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APPROXIMATE CONTROLLABILITY OF STOCHASTIC IMPULSIVE INTEGRO-DIFFERENTIAL SYSTEMS WITH INFINITE DELAY

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Abstract

This paper is concerned with the approximate controllability of impulsive stochastic integro-differential systems with infinite delay in Hilbert spaces. By using the contraction mapping principle, some sufficient conditions are given with no compactness condition imposed on the semigroup generated by the linear part of the system. The results are obtained by using the Nussbaum's fixed point theorem.

1. Introduction

The controllability is one of the fundamental concepts in linear and

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nonlinear control theory, and plays a crucial role in both deterministic and stochastic control systems (see, e.g., Zabczyk [30]). These systematic studies have started at the beginning of the 1960s, when the theory of controllability based on the state space description for both time-invariant and time-varying linear control systems was developed.

There are many different definitions of controllability in the literature for both linear (Klamka [6, 7], Mahmudov [12, 13] and Mahmudov and Denker [11]) and nonlinear dynamic systems (Klamka [9], Mahmudov [14], Mahmudov [15] and Mahmudov and Zorlu [17]), which do depend on the class of dynamic control systems and the set of admissible controls (Klamka [6, 8]). For stochastic system, the controllability problem has been studied by several authors (e.g., Mahmudov [12, 13]) for the linear case and Mahmudov [15], Mahmudov and Zorlu [17] and Wang et al. [27] for the nonlinear one. The approximate controllability of stochastic systems in abstract space has been extensively considered in many publications and monographs, an extensive list of these publications can be found in the monograph of Mahmudov [12, 13], Hu and Li [5], Rathinasam [19] and Subalakshmi and Balachandran [26]. The controllability of nonlinear stochastic system in infinite-dimensional spaces has been studied by many authors, Mahmudov [14] investigated the sufficient conditions for approximate controllability of nonlinear systems in Hilbert spaces by using the Nussbaum's fixed point theorem. Balachandran and Sakthivel [2] discussed the controllability of neutral functional integro-differential systems in Banach spaces by using the semigroup theory and Schaefer's fixed point theorem. Karthikeyan and Balachandran [10] and Sakthivel et al. [22], respectively, investigated the controllability of impulsive stochastic control systems by using contraction mapping principle, and Subalakshmi and Balachandran [26] studied the approximate controllability of nonlinear stochastic impulsive systems in Hilbert spaces by using Nussbaum's fixed point theorem. Balachandran and Karthikeyan [1] derived sufficient conditions for the controllability of stochastic integro-differential systems in finite dimensional spaces using the resolvent matrix and the Banach fixed point theorem. Shen et al. [24] investigated the complete controllability problem of impulsive stochastic integro-differential systems using Schaefer's fixed point theorem, Sakthivel et al. [20] studied the approximate controllability of nonlinear impulsive differential systems. Shen and Sun [23] studied the approximate controllability of stochastic impulsive functional system with infinite delay in abstract space. However, for our knowledge, the problem of approximate controllability of stochastic impulsive integro-differential systems with infinite delay has not been considered in the literature. In the present paper, we shall study approximate controllability of stochastic impulsive integro-differential systems with infinite delay in abstract space, which extends the problem considered by Shen and Sun [23]. In Section 3, we study the approximate controllability of the following impulsive stochastic integro-differential system with infinite delay. Consider the following stochastic semilinear impulsive integro-differential system:

$$\begin{cases} dx(t) = \left[Ax(t) + Bu(t) + f(t,\ x(t)) + \int_0^t f_1(t,\ s,\ x(s))ds \right]dt \\ \qquad\qquad + g(t,\ x(t))dw(t),\ t \in J, \\ \Delta x(t_k) = I_k(x(t_k)),\ k = 1,\ \dots,\ m, \\ x(t) = \phi(t),\ t \in J_1 = (-\infty,\ 0], \end{cases} \tag{1}$$

where $\phi(t) \in \mathcal{D}$ and $\mathcal{D}$ is called a *phase space* that will be defined later. U, X and E are separable Hilbert spaces. Denote by $\mathcal{L}(E)$ the space of all linear bounded operators from E to E. $w(t)$ is a Q-Wiener process on $(\Omega,\ \mathcal{F},\ P)$ with the linear bounded covariance operator $Q \in \mathcal{L}(E)$ such that $trQ < \infty$.

We assume that there exist a complete orthonormal system e_k in E, a bounded sequence of non-negative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1,\ 2,\ \dots,$ and a sequence B_k of independent Brownian motions such that

$$\langle W(t),\ e\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle e_k,\ e\rangle B_k(t),\ e \in E,\ t \in J = [0,\ T]$$

and $\mathcal{F} = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$. A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t), \ t > 0$ on X. $\mathcal{F}_t$ is the σ-algebra generated by $w(s), \ 0 \leq s \leq t$. For every $t \in [0, T]$, the history function $x_t : \Omega \to X$ is defined by $x_t(\theta) = x(t + \theta), \ \theta \in J_1$.

We assume that the histories x_t belong to the abstract phase space $\mathcal{D}$, $f : J \times \mathcal{D} \to X$, $f_1 : J \times J \times \mathcal{D} \to X$, $g : J \times \mathcal{D} \to L_0^2(Q^{\frac{1}{2}}E, X)$, and $I_k : (X, X)$, where $L_0^2(Q^{\frac{1}{2}}E, X)$ is the separable Hilbert space of all Hilbert-Schmidt operators, with the norm $\| \psi \|_{L_0^2}^2 = tr(\psi Q \psi^*) < \infty$.

B is a bounded linear operator from U into X. $\Delta x(t)$ denotes the jump of x at t, i.e.,

$$\Delta x(t) = x(t^+) - x(t^-) = x(t^+) - x(t).$$

The paper is organized as follows: In Section 2, some preliminary facts are recalled. Section 3 is devoted to sufficient conditions on the approximate controllability of (1) by the contraction mapping theorem and Nussbaum's fixed point theorem. Finally, concluding remarks are presented in Section 4.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\mathcal{F}_{t>0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all P-null sets). $E(\cdot)$ denotes the expectation with respect to the measure P. The collection of all square integrable and $\mathcal{F}_t$-adapted processes is denoted by $L_2^{\mathcal{F}_t}(J, X)$. Then we present the abstract phase space $\mathcal{D}$. Assume $h_0 : J_1 \to R^+$ is a continuous function with $l = \int_{-\infty}^{0} h_0(t)dt < \infty.$ Now consider the following spaces.

Let $L_2(\Omega, \mathcal{F}_t, X)$ be the space of all $\mathcal{F}_t$-measurable square integrable random variables with value in *X*.

$$\mathcal{D} = \Big\{ \Psi : J_1 \to X, \text{ for any } c > 0, \ \Psi|_{[-c,0]} \text{ is a bounded and measurable function satisfying } \| \Psi \|_{[-c,0]} = \sup_{s \in [-c,0]} \| \Psi(s) \|, \text{ and } \int_{-\infty}^{0} h_0(s) \| \Psi \|_{[s,0]} ds < \infty \Big\}.$$

$\mathcal{D}$ is with the norm $\| \Psi \|_{\mathcal{D}} = \int_{-\infty}^{0} h_0(s) \| \Psi \|_{[s,0]} ds.$

$$PC(J_0, L_2(\Omega, \mathcal{F}_t, P, X))$$

$$= \{x(t) : J_0 = [-\infty, T] \to L_2(\Omega, \mathcal{F}_t, X)$$

is continuous everywhere except some τ_k at which $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k) = x(t_k^-)$ satisfying $\sup_{s \in J_0} E\| x(s) \|^2 < \infty\}.$

Let H be the closed subspace of $PC(J_0, L_2(\Omega, \mathcal{F}_t, \mathcal{D}))$ consisting of $\mathcal{F}_t$-adapted measurable processes and $\mathcal{F}_0$-adapted processes $y \in L_2(\Omega, \mathcal{F}_t, \mathcal{D})$. Let $\| \cdot \|_*$ be a semi-norm in H defined by $\| y \|_*^2 = E \sup\{\| y \|_{\mathcal{D}}^2 : t \in J\}.$

Definition 2.1. The stochastic system (1) is approximately controllable on the interval J if $\overline{R(T; \phi, u)} = L_2(\Omega, \mathcal{F}_t, X),$ where the reachable set $\overline{R(T; \phi, u)}$ is defined as $R(T; \phi, u) = \{x(T; \phi, u), u(\cdot) \in L_2^{\mathcal{F}_t}(J, U)\}.$

Lemma 2.2 [3]. *Assume* $x(t) \in PC(J_0, L_2(\Omega, \mathcal{F}_t, X))$. *Then for any* $t \in J$, $x_t \in \mathcal{D}$, *one has* $l\| x(t) \| \leq \| x_t \|_{\mathcal{D}} \leq \| \phi \|_{\mathcal{D}} + l \sup_{s\in[0,t]} \| x(s) \|$, *where* $l = \int_{-\infty}^{t} h_0(t)dt < \infty$.

Throughout this article, we will employ the contraction mapping principle and the following Nussbaum's fixed point theorem to investigate the approximate controllability of (1).

Lemma 2.3 [18]. *Let S be a closed, bounded convex subset of a Banach space X, and* G_1 *and* G_2 *be continuous mappings from S into X such that* $(G_1 + G_2)S \subset S$, $\| G_2 x - G_2 y \| \leq k\| x - y \|$ *for all* $x, y \in S$, *where* $0 \leq k \leq 1$ *is a constant, and* $\overline{G_1 S}$ *is compact. Then the operator* $G_1 + G_2$ *has a fixed point in S.*

3. Results for Stochastic Impulsive Systems

Choose $\phi \in \mathcal{D}$, $h \in L_2(\Omega, \mathcal{F}_t, X)$ and consider how to determine an appropriate control $u(\cdot) \in L_2^{\mathcal{F}_t}(J, U)$ which steers the solution of (1) with initial value ϕ to a very small neighborhood of h. In order to study the approximate controllability of system (1), we introduce the following hypotheses which are assumed here and henceforth:

- (H1) The operator $\alpha R(\alpha, \Gamma_t^T) = \alpha(\alpha I + \Gamma_t^T)^{-1}$ converges to zero operator in the strong operator topology as $\alpha \to 0^+$ for any $t \in J$, where Γ_t^T is the linear controllability operator on X that is given by

$$\Gamma_t^T = \int_t^T S(T - s)BB^* S(T - s)ds,$$

  and it satisfies $\| \alpha R(\alpha, \Gamma_t^T) \| \leq 1$, for any $t \in J$.

- (H2) There exist positive constants M_1, M_2, d, and d_k, $k = 1, 2, ..., m$ such that $\| S(t) \|^2 \leq M_1$, $\| B \|^2 \leq M_2$, $\| I_k(x) \|^2 \leq d_k$, $d = \sum_{k=1}^{m} d_k$.

- (H3) There exist constants $L_1,\ L_2,\ d_{1k},\ k = 1, ..., m$ such that for all $t \in J,\ x, y \in X,\ x_t, y_t \in \mathcal{D}$ such that

$$\| f(t, x) - f(t, y) \|^2 \vee \| g(t, x) - g(t, y) \|^2_{L_2^0} \le L_1 \| x - y \|^2_{\mathcal{D}},$$

$$\| f_1(t, s, x) - f_1(t, s, y) \|^2 \le L_2 \| x - y \|^2_{\mathcal{D}},$$

$$\| I_k(x) - I_k(y) \|^2 \le d_{1k} \| x - y \|^2.$$

- (H4) $f : J \times \mathcal{D} \to X,\ f_1 : J \times J \times \mathcal{D} \to X,\ g : J \times \mathcal{D} \to L_0^2(Q^{\frac{1}{2}}E, X),$ and satisfy the usual linear growth condition, i.e., there exist constants $L_3, L_4,$

$$\| f(t, x) \|^2 \vee \| g(t, x) \|^2_{L_2^0} \le L_3(1 + \| x \|^2_{\mathcal{D}}),$$

$$\| f_1(s, t, x) \|^2 \le L_4(1 + \| x \|^2_{\mathcal{D}}).$$

- (H5) $f : J \times \mathcal{D} \to X,\ f_1 : J \times J \times \mathcal{D} \to X,\ g : J \times \mathcal{D} \to L_0^2(Q^{\frac{1}{2}}E, X),$ are uniformly bounded for $t \in J,\ x_t \in \mathcal{D},$ i.e., there exist constants L_5, L_6 such that

$$\| f(t, x) \|^2 + \| g(t, x) \|^2_{L_2^0} \le L_5,$$

$$\| f_1(t, s, x) \|^2 \le L_6.$$

It is clear that under these conditions, system (1) admits a mild solution $x(\cdot) \in PC(J_0, L_2)$ for any $\phi(t) \in H,\ u(\cdot) \in L_2^{\mathcal{F}_t}(J, U)$ and this mild solution satisfies

$$x(t) = S(t)\phi(0) + \int_0^t S(t-s)\left(B(s)u(s) + \int_0^t S(t-s) f(s, x_s) \right) ds$$

$$+\int_0^t S(t-s)\left[\int_0^s f_1(s, \tau, x_\tau)d\tau\right]ds$$

$$+\int_0^t S(t-s)g(s, x_s)dw(s) + \sum\nolimits_{0<t_k<t} S(t-t_k)I_k(x(t_k)).$$

Lemma 3.1 [13]. *For arbitrary* $h \in L_2(\Omega, \mathcal{F}_t, X)$, *there exists* $z \in L_2^{\mathcal{F}_t}(J, L_2^0)$, *and such that* $h = Eh + \int_0^t z(s)dw(s)$. *For all* $\alpha > 0$, *define the control for system* (1) *as*

$$u_\alpha(t, x) = B^*S^*(T-t)\left[R(\alpha, \Gamma_0^T)(Eh - S(T)\phi(0)) + \int_0^t R(\alpha, \Gamma_0^T)z(s)dw(s)\right]$$

$$- B^*S^*(T-t)\int_0^t R(\alpha, \Gamma_s^T)S(T-s)f(s, x_s)ds$$

$$- B^*S^*(T-t)\int_0^t R(\alpha, \Gamma_s^T)S(T-s)\times\left(\int_0^s f(s, \tau, x_\tau)d\tau\right)ds$$

$$- B^*S^*(T-t)\int_0^t R(\alpha, \Gamma_s^T)S(T-s)g(s, x_s)dw$$

$$- B^*S^*(T-t)R(\alpha, \Gamma_0^T)\sum_{k=1}^m S(T-t_k)I_k(x(t_k))$$

and the operator $\mathcal{P}_\alpha$ *on* $PC(J_0, L_2(\Omega, \mathcal{F}_t, X))$ *as follows*:

$$\mathcal{P}_\alpha x(t) = \begin{cases} \phi(t), & t \le 0, \\ S(t)\phi(0) + \int_0^t S(t-s)\left(B(s)u_\alpha(s) + \int_0^t S(t-s)f(s, x_s)\right)ds & \\ + \int_0^t S(t-s)\left[\int_0^s f_1(s, \tau, x_\tau)d\tau\right]ds & \\ + \int_0^t S(t-s)g(s, x_s)dw(s) & \\ + \sum\nolimits_{0<t_k<t} S(t-t_k)I_k(x(t_k)), & t \in J. \end{cases} \tag{2}$$

It will be shown that (1) is approximately controllable if for all $\alpha > 0$ there exists a fixed point of the operator $\mathcal{P}_\alpha$. To show that $\mathcal{P}_\alpha$ has a fixed point, we employ the contraction mapping theorem. For convenience, denote $c = (E\| h \| + M_1\| \phi \|_{\mathcal{D}})^2$.

Theorem 3.2. *Suppose* (H1)-(H4) *hold. If*

$$\frac{20M_1^3M_2^2}{\alpha}\left(2l^2T(T+1)L_1 + 2l^2T^3L_2 + m\sum_{k=1}^{m} d_{1k}\right)$$

$$+ 5M_1T(T+1)L_1 + 5M_1T^3L_2 + 5M_1m\sum_{k=1}^{m} d_{1k} \leq 1, \tag{3}$$

then operator $\mathcal{P}_\alpha$ *has a fixed point in H.*

Proof. Using Lemma 3.1, we get

$$E\| x_t \|^2 \leq 2l^2E\sup\{\| x(s) \|^2 : 0 \leq s \leq t\} + 2E\| \phi \|_{\mathcal{D}}^2.$$

We shall first study the control $u_\alpha(t, x)$. By (2), (H2) and (H4),

$$E\| u_\alpha(t, x) \|^2 \leq \frac{2}{\alpha}M_1M_2\Bigg(\| Eh - S(T)\phi(0) \| + \sqrt{M_1}\int_0^T \| f(s, x_s) \| ds$$

$$+ \sqrt{M_1}\int_0^T\left[\int_0^s \| f(s, \tau, x_\tau)d\tau \|\right]ds + \sqrt{M_1}d\Bigg)^2$$

$$+ \frac{4}{\alpha}M_1M_2E\left(\int_0^T \| z(s) \|^2 ds + M_1\int_0^T \| g(s, x_s) \|^2 ds\right)$$

$$\leq \frac{1}{\alpha}M_1M_2\Bigg(8c + 8M_1(T+1) \times \int_0^T L_3(1 + \| x_s \|_{\mathcal{D}}^2)ds$$

$$+ 8T^2M_1 \times \int_0^T L_4(1 + \| x_s \|_{\mathcal{D}}^2)ds + 8M_1d\Bigg). \tag{4}$$

Similarly, for x and $y \in H$, one can also obtain by (H2) and (H3) that

$$E\| u_\alpha(t, x) - u_\alpha(t, y) \|^2$$

$$= \Bigg\| B^* S^*(T - t) \int_0^t R(\alpha, \Gamma_s^T)$$

$$\times S(T - s)(f(s, x_s) - f(s, y_s))ds$$

$$+ B^* S^*(T - t) \int_0^t R(\alpha, \Gamma_s^T) S(T - s)$$

$$\times \left(\int_0^s [f_1(s, \tau, x_\tau) - f_1(s, \tau, y_\tau)]d\tau \right) ds$$

$$+ B^* S^*(T - t) \int_0^t R(\alpha, \Gamma_s^T) \times S(T - s)(g(s, x_s) - g(s, y_s))dw(s)$$

$$+ B^* S^*(T - t) R(\alpha, \Gamma_s^T) \times \sum_{k=1}^m S(T - t_k)(I_k(x(t_k)) - I_k(y(t_k))) \Bigg\|$$

$$\leq \frac{1}{\alpha} M_1 M_2 \Bigg[4M_1 T \int_0^t L_1 \| x_s - y_s \|_{\mathcal{D}}^2 + 4M_1 T^2 \int_0^T L_2^2 \| x_\tau - y_\tau \|_{\mathcal{D}}^2$$

$$+ 4M_1 \int_0^t L_1 \| x_s - y_s \|_{\mathcal{D}}^2 + 4M_1 m \sum_{k=1}^m d_{1k} E\| x(t_k - y(t_k)) \|^2 \Bigg]$$

$$\leq \frac{4M_1^2 M_2}{\alpha} \left(2l^2 T(T + 1)L_1 + 2l^2 T^3 L_2 + m \sum_{k=1}^m d_{1k} \right) \| x - y \|_*^2.$$

Now consider $\mathcal{P}_\alpha$,

$$E\| (\mathcal{P}_\alpha x(t)) \|^2 \leq 5c + 5M_1 M_2 T E\| u_\alpha(t, x) \|^2$$

$$+ 5M_1 T[(T + 1)L_3 + T^3 L_4](1 + 2l^2 \| x \|_*^2 + 2E\| x \|_{\mathcal{D}}^2),$$

which, together with (3), implies that $E\| (\mathcal{P}_\alpha x(t)) \|^2 \leq \infty$, that is, $\mathcal{P}_\alpha H \in H$.

To apply the contraction mapping principle, now we prove under some conditions, $\mathcal{P}_\alpha$ is a contraction on *H*. To show this, let $x, y \in H,$ then for $t \in [0, T],$ we have

$$E\| (\mathcal{P}_\alpha x(t)) - (\mathcal{P}_\alpha y(t)) \|^2$$

$$= 5M_1 M_2 E\| u_\alpha(t, x) - u_\alpha(t, y) \|^2 ds$$

$$+ 5M_1(T+1)\int_0^t L_1 \| x_s - y_s \|_{\mathcal{D}}^2 + 5M_1 T^2 \int_0^t L_2 \| x_\tau - y_\tau \|^2$$

$$+ 5M_1 m \sum_{k=1}^m d_{1k} \| x - y \|^2$$

$$\leq \left(\frac{20 M_1^3 M_2^2}{\alpha} \left(2l^2 T(T+1) L_1 + 2l^2 T^3 L_2 + m \sum_{k=1}^m d_{1k} \right) \right.$$

$$\left. + 5M_1 T(T+1) L_1 + 5M_1 T^3 L_2 + 5M_1 m \sum_{k=1}^m d_{1k} \right) \| x - y \|_*^2,$$

then it can be easily concluded that if (3) is satisfied, $\mathcal{P}_\alpha$ is a contraction mapping, and thus by the contraction mapping theorem, has a unique fixed point in *H*. The proof is completed.

Theorem 3.3. *Assume the conditions in Theorem* 3.4 *and* (H5) *are satisfied*. *Then system* (1) *is approximately controllable on* $[0, T]$.

Proof. By Theorem 3.2, $\mathcal{P}_\alpha$ has a unique fixed point *x* in *H*. By substituting (1) into (2) and using the stochastic Fubini theorem (Da Prato and Zabczyk [4] and Zabczyk [30]), we can obtain

$$x^*(t) = S(T)\phi(0) + \int_0^T S(T-s) f(s, x_s^*) ds$$

$$+ \int_0^T S(T-s) \left[\int_0^s f_1(s, \tau, x_\tau^*) d\tau \right] ds$$

$$+ \int_0^t S(T-s) g(s, x_s^*) dw(s)$$

$$+ \sum_{k=1}^{m} S(T - t_k) I_k(x^*(t_k))$$

$$+ \Gamma_0^T R(\alpha, \Gamma_0^T)(Eh - S(T)\phi(0))$$

$$+ \int_0^T \int_r^T S(T - s) BB^* S^*(T - s) \times R(\alpha, \Gamma_r^T) z(r) dw(r)$$

$$- \int_0^T \int_r^T S(T - s) BB^* S^*(T - t) \times R(\alpha, \Gamma_r^T) S(T - r) f(s, x_r^*) dsdr$$

$$- \int_0^T \int_r^T S(T - s) BB^* S^*(T - t) \times R(\alpha, \Gamma_r^T) S(T - r)$$

$$\times \int_0^s f_1(s, \tau, x_r^*) d\tau) dsdr$$

$$- \int_0^T \int_r^T S(T - s) BB^* S^*(T - t) \times R(\alpha, \Gamma_r^T) S(T - r) g(r, x_r^*) dsdw(r)$$

$$- \Gamma_0^T R(\alpha, \Gamma_0^T) \sum_{k=1}^{m} S(T - t_k) I_k(x^*(t_k))$$

$$= h - \alpha R(\alpha, \Gamma_s^T)(Eh - S(T)\phi(0))$$

$$+ \int_0^T \alpha R(\alpha, \Gamma_s^T) S(T - s) f(s, x_s^*) ds + \int_0^T \alpha R(\alpha, \Gamma_0^T)$$

$$\times S(T - s)\left[\int_0^s f_1(s, \tau, x_\tau^*) d\tau\right] ds$$

$$+ \int_0^T \alpha R(\alpha, \Gamma_s^T)(S(T - s) g(s, x_s^*) - z(s)) dw(s)$$

$$+ \alpha R(\alpha, \Gamma_0^T) \sum_{k=1}^{m} S(T - t_k) I_k(x^*(t_k)).$$

Proof. If $I_k = 0$, then condition (3.1) can be reduced to

$$\frac{20M_1^3M_2^2}{\alpha}(2l^2T(T+1)L_1 + 2l^2T^3L_2)$$

$$+ 5M_1T(T+1)L_1 + 5M_1T^3L_2 \leq 1 \tag{5}$$

which can be taken away by considering system (1) with $I_k = 0$ on $[0, \widetilde{T}]$, $[\widetilde{T}, 2\widetilde{T}]$, ..., $[(\widetilde{m}-1)\widetilde{T}, \widetilde{m}\widetilde{T}]$ with $\widetilde{T}$ satisfying (5). That completes the proof. Now we apply the Nussbaum's fixed point theorem to remove the condition (3). To start with, rewrite $\mathcal{P}_\alpha$ as $\mathcal{P}_\alpha = \mathcal{P}_{1\alpha} + \mathcal{P}_{2\alpha}$, where

$$\mathcal{P}_{1\alpha} = \int_0^t S(t-s)f(s, x_s)ds$$

$$+ \int_0^t S(T-s)\left(\int_0^s f(s, \tau, x_\tau)d\tau\right)ds$$

$$+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)),$$

$$\mathcal{P}_{2\alpha} = S(t)\phi(0) + \int_0^t S(t-s)Bu_\alpha(s)$$

$$+ \int_0^t S(T-s)(g(s, x_s))dw(s).$$

Theorem 3.4. *Assume* (H1), (H3) *and* (H5) *hold*. *If* $f(t, \cdot)$, $f_1(s, \tau, \cdot)$ *and* $I_k(\cdot)$ *are compact*, *then system* (3) *is approximately controllable on* $[0, T]$.

Proof. By (H2) and (H5),

$$E\|(P_{1\alpha}x)(t)\|^2 \leq 3M_1M_5T^2 + 3M_1M_6T^3 + 3M_1d,$$

and for $P_{2\alpha}$,

$$E\|(P_{2\alpha}x)(t)\|^2 \leq 3c + 3M_1M_5T^2E\|u_\alpha(t, x)\|^2 + 3M_1M_5T.$$

From (4), there exists a constant $r > 0$ such that $P_\alpha Y_r \subset Y_r$ with $Y_r = \{x \in H, \| x \|_* \leq r\}$. On this Y_r, we can verify

$$E\| (P_{2\alpha}x)(t) - (P_{2\alpha}y)(t) \|^2 \leq 2M_1 E\int_0^T \| g(s, x_s) \cdot g(s, y_s) \|^2 ds$$

$$+ 2M_1 M_2 E\int_0^T \| \mu_\alpha(s, x) \cdot u_\alpha(s, y) \|^2 ds$$

$$\leq L\| x - y \|_*^2,$$

where

$$L = \frac{4M_1^2 M_2}{\alpha}\left(2l^2 T(T+1)L_1 + 2l^2 T^3 L_2 + m\sum_{k=1}^m d_{1k}\right) + 2M_1 T L_1.$$

That is, if $L < 1$, $P_{2\alpha}$ is a contraction mapping, which can be removed if considering system (1) on $[0, \widetilde{T}], [\widetilde{T}, 2\widetilde{T}], \ldots, [(\widetilde{m}-1)\widetilde{T}, \widetilde{m}\widetilde{T}]$ with $\widetilde{T}$ satisfying $L < 1$.

The operator $P_{1\alpha}$ maps Y_r into a relatively compact subset of Y_r. Denote the set $V(t) = (P_{1\alpha})(t) | x \in Y_r$.

Take into account that the operators

$$S(t-s)f(s, x_s),\ \ S(t-s)\int_0^s f_1(s, \tau, \tau_s) \text{ and } \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k))$$

are compact. This is because $S(t)$ is strongly continuous and f, f_1 and I_k are compact. On the other hand, since Y_r is bounded in H, the sets

$$\{S(t-s)f(s, x_s), t, s \in J, x \in Y_r\},$$

$$\left\{S(t-s)\int_0^s f_1(s, \tau, x_\tau), t, s, \tau \in J, x \in Y_r\right\},$$

$$\left\{\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), t, t_k \in J, x \in Y_r\right\}$$

are pre-compact. That $V(T)$ is relatively compact follows directly from the fact that

$$V(t) \subset t\overline{conv}\{S(t-s)f(s, x_s), t, s \in J, x \in Y_r\}$$

$$+ t\overline{conv}\left\{S(t-s)\int_0^s f_1(s, \tau, x_\tau), t, s, \tau \in J, x \in Y_r\right\}$$

$$+ \left\{S(t-\tau_k)\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), t, \tau_k \in J, x \in Y_r\right\},$$

where $\overline{conv}$ denotes the closure of the convex hull defined in Xue [28].

Now we show $V(t)$ is equicontinuous on $[0, T]$. Let $0 < t_1 < t_2 \leq T$. Then we have

$$E\|(P_{1\alpha}x)(t) - (P_{1\alpha}y)(t)\|$$

$$\leq \left\|\int_0^{t_1}(S(t_1-s) - S(t_2-s))f(s, x_s)ds\right\|$$

$$+ \left\|\int_{t_1}^{t_2} S(t_2-s)f(s, x_s)ds\right\|$$

$$\leq \left\|\int_0^{t_1}(S(t_1-s) - S(t_2-s))\left[\int_0^s f_1(s, \tau, x_\tau)d\tau\right]ds\right\|$$

$$+ \left\|\int_{t_1}^{t_2} S(t_2-s)\left[\int_0^s f_1(s, \tau, x_\tau)d\tau\right]ds\right\|$$

$$+ \left\|\sum_{0\leq t\leq t_1}(S(t_1-t_k) - S(t_2-t_k)I_k x(t_k))\right\|$$

$$+ \left\|\sum_{t_1\leq t\leq t_2} S(t_2-t_k)I_k x(t_k)\right\|$$

$$\leq \sqrt{M_1}\left\|\int_0^{t_1}(I - S(t_2-t_1))f(s, x_s)ds\right\|$$

$$+\sqrt{M_1}\sqrt{M_5}(t_2-t_1)+\sqrt{M_1}\sqrt{M_6}(t_2-t_1)$$

$$+M_1\left\|\int_0^{t_1}(I-S(t_2-t_1))\left[\int_0^s f_1(s,\tau,x_\tau)d\tau\right]ds\right\|$$

$$+\sqrt{M_1}\sqrt{d}\,\|I-S(t_2-t_1)\|+\left\|\sum_{t_1\le t\le t_2}S(t_2-t_k)I_kx(t_k)\right\|.$$

Since $f,\ f_1$ and I_k are compact, and $S(t)$ is strongly continuous, one has that, as $t_2\to t_1$,

$$\|(I-S(t_2-t_1))f(s,x_s)\|\mapsto 0,\quad (I-S(t_2-t_1))\left[\int_0^s f_1(s,\tau,x_\tau)d\tau\right]\mapsto 0,$$

$$\|I-S(t_2-t_1)I_kx(t_k)\|\mapsto 0,$$

uniformly for $s\in[0,T],\ \ k=1,\ldots,m$. Thus, the equicontinuity of $P_{1\alpha}$ is obtained. According to the infinite dimensional version of the Ascoli-Arzela theorem (Yosida [29]), condition (3) in Lemma 2.3 is thus proved.

Now it remains to show $P_{1\alpha}$ is continuous in X. Letting $\{x_n\}\in X$ with $x_n(\cdot)\to x(\cdot),\ n\to\infty$ and using the continuity of $f,\ f_1$ and I_k, one has

$$f(s,x_{ns})\to f(s,x_s),\quad \int_0^s f(s,\tau,x_{n\tau})d\tau\to\int_0^s f(s,\tau,x_\tau)d\tau$$

and $I_k(x_n)-I_k(x)$, as $n\to\infty$. Meanwhile, $\|f(s,x_{ns})-f(s,x_s)\|\le 3\sqrt{M_5}$,

$$\left\|\left[\int_0^s[f(s,\tau,x_{n\tau})-f(s,\tau,x_\tau)]d\tau\right]\right\|\le 3\sqrt{M_6},\quad \|I_k(x)-I_k(x_n)\|\le 3d_k,$$

and by the dominated convergence theorem, $P_{1\alpha}$ is continuous in X. Therefore, by Lemma 2.3, P_α has a fixed point in Y_r. Using the same procedure as in Theorem 3.3, system (1) is approximately controllable on $[0,T]$. That completes the proof.

4. Conclusion

The approximate controllability of stochastic impulsive integro-differential systems with infinite delay has been investigated in this paper. By the contraction mapping theorem and Nussbaum's fixed point theorem, sufficient conditions on the approximate controllability or the system have been obtained.

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