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NEW RESULTS ON UPPER BOUNDS FOR THE CHROMATIC NUMBER OF FUZZY GRAPHS AND THEIR COMPLEMENTS

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Abstract

Upper bounds for sum and product of chromatic number of complementary fuzzy graphs are given in [9]. We find that these bounds do not hold for fuzzy graphs which have certain properties. This problem motivates us to investigate upper bounds for sum and product of chromatic number of several classes of fuzzy graphs and their complements. We obtain new results on upper bounds for sum

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and product of complementary fuzzy graphs. Finally, we investigate these results related to the bounds given in [9]. We add a necessary condition to fuzzy graphs so that the upper bounds given in [9] can be improved.

1. Introduction

Let $G(V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Vertex coloring of G is a mapping C from $V(G)$ to the set of natural numbers $\mathbb{N}$ such that $C(x) \neq C(y)$ for all $(x, y) \in E(G)$. Given an integer k, a k-coloring of G is a mapping C from $V(G)$ to the set of colors $\{1, 2, ..., k\}$ such that $C(x) \neq C(y)$ for all $(x, y) \in E(G)$. The chromatic number of G, denoted by $\chi(G)$, is the smallest integer k such that there is a k-coloring of G. For simplicity, we will use symbol V for $V(G)$ and E for $E(G)$.

Vertex coloring of graph G can be interpreted as a problem of special kind about partition of the vertex set, as mentioned in [1]. Therefore, there is an equivalent definition of vertex coloring as follows. A vertex coloring of $G(V, E)$ is a partition of V into non-empty subsets $V_1, V_2, ..., V_k$, which are called *color classes* such that $V = V_1 \cup V_2 \cup \cdots \cup V_k$, the subsets V_i $(1 \leq i \leq k)$ are mutually disjoint and each V_i contains no pair of adjacent vertices. The chromatic number of G is the smallest natural number k for which such partition is possible.

The notion of a fuzzy set was introduced by Zadeh in [11]. The ideas of fuzzy set theory have been introduced into graph theory by Rosenfeld in 1975 as mentioned in [5]. Rosenfeld introduced a fuzzy graph $\widetilde{G}(V, \sigma, \mu)$ that is a graph which has a fuzzy vertex set $\widetilde{V}$ with a membership function $\sigma : V \to [0, 1]$ and a fuzzy edge set $\widetilde{E}$ with a membership function $\mu : E \to [0, 1]$ such that $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$ for all $u, v \in V$. The fuzzy graph $\widetilde{G}(V, \sigma, \mu)$ is also denoted by $\widetilde{G}(\widetilde{V}, \widetilde{E})$. While Kaufmann introduced a fuzzy graph $\widetilde{G}(V, \mu)$ that is a graph with a crisp vertex set V

and a fuzzy edge set $\tilde{E}$ with a membership function $\mu : E \to [0, 1]$, as cited in [10]. The fuzzy graph $\tilde{G}(V, \mu)$ is also denoted by $\tilde{G}(V, \tilde{E})$. Note that a crisp graph $G(V, E)$ is a special case of a fuzzy graph which each vertex of $\tilde{V}$ and each edge of $\tilde{E}$ have degree of membership 1. Further, a graph $G(V, E)$ will be called a *crisp graph*.

Vertex coloring of a fuzzy graph $\tilde{G}(V, \mu)$ was introduced by some authors in two different ways. First, Munoz et al. [6] generalized the coloring function $C : V \to \mathbb{N}$ of a crisp graph $G(V, E)$ into a function $C_{d,f} : V \to S$ of a fuzzy graph $\tilde{G}(V, \mu)$, where S is the set of colors, d is a distance function defined between two colors on S, and f is a real scale function defined on image of μ. After that, Pourpasha and Soheilifar [8] generalized the coloring function $C_{d,f} : V \to S$ of a fuzzy graph $\tilde{G}(V, \mu)$ into the function $C_{d,f,g} : V \to S$ of a fuzzy graph $\tilde{G}(V, \sigma, \mu)$, where g is a real scale function defined on image of σ. Second, coloring of a fuzzy graph $\tilde{G}$ has been done based on partition of vertex set V into independent vertex sets. With respect to the second approach, Eslahchi and Onagh [2] introduced a vertex coloring of a fuzzy graph $\tilde{G}(V, \sigma, \mu)$ through a partition of the fuzzy vertex set $\tilde{V}$ into fuzzy color-classes $\tilde{V}_i$, $i = 1, \ldots, k$.

The fuzzy graph coloring problem consists of determining the chromatic number of a fuzzy graph with an associated coloring definition. Several authors have studied the problems in obtaining chromatic number of fuzzy graphs. The chromatic number for several classes of fuzzy graphs using the $C_{d,f,g}$ function has been investigated in [8]. While upper bounds for sum and product of chromatic number of complementary fuzzy graphs have been given in [9] using the definition as in [2].

In this paper, we find that the bounds given in [9] do not hold for fuzzy graphs which have certain properties. This problem motivates us to investigate upper bounds for sum and product of chromatic number of several classes of fuzzy graphs and their complements. We obtain new results on upper bounds for sum and product of complementary fuzzy graphs. Finally, we investigate these results related to the bounds given in [9]. We add a necessary condition to fuzzy graphs so that the upper bounds given in [9] can be improved.

2. Preliminaries

We review briefly some definitions in fuzzy sets as in [11]. Let *X* be a space of objects. A fuzzy set *A* on *X* is a set of the form $\{(x, \mu_A(x)) : x \in X\}$, where μ_A is a mapping: $X \to [0, 1]$. We call μ_A as a *membership function* of the fuzzy set *A* and $\mu_A(x)$ represents the grade of membership of *x* in *A*.

Other definition said that a fuzzy set *A* on *X* is a mapping $\mu : X \to [0, 1]$ as in [3]. According to the first notation, the symbol of the fuzzy set *A* is distinguished from the symbol of its membership function (μ_A). According to the second notation, there is no distinction between the two symbols. In this paper, we use the first notation.

A fuzzy set *A* on *X* is empty if and only if $\mu_A(x) = 0$ for all $x \in X$. Let *A* and *B* be two fuzzy sets on *X* with the membership functions $\mu_A : X \to [0, 1]$ and $\mu_B : X \to [0, 1]$, respectively. The union of *A* and *B*, written as $C = A \cup B$, is the fuzzy set on *X* with the membership function $\mu_C : X \to [0, 1]$ defined by $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$. The intersection of *A* and *B*, written as $C = A \cap B$, is the fuzzy set on *X* with the membership function $\mu_C : X \to [0, 1]$ defined by $\mu_C(x) = \min\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$. The fuzzy set *A* is called a *subset* of *B*, denoted by $A \subseteq B$, if $\mu_A(x) \le \mu_B(x)$ for all $x \in X$.

We review briefly some definitions in fuzzy graphs as in [5]. Let $\widetilde{G}'(V', \sigma', \mu')$ be a fuzzy graph with $\sigma' : V' \to [0, 1]$ and $\mu' : V' \times V' \to [0, 1]$.

The fuzzy graph $\tilde{G}'$ is called a *fuzzy subgraph* of $\tilde{G}(V, \sigma, \mu)$ if $V' \subseteq V$ and $\tilde{E}' \subseteq \tilde{E}$. The underlying crisp graph of a fuzzy graph $\tilde{G}(\tilde{V}, \tilde{E})$ is a graph $\tilde{G}(V^*, E^*)$, where $V^* = \{v \in V : \sigma(v) > 0\}$ and $E^* = \{(u, v) \in V \times V : \mu(u, v) > 0\}$ [10].

Furthermore, we refer to the definition of complement of a fuzzy graph as in [10].

Definition 2.1 [10]**.** The complement of a fuzzy graph $\tilde{G}(V, \sigma, \mu)$ is a fuzzy graph $\overline{\tilde{G}} = (V, \overline{\sigma}, \overline{\mu})$, where $\overline{\sigma} = \sigma$ and $\overline{\mu}(u, v) = \min\{\sigma(u), \sigma(v)\} - \mu(u, v)$ for all $u, v \in V$.

Definition 2.2 [10]**.** A fuzzy graph $\tilde{G}(V, \sigma, \mu)$ is self complementary if $\tilde{G} = \overline{\tilde{G}}$.

There are two types of adjacency in a fuzzy graph, namely strong adjacency and weak adjacency [2].

Definition 2.3 [2]**.** Two vertices *u* and *v* of a fuzzy graph $\tilde{G}(V, \sigma, \mu)$ are called *strongly adjacent* if $\mu(u, v) \geq \frac{1}{2}\min\{\sigma(u), \sigma(v)\}$, otherwise is weakly adjacent.

Sunitha [10] gave the condition for a fuzzy graph to be self complementary as follows.

Theorem 2.4 [10]**.** *Let* $\tilde{G}(V, \sigma, \mu)$ *be a fuzzy graph. If* $\mu(u, v) = \frac{1}{2}\min\{\sigma(u), \sigma(v)\}$ *for all* $u, v \in V$, *then G is self complementary.*

Theorem 2.5 [9]**.** *Let* $\tilde{G}(V, \sigma, \mu)$ *be a fuzzy graph and* $\overline{\tilde{G}} = (V, \overline{\sigma}, \overline{\mu})$ *be its complement. The vertices* $u, v \in V$ *are strongly adjacent in* $\tilde{G}$ *if and only if* u, v *are weakly adjacent in* $\overline{\tilde{G}}$. *The strongly adjacent vertices* $u, v \in V$ *in this theorem are restricted to* $\mu(u, v) > \frac{1}{2}\min\{\sigma(u), \sigma(v)\}$.

Next, Sunitha [10] gave a theorem for the complement of joint and union of two fuzzy graphs as follows.

Theorem 2.6 [10]**.** *Let* $\widetilde{G}_1(\widetilde{V}_1, \widetilde{E}_1)$ *and* $\widetilde{G}_2(\widetilde{V}_2, \widetilde{E}_2)$ *be two fuzzy graphs. Then* $\overline{(\widetilde{G}_1 + \widetilde{G}_2)} = \overline{\widetilde{G}}_1 \cup \overline{\widetilde{G}}_2$ *and* $\overline{(\widetilde{G}_1 \cup \widetilde{G}_2)} = \overline{\widetilde{G}}_1 + \overline{\widetilde{G}}_2$.

In crisp graph case, upper bounds for sum and product of chromatic number of a graph and its complement are given in [7]. For a graph $G(V, E)$ with n vertices:

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1 \quad \text{and} \quad n \leq \chi(G) \cdot \chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2.$$

The bounds given in [7] do not hold for fuzzy graphs. In case of fuzzy graph, the bounds are given in [9]. Let $\widetilde{G}(V, \sigma, \mu)$ be a fuzzy graph with n vertices and $\overline{\widetilde{G}} = (V, \overline{\sigma}, \overline{\mu})$ be its complement:

(1) $\chi(\widetilde{G}) + \chi(\overline{\widetilde{G}}) \leq 2(n-1)$ and (2) $\chi(\widetilde{G}) \cdot \chi(\overline{\widetilde{G}}) \leq n^2 + 1$.

3. Main Results

First, we modify the definition of fuzzy graph coloring which given in [2] and [9]. In [2, 9], a fuzzy set A is symbolized by its membership function μ. However, in Definition 3.1, the symbol of a fuzzy set A is distinguished from the symbol of its membership function (μ_A).

Definition 3.1 [9]**.** Let $\widetilde{G}(\widetilde{V}, \widetilde{E})$ be a fuzzy graph where the membership function of $\widetilde{V}$ is σ and the membership function of $\widetilde{E}$ is μ. A k-coloring of $\widetilde{G}$ is defined as a partition of $\widetilde{V}$ into k-fuzzy subsets $\widetilde{V}_1, \widetilde{V}_2, ..., \widetilde{V}_k$ of which membership functions are $\gamma_1, \gamma_2, ..., \gamma_k$, respectively, such that it satisfies the following conditions:

a. $\widetilde{V} = \widetilde{V}_1 \cup \widetilde{V}_2 \cup \cdots \cup \widetilde{V}_k$.

b. $\widetilde{V}_i \cap \widetilde{V}_j = \varnothing$, $i \neq j$.

c. For every strongly adjacent vertices u, v of $\tilde{G}$, $\min\{\gamma_i(u), \gamma_i(v)\} = 0$ $(1 \leq i \leq k)$.

The chromatic number of $\tilde{G}$, denoted by $\chi(\tilde{G})$, is the smallest number of k for which the fuzzy graph $\tilde{G}$ has k-coloring.

The chromatic number of complement $\overline{\tilde{G}}$ is denoted by $\chi(\overline{\tilde{G}})$. For simplicity, we usually use the symbol χ for $\chi(\tilde{G})$ and $\overline{\chi}$ for $\chi(\overline{\tilde{G}})$.

By Definition 3.1, we give a condition to a fuzzy graph so that its chromatic number is equal to 1. Next, we give a condition to a fuzzy graph with n vertices so that its chromatic number is equal to n.

Lemma 3.2. *Let $\tilde{G}(V, \sigma, \mu)$ be a fuzzy graph with n vertices.*

(1) *If every pair of two vertices in $\tilde{G}$ is weakly adjacent, then $\chi(\tilde{G}) = 1$.*

(2) *Otherwise, if every pair of two vertices in $\tilde{G}$ is strongly adjacent, then $\chi(\tilde{G}) = n$.*

Proof. Let $V = \{v_1, v_2, ..., v_n\}$.

(1) Since every pair of vertices is weakly adjacent, we only have one partition $S = \tilde{V}$ which satisfies all of the conditions in Definition 3.1. Thus, $\chi(\tilde{G}) = 1$. The fuzzy graph $\tilde{G}$ as in (1) is called *trivial*.

(2) Since every pair of two vertices is strongly adjacent, we can construct a partition $S = \{\tilde{V}_1, \tilde{V}_2, ..., \tilde{V}_n\}$ on the fuzzy vertex set $\tilde{V}$ with the membership functions $\gamma_1, \gamma_2, ..., \gamma_n$, respectively, where

$$\gamma_i(x) = \begin{cases} \sigma(x_i), & x = x_i, \\ 0, & x \neq x_i. \end{cases}$$

The partition S satisfies the properties in Definition 3.1. This is the minimal partition since any partition with less than n members does not satisfy all of the conditions in Definition 3.1. Thus, $\chi(\tilde{G}) = n$. $\square$

In this paper, we investigate upper bounds for sum and product of chromatic number of several classes of fuzzy graphs and their complements. Further, we show that the bounds given in [9] do not hold for fuzzy graphs with certain properties. First, we investigate upper bounds for sum and product of chromatic number of fuzzy cycles and their complements.

3.1. The chromatic number of fuzzy cycles and their complements

Definition of a fuzzy cycle is presented below.

Definition 3.3 [5]**.** A fuzzy cycle with length *n*, denoted by $\tilde{C}_n$, is a fuzzy graph consisting a sequence of distinct vertices $(u_0, u_1, u_2, ..., u_n)$ such that $\mu(u_i, u_{i+1}) > 0$ for $1 \leq i \leq n$ and $\mu(u_j, u_k) = 0$ for $k \neq j + 1$, where $u_0 = u_n$.

In crisp graph case, chromatic number of a cycle with even length is 2 and chromatic number of a cycle with odd length is 3. In the case of fuzzy graph, chromatic number of a fuzzy cycle with even length is 2. However, chromatic number of a fuzzy cycle with odd length is 2 or 3, that is stated in Theorem 3.4.

Theorem 3.4. *Let* $\tilde{C}_n$ *be a fuzzy cycle with length n.*

(1) *The chromatic number of* $\tilde{C}_n$ *is* 2 *if n is even.*

(2) *The chromatic number of* $\tilde{C}_n$ *is* 2 *or* 3 *if n is odd.*

Proof. Let $\tilde{C}_n = \tilde{C}(V, \sigma, \mu)$.

(1) Let $V = \{v_1, v_2, ..., v_n\}$, where *n* is even.

Since $\tilde{C}$ is a fuzzy cycle, $\mu(u_i, u_{i+1}) > 0$, for $1 \leq i \leq n$, and $\mu(u_j, u_k) = 0$ for $k \neq j + 1$. Since $\frac{1}{2}\min\{\sigma(u_j), \sigma(u_k)\} \neq 0$, $\mu(u_j, u_k) < \frac{1}{2}\min\{\sigma(u_j), \sigma(u_k)\}$. This means that all of the pair of vertices (u_j, u_k) with $k \neq j + 1$ are weakly adjacent. So that we can construct a partition $S = \{\tilde{V}_1, \tilde{V}_2\}$

on the fuzzy vertex set $\tilde{V}$, where $\tilde{V}_1 = \{v_1, v_3, v_5, ..., v_{n-1}\}$ and $\tilde{V}_2 = \{v_2, v_4, v_6, ..., v_n\}$. Every pair of two vertices in $\tilde{V}_i$ is weakly adjacent. The membership functions of $\tilde{V}_1$ and $\tilde{V}_2$ are γ_1 and γ_2, respectively, where

$$\gamma_1(v_i) = \begin{cases} \sigma(v_i), & \text{if } i \text{ is odd} \\ 0, & \text{if } i \text{ is even} \end{cases} \quad \text{and} \quad \gamma_2(v_i) = \begin{cases} 0, & \text{if } i \text{ is odd} \\ \sigma(v_i), & \text{if } i \text{ is even.} \end{cases}$$

We can see that the partition S satisfies all of the conditions in Definition 3.1. This is the minimal partition since any partition with less than 2 subsets does not satisfy all of the conditions in Definition 3.1. Thus, $\chi(\tilde{C}_n) = 2$.

(2) Let $V = \{v_1, v_2, ..., v_n\}$, where n is odd. We consider two cases:

Case 1. There exists at least a pair of vertices (u_k, u_{k+1}) which is weakly adjacent. Without loss of generality, we assume that v_i and v_{i+1} are weakly adjacent. If i is even, then we can construct a partition $S = \{\tilde{V}_1, \tilde{V}_2\}$ on the fuzzy vertex set $\tilde{V}$, where $\tilde{V}_1 = \{v_1, v_3, v_5, ..., v_{n-2}, v_n\}$ and $\tilde{V}_2 = \{v_2, v_4, ..., v_i, v_{i+1}, ..., v_{n-3}, v_{n-1}\}$. While if i is odd, then we can construct a partition $S = \{\tilde{V}_1, \tilde{V}_2\}$ on the fuzzy vertex set $\tilde{V}$, where $\tilde{V}_1 = \{v_1, v_3, v_5, ..., v_i, v_{i+1}, ..., v_{n-2}, v_n\}$ and $\tilde{V}_2 = \{v_2, v_4, ..., v_{n-3}, v_{n-1}\}$. Every pair of two vertices in $\tilde{V}_i$ is weakly adjacent. The sets $\tilde{V}_1$ and $\tilde{V}_2$ have the membership functions γ_1 and γ_2, respectively, where

$$\gamma_1(v) = \begin{cases} \sigma(v), & \text{if } v \in V_1, \\ 0, & \text{if } v \in V_2, \end{cases}$$

$$\gamma_2(v) = \begin{cases} 0, & \text{if } v \in V_1, \\ \sigma(v), & \text{if } v \in V_2. \end{cases}$$

The partition S satisfies all of the conditions in Definition 3.1. This is the minimal partition since any partition with less than 2 subsets does not satisfy all of the conditions in Definition 3.1. Thus, $\chi(\tilde{C}_n) = 2$.

Case 2. Every pair of two vertices (u_i, u_{i+1}) is strongly adjacent for $i = 1, 2, ..., n$. Since the fuzzy cycle $\tilde{C}_n$ has an odd length n, we can construct a partition $S = \{\tilde{V}_1, \tilde{V}_2, \tilde{V}_3\}$, where $\tilde{V}_1 = \{v_1, v_3; v_5, ..., v_{n-2}\}$, $\tilde{V}_2 = \{v_2, v_4; v_6, ..., v_{n-1}\}$ and $\tilde{V}_3 = \{v_n\}$. Every pair of two vertices (u_j, u_k) with $k \neq j + 1$ is weakly adjacent in $\tilde{C}_n$. This means that every pair of two vertices in $\tilde{V}_i$ is weakly adjacent. The sets $\tilde{V}_1, \tilde{V}_2, \tilde{V}_3$ have the membership functions γ_1, γ_2 and γ_3, respectively, where

$$\gamma_k(v) = \begin{cases} \sigma(v), & \text{if } v \in V_k \\ 0, & \text{if } v \notin V_k \end{cases} \quad \text{for all } k = 1, 2, 3.$$

The partition S satisfies all of the conditions in Definition 3.1 and this is the minimal partition. Thus, $\chi(\tilde{C}_n) = 3$. □

Next, we give a theorem on upper bounds for chromatic number of fuzzy cycles and their complements.

Theorem 3.5. *Let $\tilde{C}_n$ be a fuzzy cycle with n vertices $(n \geq 4)$. For n is even, $\chi(\tilde{C}_n) + \chi(\overline{\tilde{C}_n}) \leq 2 + n$ and $\chi(\tilde{C}_n) \cdot \chi(\overline{\tilde{C}_n}) \leq 2n$. For n is odd, $\chi(\tilde{C}_n) + \chi(\overline{\tilde{C}_n}) \leq 3 + n$ and $\chi(\tilde{C}_n) \cdot \chi(\overline{\tilde{C}_n}) \leq 3n$.*

Proof. Let $\tilde{C}_n = \tilde{C}(V, \sigma, \mu)$ be a fuzzy cycle and $\overline{\tilde{C}_n}$ be its complement. Let $V = \{v_1, v_2, ..., v_n\}$. We consider two cases:

Case 1. $\mu(v_i, v_{i+1}) = \frac{1}{2}\min\{\sigma(v_i), \sigma(v_{i+1})\}$ for all $i = 1, 2, ..., n$.

Then $\overline{\mu}(v_i, v_{i+1}) = \frac{1}{2}\min\{\sigma(v_i), \sigma(v_{i+1})\}$ for all $i = 1, 2, ..., n$. In other words, the pair of vertices (v_i, v_{i+1}) is strongly adjacent in $\overline{\tilde{C}_n}$. On the other hand, every pair of two vertices (v_j, v_k) with $k \neq j + 1$ is weakly adjacent in $\tilde{C}_n$. Then these vertices are strongly adjacent in $\overline{\tilde{C}_n}$. Thus, every pair of two vertices is strongly adjacent in $\overline{\tilde{C}_n}$. By Lemma 3.2, $\chi(\overline{\tilde{C}_n}) = n$.

By Theorem 3.4, if n is even $(n \geq 4)$, then $\chi(\tilde{C}_n) + \chi(\overline{\tilde{C}_n}) \leq 2 + n$ and $\chi(\tilde{C}_n) \cdot \chi(\overline{\tilde{C}_n}) \leq 2n$. If n is odd $(n \geq 5)$, then $\chi(\tilde{C}_n) + \chi(\overline{\tilde{C}_n}) \leq 3 + n$ and $\chi(\tilde{C}_n) \cdot \chi(\overline{\tilde{C}_n}) \leq 3n$.

Case 2. $\tilde{C}_n$ has at least a pair of vertices (v_k, v_{k+1}) such that $\mu(v_k, v_{k+1}) \neq \frac{1}{2} \min\{\sigma(v_k), \sigma(v_{k+1})\}$. This means that there is at least a pair of vertices v_k and v_{k+1} which are weakly adjacent in $\overline{\tilde{C}_n}$. We can construct a partition $S = \{\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_k\}$, $k \leq n-1$ in $\overline{\tilde{C}_n}$. The membership functions of $\tilde{V}_i$, $i = 1, \ldots, k$ are $\gamma_1, \gamma_2, \ldots, \gamma_k$, respectively, where

$$\gamma_i(u) = \sigma(u), \text{ and } \gamma_i(v) = \sigma(v) \text{ if } u \text{ and } v \text{ are weakly adjacent in } \overline{\tilde{C}_n},$$

$$\gamma_k(u) = \sigma(u), \gamma_k(v) = 0, k \neq i, \text{ if } u \text{ and } v \text{ are strongly adjacent in } \overline{\tilde{C}_n}.$$

The partition S satisfies all of the conditions in Definition 3.1. This is the minimal partition, thus $\chi(\overline{\tilde{C}_n}) \leq n-1$. By Theorem 3.4, if n is even $(n \geq 4)$, then $\chi(\tilde{C}_n) + \chi(\overline{\tilde{C}_n}) \leq 2 + (n-1) < 2 + n$ and $\chi(\tilde{C}_n) \cdot \chi(\overline{\tilde{C}_n}) \leq 2(n-1) < 2n$. If n is odd $(n \geq 5)$, then $\chi(\tilde{C}_n) + \chi(\overline{\tilde{C}_n}) \leq 3 + (n-1) < 3 + n$ and $\chi(\tilde{C}_n) \cdot \chi(\overline{\tilde{C}_n}) \leq 3(n-1) < 3n$. $\square$

We give some remarks as follows:

(a) The upper bound for $\chi(\tilde{C}_n) + \chi(\overline{\tilde{C}_n})$ satisfies the upper bound (1). Since, if n is even and $n \geq 4$, then $\chi(\tilde{C}_n) + \chi(\overline{\tilde{C}_n}) \leq 2 + n \leq (n-2) + n = 2(n-1)$. Next, if n is odd and $n \geq 5$, then $\chi(\tilde{C}_n) + \chi(\overline{\tilde{C}_n}) \leq 3 + n \leq (n-2) + n = 2(n-1)$. However, the upper bound for $\chi(\tilde{C}_n) \cdot \chi(\overline{\tilde{C}_n})$ is smaller than the upper bound (2). Since, if n is even and $n \geq 4$, then $\chi(\tilde{C}_n) \cdot \chi(\overline{\tilde{C}_n}) \leq 2n \leq (n-2)n < (n-1)^2 < n^2 + 1$. Next, if n is odd and $n \geq 5$, then $\chi(\tilde{C}_n) \cdot \chi(\overline{\tilde{C}_n}) \leq 3n \leq (n-2)n < (n-1)^2 < n^2 + 1$.

(b) While upper bound for sum of chromatic number of a fuzzy cycle $\tilde{C}_3$ and its complement do not lie within the upper bound (1).

In Example 3.6, we give a fuzzy cycle $\tilde{C}_3$, where its bound does not lie within the bound (1).

Example 3.6. Let $\tilde{C}_3 = \tilde{C}(V, \sigma, \mu)$ be a fuzzy cycle given in Figure 1. There is exactly one pair of vertices (B, C) such that

$$\mu(B, C) \neq \frac{1}{2}\min\{\sigma(B), \sigma(C)\}.$$

While $\mu(A, B) = \frac{1}{2}\min\{\sigma(A), \sigma(B)\}$ and $\mu(A, C) = \frac{1}{2}\min\{\sigma(A), \sigma(C)\}$.

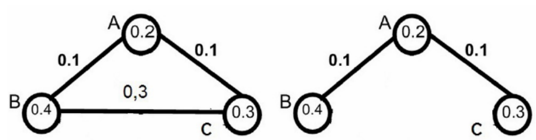

Figure 1. A fuzzy cycle $\tilde{C}_3$ and its complement.

Let $S = \{\tilde{V}_1, \tilde{V}_2, \tilde{V}_3\}$ be a partition of the fuzzy vertex set $\tilde{V}$. The membership functions of $\tilde{V}_1, \tilde{V}_2, \tilde{V}_3$ are γ_1, γ_2 and γ_3, respectively which are defined as in Table 1. The partition S satisfies all of the conditions in Definition 3.1. Thus, $\chi(\tilde{C}_3) = 3$.

Table 1. The membership functions $\gamma_1, \gamma_2, \gamma_3$ of the fuzzy subsets in $\tilde{C}_3$ (left) and the membership functions γ_1, γ_2 of the fuzzy subsets in $\overline{\tilde{C}_3}$ (right)

Vertices	γ_1	γ_2	γ_3	Max
A	0.2	0	0	0.2
B	0	0.4	0	0.4
C	0	0	0.3	0.3

Vertices	γ_1	γ_2	Max
A	0.2	0	0.2
B	0	0.4	0.4
C	0	0.3	0.3

On the other hand, we can construct a partition $S' = \{\tilde{V}_1, \tilde{V}_2\}$ in $\overline{\tilde{C}}_3$. The membership functions of $\tilde{V}_1$ and $\tilde{V}_2$ are γ_1 and γ_2, respectively, which are defined as in Table 1. The chromatic number $\chi(\overline{\tilde{C}}_3) = 2$. Thus, $\chi(\tilde{C}_3) + \chi(\overline{\tilde{C}}_3)$ $= 5 > 2(n-1)$ and $\chi(\tilde{C}_3) \cdot \chi(\overline{\tilde{C}}_3) = 6 < n^2 + 1$. We can see that the upper bound for sum of chromatic number of fuzzy cycle $\tilde{C}_3$ and its complement does not lie within the upper bound (1).

Then we investigate upper bounds for sum and product of chromatic number of fuzzy wheels and their complements.

3.2. The chromatic number of fuzzy wheels and their complements

In order to define a fuzzy wheel, a definition of union and joint of two fuzzy graphs are presented.

Definition 3.7 [10]**.** Let $\tilde{G}_1(\tilde{V}_1, \tilde{E}_1)$ and $\tilde{G}_2(\tilde{V}_2, \tilde{E}_2)$ be two fuzzy graphs where the membership functions of $\tilde{V}_1, \tilde{E}_1, \tilde{V}_2, \tilde{E}_2$ are $\sigma_1, \mu_1, \sigma_2, \mu_2$ respectively.

Assume that $\tilde{V}_1 \cap \tilde{V}_2 = \varnothing$. The union of two fuzzy graphs $\tilde{G}_1$ and $\tilde{G}_2$ is a fuzzy graph $\tilde{G}_1 \cup \tilde{G}_2 = \tilde{G}(\tilde{V}_1 \cup \tilde{V}_2, \tilde{E}_1 \cup \tilde{E}_2)$, where the membership function of $\tilde{V}_1 \cup \tilde{V}_2$ is $\sigma_{\tilde{V}_1 \cup \tilde{V}_2}$ which is defined by

$$\sigma_{\tilde{V}_1 \cup \tilde{V}_2}(u) = \begin{cases} \sigma_1(u), & \text{if } u \in \tilde{V}_1 - \tilde{V}_2; \\ \sigma_2(u), & \text{if } u \in \tilde{V}_2 - \tilde{V}_1 \end{cases}$$

and the membership function of $\tilde{E}_1 \cup \tilde{E}_2$ is $\mu_{\tilde{E}_1 \cup \tilde{E}_2}$ which is defined by

$$\mu_{\tilde{E}_1 \cup \tilde{E}_2}(u, v) = \begin{cases} \mu_1(u, v), & \text{if } (u, v) \in \tilde{E}_1 - \tilde{E}_2; \\ \mu_2(u, v), & \text{if } (u, v) \in \tilde{E}_2 - \tilde{E}_1. \end{cases}$$

The joint of two fuzzy graphs $\widetilde{G}_1$ and $\widetilde{G}_2$ is a fuzzy graph $\widetilde{G}_1 + \widetilde{G}_2 = \widetilde{G}(\widetilde{V}_1 \cup \widetilde{V}_2, \widetilde{E}_1 \cup \widetilde{E}_2 \cup \widetilde{E}')$, where $\widetilde{E}'$ is the set of all edges joining the vertices of $\widetilde{V}_1$ and $\widetilde{V}_2$. The membership function of fuzzy vertex set $\widetilde{V}_1 \cup \widetilde{V}_2$ is $\sigma_{\widetilde{V}_1 \cup \widetilde{V}_2}$.

While the membership function of fuzzy edge set $\widetilde{E}_1 \cup \widetilde{E}_2 \cup \widetilde{E}'$ is $\mu_{\widetilde{E}_1 \cup \widetilde{E}_2 \cup \widetilde{E}'}$ which is defined by

$$\mu_{\widetilde{E}_1 \cup \widetilde{E}_2 \cup \widetilde{E}'}(u, v) = \begin{cases} \mu_{\widetilde{E}_1 \cup \widetilde{E}_2}(u, v), & \text{if } (u, v) \in \widetilde{E}_1 \cup \widetilde{E}_2, \\ \min\{\sigma_1(u), \sigma_2(u)\}, & \text{if } (u, v) \in \widetilde{E}'. \end{cases}$$

After that, we give the concept of a fuzzy wheel.

Definition 3.8. A fuzzy wheel $\widetilde{W}_n$ is a fuzzy graph with n vertices $(n \geq 4)$, formed by connecting a single vertex x to all vertices in a fuzzy cycle with $n - 1$ vertices. In other words, a fuzzy wheel $\widetilde{W}_n$ $(n \geq 4)$ can also be defined as joint $\widetilde{G}_1 + \widetilde{C}_{n-1}$, where $\widetilde{G}_1$ consists of a single vertex.

In order to find the chromatic number of a fuzzy wheel and its complement, the chromatic number of union of two fuzzy graphs and the chromatic number of joint of two fuzzy graphs are presented.

Theorem 3.9. *The union of two fuzzy graphs* $\widetilde{G}_1 \cup \widetilde{G}_2$ *has the chromatic number* $\chi(\widetilde{G}_1 \cup \widetilde{G}_2) = \max\{\chi(\widetilde{G}_1), \chi(\widetilde{G}_2)\}$.

Proof. Based on definition, $\widetilde{G}_1(\widetilde{V}_1, \widetilde{E}_1)$ and $\widetilde{G}_2(\widetilde{V}_2, \widetilde{E}_2)$ are fuzzy subgraphs of $\widetilde{G}_1 \cup \widetilde{G}_2$, thus $\chi(\widetilde{G}_1 \cup \widetilde{G}_2) \geq \max\{\chi(\widetilde{G}_1), \chi(\widetilde{G}_2)\}$. We will prove the upper bound: $\chi(\widetilde{G}_1 \cup \widetilde{G}_2) \leq \max\{\chi(\widetilde{G}_1), \chi(\widetilde{G}_2)\}$. Let $P_1 = \{\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_{k_1}\}$ be a partition of $\widetilde{V}_1$ which gives $\chi(\widetilde{G}_1) = k_1$. The membership functions of $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_{k_1}$ are $\alpha_1, \alpha_2, ..., \alpha_{k_1}$, respectively. Let $P_2 =$

$\{\tilde{Y}_1, \tilde{Y}_2, ..., \tilde{Y}_{k_2}\}$ be a partition of $\tilde{V}_2$ which gives $\chi(\tilde{G}_2) = k_2$. The membership functions of $\tilde{Y}_1, \tilde{Y}_2, ..., \tilde{Y}_{k_2}$ are $\beta_1, \beta_2, ..., \beta_{k_2}$, respectively. Let $k = \max\{k_1, k_2\}$. We can construct a partition $P = \{\tilde{W}_1, \tilde{W}_2, ..., \tilde{W}_k\}$ of $\tilde{V}_1 \cup \tilde{V}_2$, where the membership functions of $\tilde{W}_1, \tilde{W}_2, ..., \tilde{W}_k$ are $\gamma_1, \gamma_2, ..., \gamma_k$.

If $k = k_1$, then the membership functions $\gamma_1, \gamma_2, ..., \gamma_k$ are defined by $\gamma_i(v) = \alpha_i(v)\ (i = 1, ..., k)$ for all $v \in \tilde{G}_1$ and $\gamma_l(v) = \beta_l(v)\ (l = 1, ..., k_2)$ for all $v \in \tilde{G}_2$. Otherwise, if $k = k_2$, then the membership functions $\gamma_1, \gamma_2, ..., \gamma_k$ are defined by $\gamma_i(v) = \beta_i(v)\ (i = 1, ..., k)$ for all $v \in \tilde{G}_2$ and $\gamma_l(v) = \alpha_l(v)\ (l = 1, ..., k_1)$ for all $v \in \tilde{G}_1$. The partition P satisfies the properties in Definition 3.1 and this is the minimal partition. Thus, $\chi(\tilde{G}_1 \cup \tilde{G}_2) \leq \max\{\chi(\tilde{G}_1), \chi(\tilde{G}_2)\}$ which completes the proof. □

Theorem 3.10. *The joint of two fuzzy graphs* $\tilde{G}_1 + \tilde{G}_2$ *has the chromatic number* $\chi(\tilde{G}_1 + \tilde{G}_2) = \chi(\tilde{G}_1) + \chi(\tilde{G}_2)$.

Proof. Let $\tilde{G}_1(\tilde{V}_1, \tilde{E}_1)$ and $\tilde{G}_2(\tilde{V}_2, \tilde{E}_2)$ be two fuzzy graphs. The joint $\tilde{G}_1 + \tilde{G}_2$ has $\tilde{G}_1$ and $\tilde{G}_2$ as fuzzy subgraphs. Since there are edges joining all of vertices in $\tilde{V}_1$ and $\tilde{V}_2$, $\chi(\tilde{G}_1 + \tilde{G}_2) \geq \chi(\tilde{G}_1) + \chi(\tilde{G}_2)$. We will prove the upper bound: $\chi(\tilde{G}_1 + \tilde{G}_2) \leq \chi(\tilde{G}_1) + \chi(\tilde{G}_2)$. Let $P_1 = \{\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_{k_1}\}$ be a partition of $\tilde{V}_1$ which gives $\chi(\tilde{G}_1) = k_1$. The membership functions of $\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_{k_1}$ are $\alpha_1, \alpha_2, ..., \alpha_{k_1}$, respectively. Let $P_2 = \{\tilde{Y}_1, \tilde{Y}_2, ..., \tilde{Y}_{k_2}\}$ be a partition of $\tilde{V}_2$ which gives $\chi(\tilde{G}_2) = k_2$. The membership functions of $\tilde{Y}_1, \tilde{Y}_2, ..., \tilde{Y}_{k_2}$ are $\beta_1, \beta_2, ..., \beta_{k_2}$, respectively. We can construct a partition $P = \{\tilde{W}_1, \tilde{W}_2, ..., \tilde{W}_{k_1}, \tilde{W}_{k_1+1}, \tilde{W}_{k_1+2}, ..., \tilde{W}_{k_1+k_2}\}$ of $\tilde{V}_1 \cup \tilde{V}_2$. The membership functions of $\tilde{W}_1, \tilde{W}_2, ..., \tilde{W}_{k_1}, \tilde{W}_{k_1+1}, \tilde{W}_{k_1+2}, ..., \tilde{W}_{k_1+k_2}$ are $\gamma_1, \gamma_2, ..., \gamma_{k_1}, \gamma_{k_1+1}, \gamma_{k_1+2}, ..., \gamma_{k_1+k_2}$, respectively, where: $\gamma_i(v) = \alpha_i(v)$ for $i = 1, ..., k_1$ and $\gamma_{k_1+1}(v) = \beta_1(v)$, $\gamma_{k_1+2}(v) = \beta_2(v), ..., \gamma_{k_1+k_2}(v) = \beta_{k_2}(v)$.

The partition P satisfies the properties in Definition 3.1 and this is the minimal partition. Thus, $\chi(\tilde{G}_1 + \tilde{G}_2) \leq \chi(\tilde{G}_1) + \chi(\tilde{G}_2)$ which completes the proof. □

In crisp graph case, the chromatic number of odd wheel is 3 and the chromatic number of even wheel is 4. In the case of fuzzy graph, the chromatic number of even fuzzy wheel is 3 or 4.

Lemma 3.11. *Let* $\tilde{W}_n$ *be a fuzzy wheel. The chromatic number of* $\tilde{W}_n$ *is* 3 *or* 4.

Proof. $\chi(\tilde{W}_n) = \chi(\tilde{G}_1 + \tilde{C}_{n-1})$. If n is odd, then the fuzzy cycle $\tilde{C}_{n-1}$ has even number of vertices. Thus, $\chi(\tilde{W}_n) = \chi(\tilde{G}_1) + \chi(\tilde{C}_{n-1}) = 1 + 2 = 3$.

If n is even, then the fuzzy cycle $\tilde{C}_{n-1}$ has odd number of vertices. By Theorem 3.4, the chromatic number $\chi(\tilde{W}_n) = \chi(\tilde{G}_1) + \chi(\tilde{C}_{n-1})$ is equal to 3 or 4, since the chromatic number of a fuzzy cycle with odd length is 2 or 3.

Based on Theorem 2.6, upper bounds for sum and product of chromatic number of fuzzy wheels and their complements are given in Theorem 3.12.

Theorem 3.12. *If* $\tilde{W}_n$ *is a fuzzy wheel with n vertices* $(n \geq 5)$, *then*

$$\chi(\tilde{W}_n) + \chi(\overline{\tilde{W}_n}) \leq 3 + n \quad \textit{and} \quad \chi(\tilde{W}_n) \cdot \chi(\overline{\tilde{W}_n}) \leq 4(n-1).$$

Proof. Let $\tilde{W}_n$ be a fuzzy wheel with n vertices and $\overline{\tilde{W}_n}$ be its complement. The complement $\overline{\tilde{W}_n} = \overline{\tilde{G}_1 + \tilde{C}_{n-1}}$, where $\tilde{G}_1$ has a single vertex. By Theorem 2.6, the complement $\overline{\widetilde{W}_n} = \overline{\widetilde{G}_1} \cup \overline{\widetilde{C}_{n-1}}$. The chromatic number $\chi(\overline{\widetilde{W}_n}) = \max\{\chi(\overline{\widetilde{G}_1}), \chi(\overline{\widetilde{C}_{n-1}})\}$. By Theorem 3.5, $\chi(\overline{\widetilde{C}_{n-1}}) \leq n-1$. By Lemma 3.11, $\chi(\widetilde{W}_n) + \chi(\overline{\widetilde{W}_n}) \leq 4 + (n-1) = 3 + n$ and $\chi(\widetilde{W}_n) \cdot \chi(\overline{\widetilde{W}_n}) \leq 4(n-1)$. □

We give some remarks as follows:

(a) The bounds given in Theorem 3.12 lie within the bounds (1) and (2) for $n \geq 5$. Since, if $n \geq 5$, then $\chi(\widetilde{W}_n) + \chi(\overline{\widetilde{W}_n}) \leq 3 + n \leq (n-2) + n = 2(n-1)$ and $\chi(\widetilde{W}_n) \cdot \chi(\overline{\widetilde{W}_n}) \leq 4(n-1) \leq (n-1) \cdot (n-1) = (n-1)^2 < n^2 + 1$.

(b) Upper bounds for sum and product of complementary fuzzy wheels always lie within the upper bounds (1) and (2), because all of fuzzy wheels $\widetilde{W}_n$ do not have the following properties:

(i) there is exactly one pair of vertices $u, v \in V$ such that $\mu(u, v) \neq \frac{1}{2}\min\{\sigma(u), \sigma(v)\}$,

(ii) $\mu(x, y) = \frac{1}{2}\min\{\sigma(x), \sigma(y)\}$ for all $x, y \in V - \{u, v\}$.

(3) A fuzzy wheel $\widetilde{W}_4$ also does not have the properties (i) and (ii) above. However, a fuzzy wheel $\widetilde{W}_4$ contains a fuzzy subgraph $\widetilde{C}_3$ which does not satisfy the upper bound (1). Therefore, upper bound for sum of chromatic number of a fuzzy wheel $\widetilde{W}_4$ and its complement do not satisfy upper bound (1).

3.3. The chromatic number of fuzzy graphs and their complements

First, we give a counterexample for upper bounds (1) and (2) in Section 2.

Example 3.13. Let $\widetilde{G}(V, \sigma, \mu)$ be a fuzzy graph given in Figure 2.

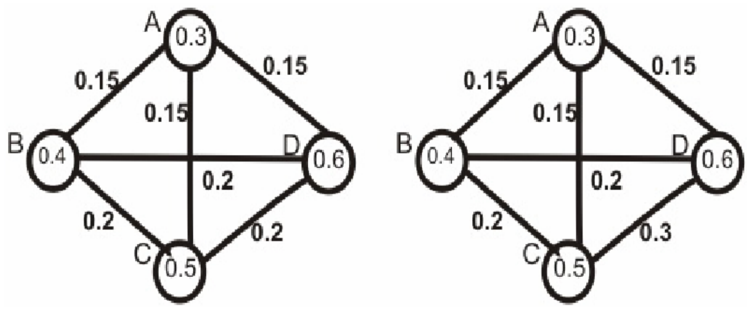

Figure 2. A fuzzy graph $\tilde{G}$ and its complement $\overline{\tilde{G}}$.

We can see that the fuzzy graph $\tilde{G}$ has the following properties:

(1) there is exactly one pair of vertices $C, D \in \tilde{G}$ such that $\mu(C, D) \neq \frac{1}{2}\min\{\sigma(C), \sigma(D)\}$ (the vertices C and D are weakly adjacent).

(2) $\mu(x, y) = \frac{1}{2}\min\{\sigma(x), \sigma(y)\}$ for all $x, y \in V - \{C, D\}$.

Let $S = \{\tilde{V}_1, \tilde{V}_2, \tilde{V}_3\}$ be a partition of $\tilde{V}$. The membership functions of $\tilde{V}_1, \tilde{V}_2, \tilde{V}_3$ are given in Table 2. The partition S satisfies all of the conditions in Definition 3.1.

Table 2. The membership functions $\gamma_1, \gamma_2, \gamma_3$ of the fuzzy subsets in $\tilde{G}$ (left) and the membership functions $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ of the fuzzy subsets in $\overline{\tilde{G}}$ (right)

Vertices	γ_1	γ_2	γ_3	Max
A	0.3	0	0	0.3
B	0	0.4	0	0.4
C	0	0	0.5	0.5
D	0	0	0.6	0.6

Vertices	γ_1	γ_2	γ_3	γ_4	Max
A	0.3	0	0	0	0.3
B	0	0.4	0	0	0.4
C	0	0	0.5	0	0.5
D	0	0	0	0.6	0.6

On the other hand, all of pairs of two vertices in $\overline{\tilde{G}}$ are strongly adjacent. Thus, we can construct a partition $S = \{\tilde{V}_1, \tilde{V}_2, \tilde{V}_3, \tilde{V}_4\}$ in $\overline{\tilde{G}}$. The membership functions of $\tilde{V}_1, \tilde{V}_2, \tilde{V}_3, \tilde{V}_4$ are given in Table 2. The chromatic number of $\tilde{G}$ is $\chi(\tilde{G}) = 3$ and the chromatic number of its complement is $\chi(\overline{\tilde{G}}) = 4$. Thus, $\chi(\tilde{G}) + \chi(\overline{\tilde{G}}) = 7 > 6 = 2(n-1)$ and $\chi(\tilde{G}) \cdot \chi(\overline{\tilde{G}}) = 12 < n^2 + 1$. These upper bounds do not lie within the upper bound (1).

We note that in order to demonstrate that the upper bound (2) can be improved, it is necessary to assume that $\tilde{G}$ does not have the following properties:

(i) there is exactly one pair of vertices $u, v \in V$ such that $\mu(u, v) \neq \frac{1}{2}\min\{\sigma(u), \sigma(v)\}$.

(ii) $\mu(x, y) = \frac{1}{2}\min\{\sigma(x), \sigma(y)\}$ for all $x, y \in V - \{u, v\}$.

Further, we can prove that the upper bound (2) can be improved. We have a new upper bound for the chromatic number of product of complementary fuzzy graphs which is smaller than the upper bound (2).

Theorem 3.14. *Let* $\tilde{G}(V\sigma, \mu)$ *be a fuzzy graph with n vertices and* $\overline{\tilde{G}} = (\overline{\sigma}, \overline{\mu})$ *be a complement of* $\tilde{G}$*. If* $\tilde{G}$ *does not have the following properties*:

(i) *there is exactly one pair of vertices* $u, v \in V$ *such that* $\mu(u, v) \neq \frac{1}{2}\min\{\sigma(u), \sigma(v)\}$,

(ii) $\mu(x, y) = \frac{1}{2}\min\{\sigma(x), \sigma(y)\}$ *for all* $x, y \in V - \{u, v\}$,

then $\chi(\tilde{G}) + \chi(\overline{\tilde{G}}) \leq 2(n-1)$ *and* $\chi(\tilde{G}) \cdot \chi(\overline{\tilde{G}}) \leq (n-1)^2$.

Proof. Let $\widetilde{G}(V\sigma, \mu)$ be a fuzzy graph with n vertices and $\overline{\widetilde{G}} = (\overline{\sigma}, \overline{\mu})$ be a complement of $\widetilde{G}$. Let $V = \{v_1, v_2, v_3, ..., v_n\}$.

Assume that $\widetilde{G}$ does not have the properties (i) and (ii). It means that $\widetilde{G}$ has at least two pairs of vertices, namely (u, v) and (y, z) such that $\mu(u, v) \neq \frac{1}{2}\min\{\sigma(u), \sigma(v)\}$ and $\mu(y, z) \neq \frac{1}{2}\min\{\sigma(y), \sigma(z)\}$.

We consider two cases:

Case 1. $\widetilde{G}$ has at least two pairs of vertices which are weakly adjacent.

Without loss of generality, we assume that there are two pairs of vertices (v_i, v_{i+1}) and (v_{i+2}, v_{i+3}) which are weakly adjacent in $\widetilde{G}$. We can construct a partition $S = \{\widetilde{V}_1, \widetilde{V}_2, ..., \widetilde{V}_k\}(k \leq n-2)$ of $\widetilde{V}$, where the membership functions of $\widetilde{V}_1, \widetilde{V}_2, ..., \widetilde{V}_k$ are γ_l $(l = 1, 2, ..., k)$.

For $l = i$, the membership function γ_l is defined by

$\gamma_l(v) = \sigma(v)$ if $v = v_i$ and $v = v_{i+1}$, while $\gamma_l(v) = 0$ for all $v \in V - \{v_i, v_{i+1}\}$.

For $l = i+1$, the membership function γ_l is defined by

$\gamma_l(v) = \sigma(v)$, if $v = v_{i+2}$ and $v = v_{i+3}$, while $\gamma_l(v) = 0$ for all $v \in V - \{v_{i+2}, v_{i+3}\}$.

For $1 \leq l < i$, the membership function γ_l is defined by

$$\gamma_l(v) = \sigma(v) \text{ for } v = v_l \text{ and } \gamma_l(v) = 0 \text{ for } v \neq v_l.$$

For $i + 2 \leq l \leq k \leq n - 2$, the membership function γ_l is defined by

$$\gamma_l(v) = \sigma(v) \text{ for } v = v_{l+2} \text{ and } \gamma_l(v) = 0 \text{ for } v \neq v_{l+2}.$$

The partition S satisfies the properties in Definition 3.1 and this is the minimal partition. Thus, $\chi(\widetilde{G}) \leq n-2$.

While in $\tilde{\overline{G}}$, every pair of two vertices can be strongly adjacent. We can construct a partition $S = \{\tilde{V}_1, \tilde{V}_2, ..., \tilde{V}_k\}(k \leq n)$ of $\tilde{V}$, where the membership functions of $\tilde{V}_l$ $(l = 1, 2, ..., k)$ are defined by:

$$\gamma_l(v) = \sigma(v) \text{ for } v = v_l \text{ and } \gamma_l(v) = 0 \text{ for } v \neq v_l.$$

The partition S satisfies the properties in Definition 3.1 and this is the minimal partition. The chromatic number $\chi(\tilde{\overline{G}}) \leq n.$ Thus, $\chi(\tilde{G}) + \chi(\tilde{\overline{G}}) \leq$ $(n-2) + n = 2(n-1)$ and $\chi(\tilde{G}) \cdot \chi(\tilde{\overline{G}}) \leq (n-2)n < (n-1)^2.$

Case 2. $\tilde{G}$ has at least a pair of vertices which is weakly adjacent and a pair of vertices which is strongly adjacent.

Without loss of generality, we assume that v_i and v_{i+1} are weakly adjacent in $\tilde{G}$. We can construct a partition $S = \{\tilde{V}_1, \tilde{V}_2, ..., \tilde{V}_k\}$ $(k \leq n-1)$ of $\tilde{V}$, where the membership functions of $\tilde{V}_1, \tilde{V}_2, ..., \tilde{V}_k$ are γ_l $(l = 1, 2, ..., k)$.

For $l = i$, the membership function γ_l is defined by

$$\gamma_l(v) = \sigma(v) \text{ if } v = v_i \text{ and } v = v_{i+1}, \text{ while } \gamma_l(v) = 0 \text{ for all } v \in V - \{v_i, v_{i+1}\}.$$

While for $1 \leq l < i$, the membership function γ_l is defined by

$$\gamma_l(v) = \sigma(v) \text{ for } v = v_l \text{ and } \gamma_l(v) = 0 \text{ for } v \neq v_l.$$

For $i + 1 \leq l \leq n-1$, the membership function γ_l is defined by

$$\gamma_l(v) = \sigma(v) \text{ for } v = v_{l+1} \text{ and } \gamma_l(v) = 0 \text{ for } v \neq v_{l+1}.$$

The partition S satisfies the properties in Definition 3.1 and this is the minimal partition. Thus, $\chi(\tilde{G}) \leq n-1.$ Furthermore, since $(\tilde{G})$ also has at least a pair of vertices which is strongly adjacent, there is at least a pair of vertices which is weakly adjacent in $\overline{\tilde{G}}$. By the same way, we have $\chi(\overline{\tilde{G}}) \leq n - 1$. Thus, $\chi(\tilde{G}) + \chi(\overline{\tilde{G}}) \leq 2(n-1)$ and $\chi(\tilde{G}) \cdot \chi(\overline{\tilde{G}}) \leq (n-1)^2$. $\square$

4. Conclusion

In this paper, we investigate upper bounds for sum and product of chromatic number of several classes of fuzzy graphs and their complements. We show that there are certain fuzzy graphs which do not satisfy the bounds given in [9], they are fuzzy graphs with the following properties:

1. there is exactly one pair of vertices $u, v \in V$ such that $\mu(u, v) \neq \frac{1}{2}\min\{\sigma(u), \sigma(v)\}$,

2. $\mu(x, y) \neq \frac{1}{2}\min\{\sigma(x), \sigma(y)\}$ for all $x, y \in V - \{u, v\}$.

Finally, we can improve the upper bounds given in [9]. By adding a necessary condition that a fuzzy graph $\tilde{G}$ does not have the properties (i) and (ii), we have $\chi(\tilde{G}) + \chi(\overline{\tilde{G}}) \leq 2(n-1)$ and $\chi(\tilde{G}) \cdot \chi(\overline{\tilde{G}}) \leq (n-1)^2$. We can see that the upper bound for the chromatic number of product of complementary fuzzy graphs is smaller than the upper bound given in [9].

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