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POSITIVE PERIODIC SOLUTIONS OF INFINITE DELAY FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we employ a fixed point theorem on a cone to study the existence of positive periodic solutions for the following higher-dimensional functional differential equation:

$$\dot{x}(t) + A(t)x(t) = f(t, x_t), \quad t \in \mathbb{R}.$$

Some existence results of multiplicity positive periodic solutions are obtained.

1. Introduction

Recently, by employing Krasnosel'skii fixed point theorem [5] on a cone, Jiang and Wei [4] have investigated the existence of one positive

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periodic solution for functional differential equations of the form

$$\dot{x}(t) = A(t)x(t) + f(t, x_t),$$

where $A(t) = diag[a_1(t), a_2(t), \ldots, a_n(t)]$, $a_j \in C(\mathbb{R}, \mathbb{R})$ is ω-periodic, $f : \mathbb{R} \times BC \to \mathbb{R}^n$, and $\omega > 0$ is a constant. In this paper, we are concerned with the following functional differential equation:

$$\dot{x}(t) + A(t)x(t) = f(t, x_t), \quad t \in \mathbb{R}, \tag{1}$$

in which $A(t) = diag[a_1(t), a_2(t), \ldots, a_n(t)]$, $a_j \in C(\mathbb{R}, \mathbb{R})$ is ω-periodic, $j = 1, 2, \ldots, n$. $f(t, x_t)$ is a function defined on $\mathbb{R} \times BC$, and $f(t, x_t)$ is ω-periodic whenever x is ω-periodic, where BC denotes the Banach space of bounded continuous functions $\phi : \mathbb{R} \to \mathbb{R}^n$ with the norm $\|\phi\| = \sum_{j=1}^{n} \sup_{\theta \in \mathbb{R}} |\phi^j(\theta)|$, where $\phi = (\phi^1, \phi^2, \ldots, \phi^n)^T$. If $x \in BC$, then $x_t \in BC$ for any $t \in \mathbb{R}$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in \mathbb{R}$. And $\omega > 0$ is a constant.

System (1) was extensively investigated in literature as bio-mathematics models. It contains many bio-mathematics models of delay differential equations or systems, such as the periodic logistic equation with several delays [7]

$$\dot{y}(t) = y(t)\left[a(t) - \sum_{i=1}^{n} b_i(t) y(t - \tau_i(t))\right]$$

and the periodic distributed delay Lotka-Volterra competition system [8]

$$\frac{du_i(t)}{dt} = u_i(t)\left[r_i(t) - a_{ii}u_i(t) - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} \int_{-T_{ij}}^{0} K_{ij}(s)u_j(t + s)ds\right], \; i = 1, 2, \ldots, n.$$

For more information about the applications of system (1) to a variety of population models, we refer to the reader to [2, 3, 6, 10] and the references cited therein. Our purpose of this paper is to study the existence of multiplicity positive periodic solutions of (1) by utilizing a fixed point theorem [1] on a cone.

For convenience, we need to introduce a few notations. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, and $\mathbb{R}_- = (-\infty, 0]$, respectively. For each $x = (x^1, x^2, ..., x^n)^T \in \mathbb{R}^n$, the norm of x is defined as $|x| = \sum_{j=1}^n |x^j|$. $\mathbb{R}_+^n = \{(x^1, x^2, ..., x^n)^T \in \mathbb{R}^n : x^j \geq 0, j = 1, 2, ..., n\}$. We say that x is *positive* when $x \in \mathbb{R}_+^n$. $BC(\mathbb{X}, \mathbb{Y})$ denotes the set of bounded continuous functions $\phi : \mathbb{X} \to \mathbb{Y}$.

2. Main Results

The objective of this section is to derive sufficient conditions for two existence results of twin positive periodic solutions of (1). It follows from (1) that

$$x(t) = \int_t^{t+\omega} g(t, s) f(s, x_s) ds, \tag{2}$$

where

$$f(s, x_s) = (f_1(s, x_s), f_2(s, x_s), ..., f_n(s, x_s))^T$$

and

$$g(t, s) = diag[g_1(t, s), g_2(t, s), ..., g_n(t, s)], g_j(t, s) = \frac{\exp\left(\int_t^s a_j(v)dv\right)}{\exp\left(\int_0^\omega a_j(v)dv\right) - 1} \tag{3}$$

for $(t, s) \in \mathbb{R}^2$, $j = 1, 2, ..., n$.

In what follows, we always assume that

(H_1) $\int_0^\omega a_j(s)ds \neq 0$ for $j = 1, 2, ..., n$.

(H_2) $f_j(t, \phi_t)\int_0^\omega a_j(s)ds \geq 0$ for all $(t, \phi) \in \mathbb{R} \times BC(\mathbb{R}, \mathbb{R}_+^n)$, $j = 1, 2, ..., n$.

(H_3) $f(t, x_t)$ is a continuous function of t for each $x \in BC(\mathbb{R}, \mathbb{R}_+^n)$.

(H_4) For any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for $\phi, \psi \in BC$, $\|\phi\| \leq L$, $\|\psi\| \leq L$, and $\|\phi - \psi\| < \delta$ for $s \in [0, \omega]$ imply

$$|f(s, \phi_s) - f(s, \psi_s)| < \varepsilon.$$

Clearly, the denominator in $g_j(t, s)$ is not zero since (H_1). Note further that

$$p_j := \frac{\exp\left(\int_0^{\omega} |a_j(v)| dv\right)}{\left|\exp\left(\int_0^{\omega} a_j(v) dv\right) - 1\right|} \leq |g_j(t, s)| \leq \frac{\exp\left(\int_0^{\omega} |a_j(v)| dv\right)}{\left|\exp\left(\int_0^{\omega} a_j(v) dv\right) - 1\right|} =: q_j, \quad (4)$$

for all $s \in [t, t + \omega]$, $j = 1, 2, \ldots, n$. And we let $p := \min_{1 \leq j \leq n} p_j$, $q := \max_{1 \leq j \leq n} q_j$. It is easy to see that $g(t, s) = g(t + \omega, s + \omega)$ for all $(t, s) \in \mathbb{R}^2$ and by (H_2),

$$g_j(t, s) f_j(u, \phi_u) \geq 0, \quad j = 1, 2, \ldots, n, \quad (5)$$

for $(t, s) \in \mathbb{R}^2$ and $(u, \phi) \in \mathbb{R} \times BC(\mathbb{R}, \mathbb{R}_+^n)$. It is not difficult to verify that any ω-periodic function $x(t)$ that satisfies (2) is also an ω-periodic solution of (1).

Next, we introduce the concerned definition and fixed point theorem that we need in this paper.

Definition. Let $\mathbb{X}$ be a Banach space and K be a closed, nonempty subset of $\mathbb{X}$. K is a *cone* if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geq 0$;

(ii) $u, -u \in K$ imply $u = 0$.

Theorem A [9]. *Let K be a cone in a Banach space E and $\mathbf{\Omega}_1$, $\mathbf{\Omega}_2$ be two bounded open sets in E such that $0 \in \mathbf{\Omega}_1$ and $\overline{\mathbf{\Omega}}_1 \subset \mathbf{\Omega}_2$. Let $T : K \cap (\overline{\mathbf{\Omega}}_2 \setminus \mathbf{\Omega}_1) \to K$ be completely continuous operator. If*

(c_1) there exists $u_0 \in K \setminus \{0\}$ such that $u - Tu \neq \lambda u_0$, $u \in K \cap \partial \mathbf{\Omega}_2$, $\lambda \geq 0$; $Tu \neq \mu u$, $u \in K \cap \partial \mathbf{\Omega}_1$, $\mu \geq 1$, or

(c_2) there exists $u_0 \in K \setminus \{0\}$ such that $u - Tu \neq \lambda u_0$, $u \in K \cap \partial \mathbf{\Omega}_1$, $\lambda \geq 0$; $Tu \neq \mu u$, $u \in K \cap \partial \mathbf{\Omega}_2$, $\mu \geq 1$,

then T has at least one fixed point in $k \cap (\overline{\mathbf{\Omega}}_2 \setminus \mathbf{\Omega}_1)$.

Let $\mathbb{X}$ be the set

$$\mathbb{X} = \{x \in C(\mathbb{R}, \mathbb{R}^n) : x(t + \omega) = x(t), t \in \mathbb{R}\}$$

with the linear structure as well as the norm

$$\| x \| = \sum_{j=1}^{n} | x^j |_0, \quad | x^j |_0 = \sup_{t \in [0, \omega]} | x^j(t) |, \quad j = 1, 2, \ldots, n,$$

where $x = (x^1, x^2, \ldots, x^n)^T \in \mathbb{R}^n$. Then $\mathbb{X}$ is a Banach space. Define

$$\sigma := \min\left\{\exp\left(-2\int_0^{\omega} | a_j(s) | ds\right), j = 1, 2, \ldots, n\right\}$$

and

$$K = \{x \in \mathbb{X} : x^j(t) \geq \sigma | x^j |_0, x = (x^1, x^2, \ldots, x^n)^T, t \in [0, \omega]\},$$

one may readily verify that K is a cone in $\mathbb{X}$.

Now, we define an operator $\Phi : K \to K$ as

$$(\Phi x)(t) = \int_t^{t+\omega} g(t, s) f(s, x_s) ds$$

for $x \in K$, $t \in \mathbb{R}$, where $g(t, s)$ is defined as that in (3), and $(\Phi x) = (\Phi_1 x, \Phi_2 x, \ldots, \Phi_n x)^T$. It is easy to see that a function $x \in \mathbb{X}$ is a solution of (1) if and only if x is a fixed point of the operator equation $x = \Phi x$ in $\mathbb{X}$.

Lemma 1. *The mapping* Φ *maps* K *into* K.

Proof. For any $x \in K$ and $t \in [0, \omega]$, it follows from (H_2) that $(\Phi x)(t)$ is continuous in t and

$$\begin{aligned}(\Phi x)(t + \omega) &= \int_{t+\omega}^{t+2\omega} g(t + \omega, s) f(s, x_s) ds \\ &= \int_t^{t+\omega} g(t + \omega, v + \omega) f(v + \omega, x_{v+\omega}) dv \\ &= \int_t^{t+\omega} g(t, v) f(v, x_v) dv = (\Phi x)(t).\end{aligned}$$

Hence, $(\Phi x) \in \mathbb{X}$. And for $x \in K$, we find

$$|\Phi_j x|_0 \leq q_j \int_0^\omega |f_j(s, x_s)| ds$$

and

$$(\Phi_j x)(t) \geq p_j \int_0^\omega |f_j(s, x_s)| ds \geq \frac{p_j}{q_j} |\Phi_j x|_0 \geq \sigma |\Phi_j x|_0$$

for $t \in [0, \omega]$, $j = 1, 2, \ldots, n$. Therefore, $(\Phi x) \in K$. This completes the proof of Lemma 1.

Lemma 2. *Suppose* (H_1)-(H_4) *hold. Let* η *be a positive number,* $\mathbf{\Omega} = \{x \in \mathbb{X} : \| x \| < \eta\}$. *Then* $\Phi : K \cap \overline{\mathbf{\Omega}} \to K$ *is completely continuous.*

By the continuity of $f(t, x_t)$ and Arzela-Ascoli (Royden [11, p. 169]) theorem, we can easily give the proof of Lemma 2 and we omit it here. The reader can refer to [9].

Theorem 1. *Suppose* (H_1)-(H_4) *hold and there are positive constants* r_1, r_2 *and* r_3 *with* $r_1 < r_3 < r_2$ *such that*

(H_5) $$\inf_{\phi \in K, \| \phi \| = r_1} \int_0^\omega |f(s, \phi_s)| ds > \frac{r_1}{p}, \quad \inf_{\phi \in K, \| \phi \| = r_2} \int_0^\omega |f(s, \phi_s)| ds > \frac{r_2}{p};$$

(H_6) $$\sup_{\phi \in K, \| \phi \| = r_3} \int_0^\omega |f(s, \phi_s)| ds < \frac{r_3}{q},$$

where $p := \min_{1 \leq j \leq n} p_j$, $q := \max_{1 \leq j \leq n} q_j$, *and* p_i, q_i *are defined in* (4). *Then system* (1) *has at least two positive periodic solutions.*

Proof. Let $\mathbf{\Omega}_1 = \{u \in \mathbb{X} : \| u \| < r_1\}$. Then for any $u \in K \cap \partial \mathbf{\Omega}_1$, we have $u - \Phi u \neq \lambda u_0$, $u_0 \in K \setminus \{0\}$, $\lambda \geq 0$. For the sake of contradiction, we choose $u_0 = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$, then $u_0 \in K \setminus \{0\}$. Suppose that there exists $\overline{u} \in K \cap \partial \mathbf{\Omega}_1$ such that $\overline{u} - \Phi \overline{u} = \lambda_0 u_0$ for some $\lambda_0 \geq 0$. Then, we have

$$\overline{u}^j(t) = (\Phi_j \overline{u})(t) + \lambda_0, \quad j = 1, 2, \ldots, n.$$

From this, the definition of Φ, (4) and (5) it follows that

$$|\bar{u}^j|_0 = \left| \int_0^\omega g_j(t, s) f_j(s, \bar{u}_s) ds + \lambda_0 \right| = \int_0^\omega g_j(t, s) f_j(s, \bar{u}_s) ds + \lambda_0$$

$$\geq p_j \int_0^\omega |f_j(s, \bar{u}_s)| ds + \lambda_0 \geq p \int_0^\omega |f_j(s, \bar{u}_s)| ds + \lambda_0$$

$$\geq p \int_0^\omega |f_j(s, \bar{u}_s)| ds.$$

Hence, we have

$$r_1 = \| \bar{u} \| \geq p \int_0^\omega |f(s, \bar{u}_s)| ds,$$

which contradicts the first inequality in assumption (H_5). Therefore, we derive that

$$u - \Phi u \neq \lambda u_0, \ \forall u_0 \in K \setminus \{0\}, \ \lambda \geq 0. \tag{6}$$

Let $\mathbf{\Omega}_2 = \{u \in \mathbb{X} : \| u \| < r_2\}$. Then for any $u \in K \cap \partial \mathbf{\Omega}_2$, applying the second inequality in (H_5), similarly to the proof of (6), we have $u - \Phi u \neq \lambda u_0$, $u_0 \in K \setminus \{0\}$, $\lambda \geq 0$.

On the other hand, let $\mathbf{\Omega}_3 = \{u \in \mathbb{X} : \| u \| < r_3\}$. Then for any $u \in K \cap \partial \Omega_3$, from the definition of Φ, (4) and (5), we have

$$|\Phi_j u|_0 \leq q_j \int_0^\omega |f_j(s, u_s)| ds \leq q \int_0^\omega |f_j(s, u_s)| ds,$$

hence, in view of (H_6), one has

$$\| \Phi u \| \leq q \int_0^\omega | f(s, u_s) | ds < r_3,$$

that is,

$$\| \Phi u \| < \| u \|, \ \forall u \in K \cap \partial \mathbf{\Omega}_3.$$

Therefore,

$$\Phi u \neq \mu u, \ \forall u \in K \cap \partial \mathbf{\Omega}_3, \ \mu \geq 1. \tag{7}$$

It is clear that $\mathbf{\Omega}_1 \subset \mathbf{\Omega}_3 \subset \mathbf{\Omega}_2$, by Theorem A, we can conclude that Φ has two fixed points $u_1 \in K \cap (\overline{\mathbf{\Omega}}_3 \setminus \mathbf{\Omega}_1)$ and $u_2 \in K \cap (\overline{\mathbf{\Omega}}_2 \setminus \mathbf{\Omega}_3)$ with $r_1 < \| u_1 \| < r_3$, $r_3 < \| u_2 \| < r_2$. Therefore, $u_1(t)$ and $u_2(t)$ are positive periodic solutions of system (1). The proof is complete.

Theorem 2. *Suppose* (H_1)-(H_4) *hold and there are positive constants* R_1, R_2 and R_3 *with* $R_1 < R_3 < R_2$ *such that*

$$(\mathrm{H}_7) \quad \sup_{\phi \in K, \|\phi\| = R_1} \int_0^{\omega} |f(s, \phi_s)| ds < \frac{R_1}{q}, \quad \sup_{\phi \in K, \|\phi\| = R_2} \int_0^{\omega} |f(s, \phi_s)| ds < \frac{R_2}{q};$$

$$(\mathrm{H}_8) \quad \inf_{\phi \in K, \|\phi\| = \mathbb{R}_3} \int_0^{\omega} |f(s, \phi_s)| ds > \frac{R_3}{p},$$

where $p := \min_{1 \le j \le n} p_j$, $q := \max_{1 \le j \le n} q_j$, *and* p_i, q_i *are defined in* (4). *Then system* (1) *has at least two positive periodic solutions.*

Proof. By condition (H_7), from the proof of Theorem 1, we know that

$$\Phi u \neq \mu u, \ \forall u \in K \cap \partial \mathbf{\Omega}_4, \ \mu \geq 1,$$

$$\Phi u \neq \mu u, \ \forall u \in K \cap \partial \mathbf{\Omega}_5, \ \mu \geq 1,$$

where $\mathbf{\Omega}_4 = \{\phi \in \mathbb{X} : \| \phi \| < R_1\}$, $\mathbf{\Omega}_5 = \{\phi \in \mathbb{X} : \| \phi \| < R_2\}$.

From condition (H_8), let $\mathbf{\Omega}_6 = \{\phi \in \mathbb{X} : \| \phi \| < R_3\}$, for any $u \in K \cap \partial \mathbf{\Omega}_6$, it is similar to the proof of (6), we have

$$u - \Phi u \neq \lambda u_0, \ \forall u_0 \in K \setminus \{0\}, \ \lambda \geq 0.$$

It is clear that $\mathbf{\Omega}_4 \subset \mathbf{\Omega}_6 \subset \mathbf{\Omega}_5$, by Theorem A, we can conclude that Φ has two fixed points $u_3 \in K \cap (\overline{\mathbf{\Omega}}_6 \setminus \mathbf{\Omega}_4)$ and $u_4 \in K \cap (\overline{\mathbf{\Omega}}_5 \setminus \mathbf{\Omega}_6)$ with $R_1 < \| u_3 \| < R_3$, $R_3 < \| u_4 \| < R_2$. Therefore, $u_3(t)$ and $u_4(t)$ are positive periodic solutions of system (1). The proof is complete.

Using the same method of this paper, one can show that

Theorem 3. *Suppose that* (H_1)-(H_4) *hold. If* $(\mathrm{H}_5), (\mathrm{H}_6)$ *hold or* $(\mathrm{H}_7), (\mathrm{H}_8)$ *hold, then the systems*

$$x'(t) - A(t)x(t) = -f(t, x_t),$$

$$x'(t) - A(t)x(t) = f(t, x_t),$$

and

$$x'(t) + A(t)x(t) = -f(t, x_t)$$

have at least two positive ω-*periodic solutions*, *respectively*.

References

[1] D. Guo and V. Lakshmikantham, Nonlinear Problem in Abstract Cones, Academic Press, New York, 1988.

[2] L. Hatvani and T. Krisztin, On the existence of periodic solutions for linear inhomogeneous and quasilinear functional differential equations, J. Differential Equations 97 (1992), 1-15.

[3] Y. Hino, S. Murakami and T. Naito, Functional Differential Equations with Infinite Delay, Springer-Verlag, New York, 1991.

[4] D. Q. Jiang and J. J. Wei, Positive periodic solutions of functional differential equations and population models, Electronic J. Differential Equations 2002, 71 (2002), 1-13.

[5] M. A. Krasnosel'skii, Positive Solutions of Operator Equations, Groningen, Noordhoff, 1964.

[6] Y. Kuang, Delay Differential Equations with Application in Population Dynamics, Academic Press, New York, 1993.

[7] Y. Li, Existence and global attractivity of positive periodic solutions for a class of delay differential equations, Science in China Series A 41(3) (1998), 273-284.

[8] Y. Li, Periodic solutions for delay Lotka-Volterra competition systems, J. Math. Anal. Appl. 246 (2000), 230-244.

[9] P. Liu and Y. K. Li, Positive periodic solutions of delay functional differential equations depending on a parameter, Applied Math. Comput. 150 (2004), 159-168.

[10] G. Makay, Periodic solutions of dissipative functional differential equations, Tohoku Math. J. 46 (1994), 417-426.

[11] H. L. Royden, Real Analysis, Macmillan Publishing Company, New York, 1988.

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