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AN ANALYTICAL EXPRESSION TO CUSUM CHART FOR SEASONAL AR(p) MODEL

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Abstract

The Cumulative Sum (CUSUM) chart provides good performance in detecting small shifts of process means compared to the traditional Shewhart chart. A common assumption of the control chart is the independent and identically distributed random variables, however, this assumption could be deviated from in practice such as chemical processes or if financial market is autocorrelated and trend stationary. In this paper, the explicit formulas of the Average Run Length (ARL) when observations form a pure seasonal autoregressive of order *p*th (seasonal AR(p)) models with exponential distribution white noise on CUSUM chart are derived. The numerical results from explicit formulas and the numerical integration approach are presented. Our results illustrate that the explicit formulas can reduce computational times to evaluate the ARL when compared with the results obtained from the numerical integration approach. According to the proposed explicit formulas for the ARL, it is very useful in practical applications in order to design an optimal CUSUM chart.

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1. Introduction

One of the primary control charts of statistical process control which has been proven to be an effective tool in reducing the variability and improving the quality of a process is Cumulative Sum (CUSUM) chart. In 1954, Page first proposed this chart and it has been successfully used in many manufacturing and service systems. Traditionally, a main assumption concerning CUSUM chart is that the observations are independent and identically distributed (i.i.d.) random variables. However, the independency assumption does not always happen in actual practice, such as chemical processing, where the observations are mostly autocorrelated and trend stationary (see Montgomery [12]. Recently, many uses of CUSUM charts have been developed and then improved for different processes by several new approaches such as Bohm [1], Lu and Reynolds [7], Lucas and Saccucci [8], Montgomery and Mastrangelo [11], Sukparungsee and Novikov [14], VanBrackle and Reynolds [15] and Woodall and Faltin [18].

The first passage times usually are called the *Average Run Length* (*ARL*) and the *Average Delay Time* (*ADT*) for the in-control and out-of-control processes, respectively. They are commonly the major criterion for measuring the performance of control charts. The ARL is used as a measure of the time before a process that is still in-control is signaled as being out-of-control; it is always desirable to have a large ARL. The ADT is used as a measure of the time before a process that has gone out-of-control is signaled as being out-of-control; which should be small. VanBrackle and Reynolds [15] found that the performance of a control chart is significantly affected by autocorrelated data. Lorden [6] indicated the reaction to a change in control chart. Woodall and Faltin [18] showed that the correlation should be eliminated if possible. However, because autocorrelation is often an inherent part of a process, it must be properly modeled and monitored. Lu and Reynolds [7] pointed out that control charts using residual-based schemes are not necessarily better than those based on the original observations with adjusted control limit, unless the level of autocorrelation is quite high. It is better to use a chart based on the original observations rather than on residuals since it is much easier to understand and interpret for an operator.

In 1978, the Integral Equation (IE) approach was first studied by Crowder for approximating the ARL of a Gaussian distribution. He derived and used a Fredholm integral equation of the second type. Indeed, there are many approaches to evaluate the ARL and ADT such as Monte Carlo Simulation (MC), Markov Chain Approach (MCA) and Martingale approach. However, these approaches provide only closed-form formulas, while the IE approach gives the explicit formulas. Mititelu et al. [10] used IE approach to solve the explicit formulae of the ARLs for CUSUM chart when observations are exponential data. Busaba et al. [3] used IE approach to solve the explicit formulas for the ARL and ADT for CUSUM chart when observations are negative exponential data. Busaba et al. [3, 4] used Integral Equations (IE) approach to solve the explicit formulae of the ARLs for CUSUM chart for the case of trend and no-trend stationary *p*th order autoregressive model with exponential distribution white noise. This is a motivation to propose the explicit formulae for ARL and ADT for the case of a seasonal AR(p) model with exponential distribution white noise. Furthermore, the equations numerically are proved by using the Gauss-Legendre Quadrature rule and then compare the results from both approaches.

2. Cumulative Sum and its Property

According to the major assumption that $\xi_1, \xi_2, \ldots$ are i.i.d. random variables with a distribution function $F(x, \alpha)$, the parameter α has the value α_0 in the in-control state, and $\alpha \neq \alpha_0$ in an out-of-control state. In this paper, we consider that the observations are a seasonal AR(p) with an exponential distribution white noise. We assume that the parameters, α and α_0, are known.

Under the assumptions that $F(x, \alpha)$ is absolute continuous with respect to $F(x, \alpha_0)$. The CUSUM chart is based on use of the first passage times τ

(see, i.e., Busaba et al. [2, 3] and Sukparungsee and Novikov [14]), which is for a statistic defined as in equation (1) as

$$\tau_h = \inf\{t \geq 0,\ X_t \geq h\}, \tag{1}$$

where h is a control limit on the value of X_t.

The typical conditions on choice of the first passage times τ are the Average Run Length (ARL) and the Average Delay Time (ADT) as in equations (2) and (3). We define $E_\theta(\cdot)$ as the expectation under distribution $F(x,\ \alpha_0)$ that the change-point occurs at time θ.

$$ARL \simeq E_\theta(\tau_h) \geq A \quad (\theta = \infty), \tag{2}$$

where A is given (usually large) and

$$ADT \simeq E_\theta(\tau_h) \geq (\tau \mid \tau \geq 1) \quad (\theta = 1). \tag{3}$$

2.1. The seasonal AR(p) on exponential CUSUM

The CUSUM is designed to detect a process mean shift of an i.i.d. observed sequence of random variables. The statistics X_t satisfies the following recursive equation as:

$$X_t = (X_{t-1} + Z_t - a)^+, \quad t = 1, 2, \ldots, X_0 = x,$$

where X_t is the CUSUM value of statistics after n observations, x is an initial value for X_t, $y^+ = \max(0,\ y)$ and a is a constant. Many discussions have led to this recursive presentation by Mazalov and Zhuravlev [9] and Venkateshwara et al. [17].

In this paper, we consider CUSUM chart for a seasonal AR(p) model with exponential distribution white noise. Thus, we define the statistics as:

$$X_t = X_{t-1} + Z_t - a, \quad t = 1, 2, \ldots;\ X_0 = x, \tag{4}$$

where

$$Z_t = \phi_1 Z_{t-s} + \phi_2 Z_{t-2s} + \cdots + \phi_p Z_{t-ps} + \xi_t$$

when $-1 < \varphi_i < 1,\ i = 1, 2, \ldots, p,\ s$ is periodicity and $\xi_t \sim \exp(\lambda)$.

For seasonal AR(1) with *s*-periodicity model, the observations Z_t, $t = 1, 2, \ldots,$ satisfy the following relationship:

$$Z_t = \phi Z_{t-s} + \xi_t,$$

where ϕ is autoregressive parameters $-1 < \phi_1 < 1$ and $\xi_t \sim \exp(\lambda)$.

For seasonal AR(2) with *s*-periodicity model, the observations Z_t, $t = 1, 2, \ldots,$ satisfy the following relationship:

$$Z_t = \phi_1 Z_{t-s} + \phi_2 Z_{t-2s} + \xi_t,$$

where ϕ_1 and ϕ_2 are autoregressive parameters that have the following restrictions:

$$\phi_1 + \phi_2 < 1, \quad \phi_1 - \phi_2 < 1, \quad -1 < \phi_2 < 1.$$

2.2. The uniqueness of solution to the ARL and ADT integral equation

According to Banach's Fixed Point Theorem, we present the existence and uniqueness of our solutions. We evaluate the ARL of CUSUM chart defined as a function $j(x) = \mathbb{E}_X \tau_h$. Let $\mathbb{P}_X$ and $\mathbb{E}_X$ be the probability measure and the induced expectation corresponding to the initial value $X_0 = x$. Vardeman and Ray [16] and Venkateshwara et al. [17] showed that the ARL for CUSUM at a given level, defined as $j(x) = ARL = \mathbb{E}_X \tau_h < \infty$, is a solution of the following integral equation:

$$j(x) = 1 + \mathbb{E}_X [I\{0 < X_1 < h\} j(X_1)] + \mathbb{P}_X \{X_1 = 0\} j(0). \tag{5}$$

For this case, ξ_n are exponential distributed observations which have been shown by Busaba et al. [2] and Mititelu et al. [10]. We also define ξ_n as exponential distribution white noise in the seasonal AR(p) model as in (4) so (5) can be written as

$$j(x) = 1 + \lambda e^{\lambda(x - a + \phi_1 Z_{t-s} + \phi_2 Z_{t-2s} + \cdots + \phi_p Z_{t-ps})} \int_0^h j(y) e^{-\lambda y} dy$$
$$+ (1 - e^{\lambda(x - a + \phi_1 Z_{t-s} + \phi_2 Z_{t-2s} + \cdots + \phi_p Z_{t-ps})}) j(0), \quad x \in [0, a). \tag{6}$$

It is shown that solutions of the integral equation (6) are continuous functions because the right hand side of (6) contains only continuous functions.

As on the metric space of all continuous functions $(\mathbb{C}(\mathbb{I}), \|\ \|_1)$, where $\mathbb{I}$ is a compact interval, and the norm is defined as $\| j \| = \operatorname{Sup}_{x \in \mathbb{I}} | j(x) |$, the operator T is named a contraction if there exists a number $0 \le q < 1$ such that $\| T(j_1) - T(j_2) \| \le q \| j_1 - j_2 \|$ for all $j_1, j_2 \in X$. Now, define the operators T as

$$T(j(x)) = 1 + \lambda e^{\lambda(x - a + \phi_1 Z_{t-s} + \phi_2 Z_{t-2s} + \cdots + \phi_p Z_{t-ps})} \int_0^h j(y) e^{-\lambda y} dy$$

$$+ (1 - e^{\lambda(x - a + \phi_1 Z_{t-s} + \phi_2 Z_{t-2s} + \cdots + \phi_p Z_{t-ps})}) j(0), \quad x \in [0, a). \quad (7)$$

Then the integral equations in (6) can be written as $T(j(x)) = j(x)$. Recalling Banach's Fixed Point Theorem if the operator T is contraction, then the fixed-point equation $T(j(x)) = j(x)$ has a unique solution. To show the uniqueness of the solution of (6), we prove in Theorem 2.1 that T is a contraction. Define the norms $\| j \|_1 = \operatorname{Sup}_{x \in \mathbb{I}_1} | j(x) |$.

Theorem 2.1. *On the metric spaces* $(\mathbb{C}(\mathbb{I}_1), \|\ \|_1)$, *the operator T is a contraction.*

Proof. First, to prove T is a contraction, we may check that for any $x \in \mathbb{I}_1$, and $j_1, j_2 \in \mathbb{C}(\mathbb{I}_1)$, we have the inequality $\| T(j_1) - T(j_2) \|_1 \le q \| j_1 - j_2 \|_1$, where q is a positive constant, $0 \le q < 1$. According to (7), we have that:

$$\| T(j_1) - T(j_2) \|$$

$$= \operatorname{Sup} | j(x) |$$

$$= \operatorname{Sup}_{x \in [0, a)} \Big| (j_1(0) - j_2(0))(1 - e^{\lambda(x - a + \phi_1 Z_{t-s} + \phi_2 Z_{t-2s} + \cdots + \phi_p Z_{t-ps})})$$

$$+ \lambda e^{\lambda(x-a+\phi_1 Z_{t-s}+\phi_2 Z_{t-2s}+\cdots+\phi_p Z_{t-ps})} \int_0^h (j_1(y) - j_2(y)) e^{-\lambda y} dy \Bigg|$$

$$\leq \sup_{x\in[0,a)} \Bigg| \| j_1(0) - j_2(0) \|_1 (1 - e^{\lambda(x-a+\phi_1 Z_{t-s}+\phi_2 Z_{t-2s}+\cdots+\phi_p Z_{t-ps})})$$

$$+ \| j_1 - j_2 \|_1 \lambda e^{\lambda(x-a+\phi_1 Z_{t-s}+\phi_2 Z_{t-2s}+\cdots+\phi_p Z_{t-ps})} \int_0^h e^{-\lambda y} dy \Bigg|$$

$$= \| j_1 - j_2 \|_1 \sup_{x\in[0,a)} [1 - e^{\lambda(x-a+\phi_1 Z_{t-s}+\phi_2 Z_{t-2s}+\cdots+\phi_p Z_{t-ps})} - \lambda h]$$

$$= [1 - e^{\lambda(\phi_1 Z_{t-s}+\phi_2 Z_{t-2s}+\cdots+\phi_p Z_{t-ps})} - \lambda h] \| j_1 - j_2 \|_1$$

$$= q_1 \| j_1 - j_2 \|, \text{ where } q_1 = [1 - e^{\lambda(x-a+\phi_1 Z_{t-s}+\phi_2 Z_{t-2s}+\cdots+\phi_p Z_{t-ps})} - \lambda h] < 1.$$

We have used the triangular inequality and the fact that

$$| j_1(0) - j_2(0) | \leq \sup_{x\in[0,a)} | j_1(x) - j_2(x) | = \| j_1 - j_2 \|.$$

3. The ARL and ADT to CUSUM Chart Seasonal AR(p) Model

We consider the explicit formulae and the numerical integral equation to solve the solutions for the seasonal AR(p) model. The explicit formulas are based on an integral equation approach, "Fredholm integral equation of the second type". In Theorem 3.1, we derive and propose explicit solutions which are guaranteed existence and uniqueness by Theorem 2.1.

Theorem 3.1. *The solution of* (6) *is*

$$j(x) = (1 + e^{\lambda(a-\phi_1 Z_{t-s}-\phi_2 Z_{t-2s}-\cdots-\phi_p Z_{t-ps})} - \lambda h) e^{\lambda h} - e^{\lambda x}, \; x \geq 0 \textit{ and } a < h.$$

Proof.

$$j(x) = 1 + \lambda e^{\lambda(x-a+\phi_1 Z_{t-s}+\phi_2 Z_{t-2s}+\cdots+\phi_p Z_{t-ps})} \int_0^h j(y) e^{-\lambda y} dy$$

$$+ (1 - e^{\lambda(x-a+\phi_1 Z_{t-s}+\phi_2 Z_{t-2s}+\cdots+\phi_p Z_{t-ps})}) j(0), \quad x \in [0, a).$$

Set $d = \int_0^h j(y) e^{-\lambda y} dy$. Now, we have

$$j(x) = 1 + \lambda e^{\lambda(x - a + \phi_1 Z_{t-s} + \phi_2 Z_{t-2s} + \cdots + \phi_p Z_{t-ps})} d$$

$$+ (1 - e^{\lambda(x - a + \phi_1 Z_{t-s} + \phi_2 Z_{t-2s} + \cdots + \phi_p Z_{t-ps})}) j(0). \tag{8}$$

If $x = 0$, then

$$j(0) = 1 + \lambda e^{\lambda(-a + \phi_1 Z_{t-s} + \phi_2 Z_{t-2s} + \cdots + \phi_p Z_{t-ps})} d$$

$$+ (1 - e^{\lambda(-a + \phi_1 Z_{t-s} + \phi_2 Z_{t-2s} + \cdots + \phi_p Z_{t-ps})}) j(0)$$

$$= e^{\lambda(a - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps})} + \lambda d.$$

Substituting $j(0)$ into (8), we found that

$$j(x) = 1 + \lambda d + e^{\lambda(a - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps})} - e^{\lambda x}. \tag{9}$$

Now the constant *d* can be found as

$$d = \int_0^h j(1 + \lambda d + e^{\lambda(a - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps})} - e^{\lambda y}) e^{-\lambda y} dy$$

$$= \frac{e^{\lambda h}}{\lambda}(1 - e^{-\lambda h})(1 + e^{\lambda(a - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps})}) - h e^{\lambda h}.$$

Substituting the constant *d* into (9), we have

$$j(x) = (1 + e^{\lambda(a - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps})} - \lambda h) e^{\lambda h} - e^{\lambda x}, \; x \geq 0 \text{ and } a < h.$$

The explicit formulas for the ARL and ADT are presented as follows:

$$ARL = j_0(x) = (1 + e^{(a - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps})} - h) e^h - e^x \tag{10}$$

and

$$ADT = j_1(x) = (1 + e^{\lambda(a - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps})} - \lambda h) e^{\lambda h} - e^{\lambda x}, \tag{11}$$

where λ is a parameter of the exponential distribution, α is a constant, δ is a trend parameter, ϕ_i is a smoothing parameter for $i = 1, 2, \ldots, p$, p is an order of autoregressive observations model, *h* is boundary value, and *a* is reference value.

4. Comparisons of the Results with Numerical Integration

We apply the numerical integration approach to the CUSUM chart for the seasonal AR(p) model. First, we assume that the system is in-control at time *n* if the CUSUM statistic X_n is in the range $H_L \le X_n \le H_U$ and out-of-control if $X_n > H_U$ or $X_n < H_L$, where H_L is a constant lower boundary, $(H_L = 0)$ and H_U is a constant upper boundary $(H_L = h)$. Second, we also assume that the system is initially in an in-control state *x*, i.e., $X_0 = x$ and $0 \le x \le h$. We then define function $j^{IE}(x)$ as follows:

$$\begin{aligned} j^{IE}(x) &= \mathbb{E}_x \tau_h < \infty \\ &= 1 + \mathbb{E}_x[I\{0 < X_1 < h\} j(X_1)] + \mathbb{P}_x\{X_1 = 0\} j(0) \\ &= 1 + \int_0^h j(y) f(y + a_1 - x) dy + F(a - x) j(0), \end{aligned} \tag{12}$$

where τ_h is the first passage time defined in (1). Then $j^{IE}(x)$ is the ARL for initial value *x*.

We present a numerical integration scheme for evaluating solutions (5) for the CUSUM chart which can be written as follows:

$$\begin{aligned} j^{IE}(x) = 1 + j(0) F(a - x - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps}) \\ + \int_0^h j(y) f(a - x - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps} + y) dy, \end{aligned}$$

where $F(x) = 1 - e^{-\lambda x}$ and $f(x) = \dfrac{dF(x)}{dx} = \lambda e^{-\lambda x}$.

Recalling a given quadrature rule for integrals on $[0, h]$, the integral equation can be approximated by

$$j(a_i) \approx 1 + j(a_1) F(a - a_i - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps})$$

$$+ \sum_{k=1}^{m} w_k j(a_k) f(a_k + a - a_i - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps}),$$

$$i = 1, 2, \ldots, m. \tag{13}$$

Without loss of generality, we can approximate the integral by a sum of areas of rectangles with bases $\frac{h}{m}$ with heights chosen as the values of $f(a_k)$ at the midpoints of intervals of length $\frac{h}{m}$ beginning at zero, i.e., on the interval $[0, h]$ with the division points $0 \leq a_1 \leq a_2 \leq \cdots \leq a_m \leq h$ and weights w_k. We obtain $\int_0^h j(y) dy \approx \sum_{k=1}^{m} w_k f(a_k)$, where $a_k = \frac{h}{m}\left(k - \frac{1}{2}\right)$, $k = 1, 2, \ldots, m$.

Because equation (13) is a system of m linear equations in the m unknowns $j(a_1), j(a_2), \ldots, j(a_m)$, it can be written in matrix form as

$$J_{m\times 1} = 1_{m\times 1} + R_{m\times m} J_{m\times 1},$$

$$(I_m - R_{m\times m}) J_{m\times 1} = 1_{m\times 1},$$

where

$$\boldsymbol{J}_{\boldsymbol{m}\times 1} = \begin{pmatrix} \boldsymbol{j}(\boldsymbol{a}_1) \\ \boldsymbol{j}(\boldsymbol{a}_2) \\ \vdots \\ \boldsymbol{j}(\boldsymbol{a}_m) \end{pmatrix}, \quad \mathbf{1}_{m\times \mathbf{1}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \vdots \\ \mathbf{1} \end{pmatrix},$$

$$\boldsymbol{R}_{\boldsymbol{m}\times\boldsymbol{m}} = \begin{pmatrix} \boldsymbol{F}(\boldsymbol{a} - \boldsymbol{a}_1 - \Delta) + \boldsymbol{w}_1 \boldsymbol{f}(\boldsymbol{a}) & \boldsymbol{w}_2 \boldsymbol{f}(\boldsymbol{a}_2 + \boldsymbol{a} - \boldsymbol{a}_1 - \Delta) & \cdots & \boldsymbol{w}_{\boldsymbol{m}} \boldsymbol{f}(\boldsymbol{a}_{\boldsymbol{m}} + \boldsymbol{a} - \boldsymbol{a}_1 - \Delta) \\ \boldsymbol{F}(\boldsymbol{a} - \boldsymbol{a}_1 - \Delta) + \boldsymbol{w}_1 \boldsymbol{f}(\boldsymbol{a}_1 + \boldsymbol{a} - \boldsymbol{a}_2 - \Delta) & \boldsymbol{w}_2 \boldsymbol{f}(\boldsymbol{a}) & \cdots & \boldsymbol{w}_{\boldsymbol{m}} \boldsymbol{f}(\boldsymbol{a}_{\boldsymbol{m}} + \boldsymbol{a} - \boldsymbol{a}_2 - \Delta) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{F}(\boldsymbol{a} - \boldsymbol{a}_{\boldsymbol{m}} - \Delta) + \boldsymbol{w}_1 \boldsymbol{f}(\boldsymbol{a}_1 + \boldsymbol{a} - \boldsymbol{a}_{\boldsymbol{m}} - \Delta) & \boldsymbol{w}_2 \boldsymbol{f}(\boldsymbol{a}_2 + \boldsymbol{a} - \boldsymbol{a}_{\boldsymbol{m}} - \Delta) & \cdots & \boldsymbol{w}_{\boldsymbol{m}} \boldsymbol{f}(\boldsymbol{a}) \end{pmatrix}$$

with $\Delta = \phi_1 Z_{t-s} + \phi_2 Z_{t-2s} + \cdots + \phi_p Z_{t-ps}$ and $\boldsymbol{I_m} = \mathbf{diag}(1, 1, ..., 1)$ is the unit matrix of order *m*. If there exists $(\boldsymbol{I_m} - \boldsymbol{R_{m\times m}})^{-1}$, then the solution of the matrix equation is as follows:

$$\boldsymbol{J_{m\times 1}} = (\boldsymbol{I_m} - \boldsymbol{R_{m\times m}})^{-1} 1_{\boldsymbol{m}\times 1}.$$

To solve this set of equations for the approximate values of $j(a_1), j(a_2), ..., j(a_m)$, we may approximate the function $j^{IE}(x)$ as

$$j^{IE}(x) \approx 1 + j(a_1)F(a - x - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps}) + \sum_{k=1}^{m} w_k j(a_k) f(a_k + a - a_i - \phi_1 Z_{t-s} - \phi_2 Z_{t-2s} - \cdots - \phi_p Z_{t-ps}) \tag{14}$$

with $w_k = \frac{h}{m}$ and $a_k = \frac{h}{m}\left(k - \frac{1}{2}\right)$.

We display numerical scheme to evaluate solutions of the integral equations in (10) and (11) from Section 3, which are compared with the approximate function $j(x)$ as in (14), by Gauss-Legendre quadrature rule. All results give a comparison of the approximated solutions $j^{IE}(x)$, the exact solutions $j(x)$, the absolute percentage difference

$$\varepsilon_r(\%) = \frac{|\, j(x) - j^{IE}(x)\,|}{j(x)} \times 100\%$$

for several values of *a*, *h* and the number of divisions *m*.

To study the ARL for CUSUM charts for the seasonal AR(p) model, we consider CUSUM chart for the seasonal AR(1) and seasonal AR(2) models. We present some numerical results in Table I in order to compare the explicit and numerical values of the ARL for the seasonal AR(1) model with exponential distribution white noise. The parameters ϕ are equal to 0.3, 0.5, 0.7 and –0.3, –0.5, –0.7. For in-control state, the ARLs are equal to 100, 370 and 500 with boundary value of 4 and then the constant value is dependent on the initial value of *x*.

Table I. ARL of CUSUM chart for the seasonal AR(1) with 12-periodicity model with exponential distribution white noise when $h = 4$

ϕ	a	Value	$x = 0$	ϕ	a	Value	$x = 0$
0.3	2.082	$j_0(x)$	100.808	–0.3	1.48	$j_0(x)$	100.2770
		$j^{IE}(x)$	100.6717			$j^{IE}(x)$	99.9407
			821.2050[1]				998.0320
		$\varepsilon_r(\%)$	0.1352[2]			$\varepsilon_r(\%)$	0.3354
	2.782	$j_0(x)$	370.0630		2.183	$j_0(x)$	370.5980
		$j^{IE}(x)$	368.6945			$j^{IE}(x)$	369.0626
			928.3780				816.7590
		$\varepsilon_r(\%)$	0.3698			$\varepsilon_r(\%)$	0.4143
	3	$j_0(x)$	500.3470		2.4	$j_0(x)$	500.3470
		$j^{IE}(x)$	498.3961			$j^{IE}(x)$	498.3391
			1168.6000				801.7350
		$\varepsilon_r(\%)$	0.3899			$\varepsilon_r(\%)$	0.4013
0.5	2.28	$j_0(x)$	100.2770	–0.5	1.28	$j_0(x)$	100.2770
		$j^{IE}(x)$	100.0467			$j^{IE}(x)$	99.9202
			841.7970				830.9870
		$\varepsilon_r(\%)$	0.2297			$\varepsilon_r(\%)$	0.3558
	2.983	$j_0(x)$	370.5980		1.983	$j_0(x)$	370.5980
		$j^{IE}(x)$	369.2794			$j^{IE}(x)$	369.1256
			803.3420				801.5800
		$\varepsilon_r(\%)$	0.3558			$\varepsilon_r(\%)$	0.3973
	3.2	$j_0(x)$	500.3470		2.2	$j_0(x)$	500.3470
		$j^{IE}(x)$	498.4272			$j^{IE}(x)$	498.3251
			801.7670				798.8790
		$\varepsilon_r(\%)$	0.3837			$\varepsilon_r(\%)$	0.4041
0.7	2.48	$j_0(x)$	100.2770	–0.7	1.082	$j_0(x)$	100.8080
		$j^{IE}(x)$	100.0724			$j^{IE}(x)$	100.4149
			856.2590				803.0460
		$\varepsilon_r(\%)$	0.2040			$\varepsilon_r(\%)$	0.3899

	3.183	$j_0(x)$	370.5980		3.195	$j_0(x)$	370.0630
		$j^{IE}(x)$	369.2716			$j^{IE}(x)$	368.5820
			803.8880				803.9830
		$\varepsilon_r(\%)$	0.3579			$\varepsilon_r(\%)$	0.4002
	3.4	$j_0(x)$	500.3470		3.41	$j_0(x)$	500.3470
		$j^{IE}(x)$	498.4702			$j^{IE}(x)$	498.3166
			799.3490				802.4690
		$\varepsilon_r(\%)$	0.3751			$\varepsilon_r(\%)$	0.4058

[1]This is the CPU time used (seconds). [2]This is the absolute percentage difference.

It is noticed that the results obtained from the explicit formulas are much closer to the numerical approximation. The explicit formulas are faster computational times. As an example, if $\phi = 0.3$, $a = 2.082$ and $x = 0$, then this gives the $\text{ARL} = 100.808$. The computational time of this ARL which is based on the IE approach takes less than 1 second while the NI approach takes 821.2050 seconds.

Table II. ARL and ADT of CUSUM Chart for the seasonal AR(1) with 12-periodicity model with exponential distribution white noise when $h = 4$

λ	$x = 0,\ a = 2.733,\ \phi = 0.25$		$\varepsilon_r(\%)$ [1]	$x = 0,\ a = 2.232,\ \phi = -0.25$		$\varepsilon_r(\%)$
	$j(x)$	$j^{IE}(x)$		$j(x)$	$j^{IE}(x)$	
1.00	**370.5980**	**369.9061**	0.1867	**370.0630**	**369.1382**	0.2499
1.01	346.7460	346.1125	0.1827	364.2480	363.3530	0.2457
1.05	269.1590	268.7076	0.1677	268.7810	268.1588	0.2315
1.07	238.8610	238.4776	0.1605	238.5290	237.9930	0.2247
1.10	201.3610	201.0567	0.1511	201.0860	200.6541	0.2148
1.20	121.4780	121.3285	0.1231	121.3220	121.0962	0.1861
2	14.7495	14.7442	0.0360	14.7379	14.7273	0.0721
3	5.8554	5.8545	0.0158	5.85276	5.8509	0.0326

[1]This is the absolute percentage difference.

The results are shown in Table II in order to compare the explicit and numerical values of the ARL and ADT for the seasonal AR(1) model with exponential distribution white noise when parameter values are $\varphi = 0.25$ and –0.25. For in-control state, the ARL is equal to 370, with boundary value equal to 4 and constant value is dependent on the initial value of *x*. Notice that $\lambda = 1$ is the value assumed for the in-control parameter, therefore the first row gives the value of the ARL. Rows for $\lambda > 1$ correspond to values of out-of-control parameters; therefore these rows give the values for ADT.

Table III. ARL and ADT of CUSUM chart for the seasonal AR(2) with 12-periodicity model with exponential distribution white noise when $\phi_1 = 0.4$, $\phi_2 = 0.2$ and $x = 0$

λ	Value	CUSUM (h, a)				
		(3, 2.75)	(3, 2.90)	(4, 2.382)	(4, 3.083)	(4, 3.3)
1	$j_0(x)$	**100.0040**	**122.8510**	**100.8080**	**370.5980**	**500.3470**
	$j^{IE}(x)$	99.8299	122.4853	100.6295	369.0867	498.9085
		963.6960[1]	917.0680	948.9850	950.1400	938.6550
	$\varepsilon_r(\%)$	0.1741[2]	0.2977	0.1771	0.4078	0.2875
1.5	$j_0(x)$	18.7236	21.5750	40.9483	40.7521	51.2113
	$j^{IE}(x)$	18.6830	21.5325	40.8583	40.6305	51.1024
		955.5370	916.7710	948.1900	946.4730	924.1840
	$\varepsilon_r(\%)$	0.2166	0.1972	0.2199	0.2985	0.2126
2	$j_0(x)$	8.6408	9.56626	14.7495	14.2983	17.4013
	$j^{IE}(x)$	8.6242	9.5386	14.7211	14.2700	17.3687
		920.4370	919.1580	948.1590	943.6500	920.3560
	$\varepsilon_r(\%)$	0.1916	0.2892	0.1924	0.1976	0.1875

[1]This is the CPU time in seconds. [2]This is the absolute percentage difference.

The results are shown in Tables III and IV in order to compare the ARL and ADT which were obtained from the numerical approximation and the explicit formulas for the seasonal AR(2) model with exponential distribution white noise when parameter values (ϕ_1, ϕ_2) are equal to (0.2, 0.1), (0.9, 0.05), respectively. However, for in-control state, the ARLs are equal to 100, 370 and 500 with the boundary value equal to 3 and 4. As an example, if the boundary value is equal to 4 and the constant value is 2.382, they give the ARL equal to 100.8080. Notice that the explicit solutions are in good agreement with the results obtained from the numerical approximation. Furthermore, it is obvious that $\lambda = 1$ is the value assumed for the in-control parameter, therefore, the first row gives the value of the ARL. Rows for $\lambda > 1$ correspond to values of out-of-control parameters, therefore these rows give the values for ADT.

Table IV. ARL and ADT of CUSUM Chart for the seasonal AR(2) with 12-periodicity model with exponential distribution white noise when $\phi_1 = 0.9$, $\phi_2 = 0.05$ and $x = 0$

λ	Value	CUSUM(b, a)			
		(3, 2.99)	(4, 2.937)	(4, 3.636)	(4, 3.853)
1	$j_0(x)$	**60.6550**	**100.4210**	**370.4880**	**500.6270**
	$j^{IE}(x)$	60.6137	100.3444	370.1924	500.1369
		963.9770[1]	962.6670	962.3700	962.9010
	$\varepsilon_r(\%)$	0.0681[2]	0.0763	0.0798	0.0979
1.5	$j_0(x)$	12.7137	15.5232	40.1326	50.4161
	$j^{IE}(x)$	12.6943	15.5055	40.0976	50.3544
		968.8910	975.6450	982.5100	980.7620
	$\varepsilon_r(\%)$	0.1523	0.1142	0.0873	0.1224
2	$j_0(x)$	6.2861	7.3003	14.1355	16.7913
	$j^{IE}(x)$	6.2809	7.2936	14.1220	16.7737
		996.7680	994.0380	997.9230	1009.9700
	$\varepsilon_r(\%)$	0.0826	0.0914	0.0955	0.104571

[1]This is the CPU time in seconds. [2]This is the absolute percentage difference.

In Tables I to IV, the analytical explicit solutions are in good agreement with the results obtained from the numerical integral equation approach with 500 nodes in the integration rule. It is obvious that explicit formulas give numerical results which are much closer to the numerical integral equations approach. In addition, the computational times of the numerical integral equations approach take approximately 15 minutes while the results obtained from the explicit formula take less than 1 second which is much less than the former.

5. Conclusions

We derive analytically explicit formulas of ARL and ADT for CUSUM chart when observations are the seasonal AR(p) model with exponential distribution white noise. The accuracy of the analytical results has been compared with the numerical integral equation based on Gauss-Legendre quadrature rule. They are in excellent agreement. The amount of time required for the numerical computations were approximately 10-15 minutes compared with less than one second for the explicit formulas. In addition, our results can easily be implemented in any computer program which is very useful for design of optimal CUSUM charts.

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