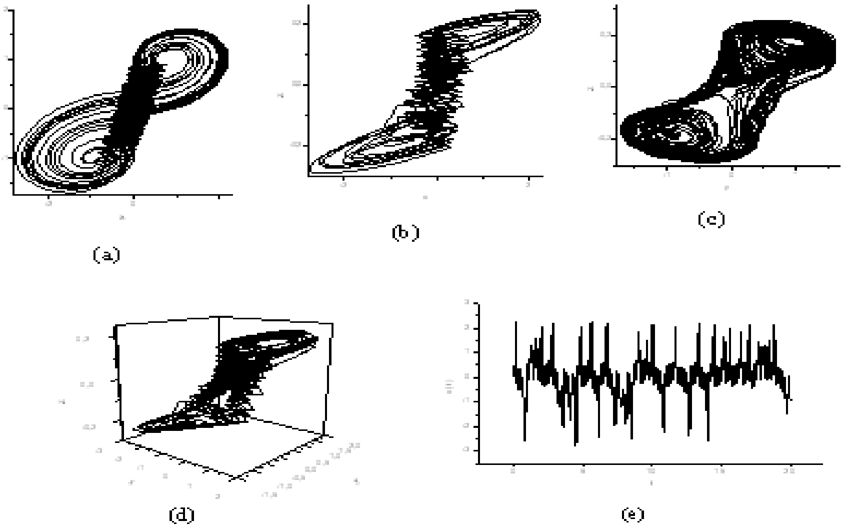

Figure 7. The chaotic attractor for the linear version of system (2) obtained for: $a = 2.$, $b = 10.01.$, $c = 10.$, $r = 42.$ (a) Projection into the x-y plane, (b) Projection into the x-z plane, (c) Projection into the y-z plane, (d) The shape of the attractor in the x-y-z space, (e) The time waveform of the time series $x(t)$

8. Conclusion

In this work we prove the existence of a new chaotic attractor obtained from simple modification of the Chen equation; it is the illustration that the chaos can be occurring in simple modified systems with only single nonlinearity.

Besides, the system (2) has some similar properties as the Chen system and others well-known modified Lorenz and Chen systems.

More detailed dynamical analysis on the new chaotic system and the piecewise-linear version and the attractor shown in Fig. 4, will also be investigated in the near future. In addition, some similar systems will be studied under a more general but unified framework, which will be reported later.

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