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ON THE AUTOMORPHISMS OF FINITE *p*-GROUPS

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Abstract

Since the 50's, there exists the conjecture:

"If G is a finite non-cyclic group of order p^n, p a prime number and n an integer greater than 2, then the order $|G|$ of G divides the order $|AG|$ of the group $A(G)$ of its automorphism".

In this paper, we prove that the conjecture is also true if G has any of the following conditions:

(i) The center Z of G is elementary abelian.

(ii) The center Z of G is cyclic or

(iii) G has class c and $c = 3$.

Introduction

Since the 50's, there exists the conjecture:

"If G is a finite non-cyclic group of order p^n, p a prime number and

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n an integer greater than 2, then the order $|G|$ of G divides the order $|A(G)|$ of the group $A(G)$ of its automorphism".

Groups which satisfy this conjecture are called *LA-groups*.

Many papers have been appeared upon this topic, but the conjecture remained open until now. For example:

Schenkman in 1955 proved that a finite non-abelian group of class two is an LA-group [26]. In this paper, some lemmas were incorrect. In 1968, Faudree proved by another way those lemmas and he proved that a finite non-abelian group of class two is an LA-group [15].

Ree in 1956 proved that any finite non-abelian group of order p^n and exponent p is an LA-group [25].

Otto in 1966 proved that any finite abelian group of order p^n is an LA-group [24].

Davitt and Otto in 1971 [5] proved that a finite p-group with the central quotient metacyclic is an LA-group. They also proved in 1972 that a finite modular p-group is an LA-group [6].

Davitt in 1970 proved that a metacyclic p-group of order p^n is an LA-group [4]. He also in 1972 proved that a non-abelian p-abelian finite group of order p^n is an LA-group [7]. He also in 1980 proved that if $\left|\frac{G}{Z}\right| \leq p^4$, then G is an LA-group [8].

Otto proved in [24, Theorem 1] that if G is a direct product $G = H \times K$, where H is abelian of order p^r and K is a PN-group, then $|A(G)| \geq p^r |A(K)|$.

The result of this type not only extends the number of groups to which the conjecture is known to be true, but also and perhaps more importantly, shows that the truth of the overall conjecture depends only on being able to prove it for a smaller class of groups. Otto's result shows that it is sufficient to consider p-groups with no non-trivial abelians direct factors.

Also, Hummel proved in [21] that if the *p*-group G is a central product $H \cdot K$, where H is abelian and non-trivial and $|K|$ divides $|A(K)|$, then $|G|$ divides $|A(G)|$. Hummel's result shows that it is sufficient to consider *p*-groups which are not central product of no non-trivial groups $H \cdot K$, where H is abelian and $|K|$ divides $|A(K)|$. It may be noted here that if $Z \not\leq \Phi(G)$, where Z is the center of G, then there exists a maximal subgroup M of G such that $Z \not\leq M$. Then $G = Z \cdot M$ and G is a central product of Z and M, where Z is abelian. If $Z = \Phi(G)$, then G is of class two. Therefore, if G is of class $c > 2$, then Z *is a proper subgroup of* $\Phi(G)$. Therefore, in trying to prove the conjecture, it is sufficient to prove it for *p*-groups for which $Z < \Phi(G)$.

Hence the truth of the overall conjecture depends only on being able to prove it for a class of groups which satisfy all the following conditions (A):

Conditions (A). (i) G has class $c > 2$.

(ii) G has more than two generators and so $t \geq 3$, where t is the number of invariants of $\frac{G}{L_1}$.

(iii) G is a PN-group.

(iv) G is not a central product of $H \cdot K$, where H is abelian and $|K|$ divides $|A(K)|$.

(v) Z is a proper subgroup of $\Phi(G)$.

Notations

Throughout this paper, G will be a PN-group which satisfies the conditions (A). Also, we shall use the following notations:

G is a finite non-abelian group of order p^n, p a prime number, $G' = [G, G]$ is the commutator subgroup of G, $Z = Z(G)$ is the center of G and $\Phi(G)$ is the Frattini subgroup of G.

We denote the lower and the upper central series of G by:

$$G = L_0 \geq L_1 \geq L_2 \geq \cdots \geq L_{c-1} > L_c = 1,$$

where c is the class of G and $L_1 = G' = [G, G]$ is the commutator subgroup of G,

$$G = Z_c \geq Z_{c-1} \geq Z_{c-2} \geq \cdots \geq Z_1 > Z_0 = 1,$$

where $Z_1 = Z$ is the center of G.

We also denote by $m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$ and $k_1 \geq k_2 \geq \cdots \geq k_s \geq 1$ the invariants of G/L_1 and Z, respectively, where t and s are the numbers of invariants of G/L_1 and Z, respectively. If $|G/L_1| = p^m$ and $|Z| = p^k$, then $m = m_1 + m_2 + \cdots + m_t$ and $k = k_1 + k_2 + \cdots + k_s$.

$A(G)$, $I(G)$, $A_c(G)$ are the groups of automorphisms, inner automorphisms and central automorphisms of G, respectively.

$Hom(G, Z)$ are the set of all homomorphisms of G into Z.

A p-group G is called *p-abelian* if $(ab)^p = a^p b^p$ for every two elements a and b of G.

The p-group G is called *metacyclic* if it has a normal subgroup H such that both H and G/H are cyclic.

We say that the p-group G has *exponent p* if $a^p = 1$ for every $a \in G$.

The p-group G is called *PN-group* if it has no non-trivial abelian direct factor.

K is a two-maximal subgroup of G, if K is a maximal subgroup of a maximal subgroup of G.

$$P(G) = P_1(G) = \{x^p \mid x \in G\} \quad \text{and} \quad E(G) = E_1(G) = \{x \in G \mid x^p = 1\}.$$

Now we will state some known results and also we shall prove some new results, which are needed and used to prove the main result of this paper.

Lemma 1. *Let G be a non-abelian group of order p^n and class c, where p is a prime number and n is a positive integer, $n > 2$. Let $G = L_0 \geq L_1 \geq L_2 \geq \cdots \geq L_{c-1} > L_c = 1$ and $G = Z_c \geq Z_{c-1} \geq Z_{c-2} \geq \cdots \geq Z_1 > Z_0 = 1$ be the lower and the upper central series of G, where $L_1 = G' = [G, G]$ is the commutator subgroup of G and $Z_1 = Z$ is the center of G. Let also $\left|\frac{G}{L_1}\right| = p^m$ and $|Z| = p^k$, $m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$ and $k_1 \geq k_2 \geq \cdots \geq k_s \geq 1$ be the invariants of $\frac{G}{L_1}$ and Z, respectively. Then $m = m_1 + m_2 + \cdots + m_t$ and $k = k_1 + k_2 + \cdots + k_s$.*

Then we have:

(a) $p^{m_2} \geq \exp \frac{L_i}{L_{i+1}} \geq \exp \frac{L_{i+1}}{L_{i+2}}$ *for all* $i = 1, 2, \ldots, c-1$. *For* $t = 2$, $\frac{L_1}{L_2}$ *is cyclic of order at most* p^{m_2}.

(b) $L_i \leq Z_{c-i}$, $L_i \nleq Z_{c-i-1}$ *and* $[L_i, L_i] \leq L_{2i}$ *for all* $i = 0, 1, 2, \ldots, c-1$.

(c) $\exp L_{i+1} \leq \left|\frac{L_i}{L_i \cap Z}\right|$.

(d) *If* $\frac{L_i}{L_{i+1}}$ *is cyclic of fixed order* p^r *for all* $i = 1, 2, \ldots, c-1$, *then* $L_i \cap Z_{c-i-1} = L_{i+1}$, $Z_{c-i-1} < L_i \leq Z_{c-i}$, $\exp \frac{G}{L_1} \leq p^r$ *and* $\left|\frac{G}{Z_2}\right| = p^{2r}$.

(e) $\exp \frac{Z_{i+1}}{Z_i} = \exp Z = p^{k_1}$ *for all* $i = 0, 1, \ldots, c-1$.

(f) *In the invariants of* $\frac{G}{L_1}$ *and* Z, *we get that* $m_2 \geq k_1$.

Proof. Case (a) has been proved by Blackburn in [2].

Case (b) and Case (c) have been proved in [13, Lemma 1].

Case (d) has been proved by Gallian [16, Lemma 2.1].

Now we shall prove the Cases (e) and (f).

(e) By definition, the upper central series of G is $\frac{Z_{i+1}}{Z_i} = Z\left(\frac{G}{Z_i}\right)$. Hence it is enough to show that $\exp\frac{Z_2}{Z} = \exp Z$. The rest of the proof follows by induction. Let $\exp\frac{Z_2}{Z} = p^x$. Take $a \in G$ and $b \in Z_2$. Then $[a, b] \in [G, Z_2] \leq Z$. So $[a, b]$ commutes with both a and b. Then $[a^r, b] = [a_1, b^r] = [a, b]^r$ for any positive integer r. Hence $[a, b^{p^x}] = [a, b]^{px} = 1$, as $b^{px} \in Z$. So

$$k_1 \leq x. \tag{1}$$

Also, $[a, b^{p^{k_1}}] = [a, b]^{p^{k_1}} = 1$, as $[a, b] \in Z$. So $b^{p^{k_1}} \in Z$ for every $b \in Z_2$. Then

$$x \leq k_1. \tag{2}$$

From (1) and (2), we get that $x = k_1$.

(f) Let $\exp\frac{L_{c-2}}{L_{c-1}} = p^x$. Take $a \in G$, $b \in L_{c-2}$. Then $[a, b] \in [G, L_{c-2}] = L_{c-1} \leq Z$. Hence $[a, b]$ commutates with both a and b. This gives that for any positive integer n, we have $[a^n, b] = [a, b^n] = [a, b]^n$. So $[a, b^{p^x}] = [a, b]^{p^x} = 1$, as $b^{p^x} \in L_{c-1} \leq Z$. Therefore, $[a, b]^{p^x} = 1$ for $[a, b] \in Z$. Since $\exp Z = p^{k_1}$, we get that $x \geq k_1$. On the other hand, by Lemma 1(a), we have that $p^{m_2} \geq \exp\frac{L_{c-2}}{L_{c-1}} = p^x$. Hence $m_2 \geq x \geq k_1$.

Lemma 2. *Let G be a group of order p^n and class c. Also, let $G = L_0 \geq L_1 \geq L_2 \geq \cdots \geq L_{c-1} > L_c = 1$ and $G = Z_c \geq Z_{c-1} \geq Z_{c-2} \geq \cdots \geq Z_1 > Z_0 = 1$ be the lower and the upper central series of G, where $L_1 = G' = [G, G]$ is the commutator subgroup of G and $Z_1 = Z$ is the center of G. Let $\left|\frac{G}{L_1}\right| = p^m$, $|Z| = p^k$ and $m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$ and $k_1 \geq k_2 \geq \cdots \geq k_s \geq 1$ be the invariants of $\frac{G}{L_1}$ and Z, where t and s are the numbers of invariants of $\frac{G}{L_1}$ and Z, respectively.* $\exp \frac{G}{L_1} = p^{m_1}$, $\exp Z = p^{k_1}$. *In such a group, all the following conditions hold*:

(i) $|A_c G| = p^a$, *where* $a = \sum_{i,j}^{t,s} \min(m_i, k_j)$.

(ii) *If* $m_i \geq k_1$ *for some* i *with* $1 \leq i \leq t$, *then* $a \geq ik + (t-i)s$.

(iii) *If* $k_j \geq m_1$ *for some* j *with* $1 \leq j \leq s$, *then* $a \geq j \cdot m + (s-j)t$.

(iv) *If* $m_1 < k$, *then* $a \geq m$, *where* $\left|\frac{G}{L_1}\right| = p^m$.

(v) *If* $k > m_1 \geq k_1 \geq m_t$, *then* $a \geq k + m + s - m_1 - 1$.

(vi) *If* $k_j \geq m_1 \geq k_{j+1}$, *then*

$$a \geq j \cdot m + k - (k_1 + k_2 + \cdots + k_j) + (s-j)(t-1).$$

(vii) *If* $|A(G)| = p^A$, *then* $A \geq a + b$, *where* $A_c(G) = p^a$, $\left|\frac{G}{Z_2}\right| = p^b$.

(viii) $A_c(G) \cap I(G)$ *is isomorphic to the center of* $\frac{G}{Z}$.

(ix) *If* $\left|\frac{G}{Z_2}\right| = p^b$, *then* $b \geq k_1 c - 2k_1 + 1$.

Proof. Cases (i), (ii), (iii) and (vi) have been proved by Exarchakos [9, Lemma 1]. Also, Case (iv) has been proved by Exarchakos [12, Lemma 2]. Case (viii) has been proved by Hall, Jr. in [18].

We now proceed to prove the Cases (v), (vii) and (ix).

(v) If $k_1 \leq m_i$ for all i, then $k_1 \leq m_t$ and so $a \geq m \cdot s$. Therefore, we may assume that $k \geq m_i$ for some i with $2 \leq i \leq s$. From Case (i), we have

$$|A_c(G)| = \left|Hom\left(\frac{G}{L_1}, Z\right)\right| = p^a, \text{ where } a = \sum_{i,\, j}^{t,\, s} \min(m_i, k_j). \tag{1}$$

Let a_x, b_x be the number of times x appears among the invariants of $\frac{G}{L_1}$ and Z, respectively. In (1), summing powers over $m_i = 1, 2, \ldots, m_t$ for $k_j = k_s, \ldots, k_1$, we get

$$a = \sum_{x=1}^{ks} xa_x + k_s \sum_{x>k_s}^{k_1} ax + \cdots + \left(\sum_{x=1}^{k_i} xa_x + k_i \sum_{x>k_i}^{k_1} xa_x\right).$$

Thus

$$a = \sum_{j=2}^{s}\left(\sum_{x=1}^{k_i} xa_x + k_i \sum_{x>k_i}^{k_1} a_x\right) + \sum_{i=1}^{s} k_j \left(\sum_{x>k_j}^{m_1} a_x + \sum_{x=1}^{k_1} xa_x\right). \tag{2}$$

Since

$$\sum_{i=1}^{s} k_j \left(\sum_{x>k_i}^{m_1} a_x + \sum_{x=1}^{k_1} xa_x\right) \geq \sum_{x=1}^{m_1} xa_x = m,$$

putting $r = \sum_{j=2}^{s}\left(\sum_{x=1}^{k_i} xa_x + k_i \sum_{x>k_i}^{k_1} ax\right)$, from (2), we get $a \geq m + r$.

Let $\varphi_i = \sum_{x=1}^{k_i} xa_x + k_i \sum_{x>k_i}^{k_1} a_x$. We claim that $\varphi_i \geq 1$.

If $\sum_{x=1}^{k_i} xa_x = 0$, then $k_i \leq m_t$ and so $k_1 \geq m_t \geq k_i$. Then $\sum_{x>k_i}^{k_1} a_x \geq 1$. This gives $\varphi_i \geq 1$.

If $\sum_{x>k_i}^{k_1} a_x = 0$, then $k_i \geq m_t$ and then $\sum_{x=1}^{k_i} xa_x \geq 1$ and we have again $\varphi_i \geq 1$. So our claim has been established.

For $\varphi_i \geq 1$, we set $r \geq s - 1$ and so $a \geq m + s - 1$.

Since $k \geq k_1 \geq m_1$, there exists a non-negative integer b such that $k = m_1 + b$. By (2), we get

$$a \geq \sum_{i=2}^{s} \varphi_i + \sum_{x=1}^{k_i} xa_x + k \sum_{x\geq k_i}^{m_1} a_x \geq s - 1 + \sum_{x=1}^{k_i} xa_x + m_1 \sum_{x>k_1}^{m_1} a_x + b \sum_{x>k_i}^{m_1} a_x.$$

Since $\sum_{x=1}^{k_i} xa_x + m_1 \sum_{x>k_1}^{m_1} a_x \geq \sum_{x=1}^{m_1} xa_x = m$ and $\sum_{x>k_i}^{m_1} a_x \geq 1$, as $m_1 \geq k_i$, we get $a \geq s - 1 + m + b$, where $b = k - m_1$. Therefore, $a \geq k + m + s - m_1 - 1$.

(vii) Let $|A(G)|_p = p^A$. Then

$$p^A = |A(G)| \geq |A_c(G)| \cdot |I(G)| = \frac{|A_c(G)| \cdot |I(G)|}{|A_c(G)| \cap |I(G)|}.$$

Since $|I(G)| = \left|\frac{G}{Z}\right|$ and by (b), $|A_c \cap I(G)| = \left|Z\left(\frac{G}{Z}\right)\right| = \left|\frac{Z_2}{Z}\right|$, we have $p^A \geq \frac{|A_c(G)| \cdot |I(G)|}{|A_c(G)| \cap |I(G)|} = |A_c(G)| \cdot \left|\frac{G}{Z_2}\right| \geq p^a \cdot p^b = p^{a+b}$. Hence $A \geq a + b$.

(ix) Since $\exp \frac{Z_{i+1}}{Z_i} = p^{k_1}$ for all $i = 0, 1, 2, ..., c-1$ and $\frac{G}{Z_{c-1}}$ cannot be cyclic, we get $\left|\frac{G}{Z_{c-1}}\right| \geq p^{k_1+1}$ and $\left|\frac{Z_{c-i}}{Z_{c-i-1}}\right| \geq p^{k_1}$ for all $i = 1, 2, ..., c-3$. This gives that $\left|\frac{G}{Z_2}\right| = \left|\frac{G}{Z_{c-1}}\right| \cdot \left|\frac{Z_{c-1}}{Z_2}\right| \geq p^{k_1+1} \cdot p^{k_1(c-3)} = p^{k_1c-2k_1+1}$. Hence $b \geq k_1c - 2k_1 + 1$.

Theorem 1. *Let G be a non-abelian group of order* p^n *and class c. If* $\frac{L_i}{L_{i+1}}$ *is cyclic of fixed order* p^r *for all* $i = 1, 2, ..., c-1$, *then G is an LA-group.*

Proof. Let $\left|\frac{G}{L_1}\right| = p^m$. Since $\left|\frac{L_i}{L_{i+1}}\right| = p^r$ for all i, we get

$$n \geq m + r(c-1). \tag{1}$$

By Lemma 1, we have $L_i \leq Z_{c-i}$ and that $L_i \not\leq Z_{c-i-1}$. Also, by Gallian [16], $L_i \cap Z_{c-i-1} = L_{i+1}$, $\left|\frac{Z_{c-i}}{Z_{c-i-1}}\right| \geq \left|\frac{L_i}{L_{i+1}}\right| \geq p^r$, $\left|\frac{G}{Z_{c-1}}\right| \geq p^{2r}$ and $\exp\frac{G}{L_1} \leq p^r = |L_{c-1}| \leq |Z|$. Let $\left|\frac{Z_2}{Z}\right| = p^x$. Then $|Z_2| = p^{x+k}$ and so $n \geq 2r + r(c-3) + x + k$ and this is the minimum value of n [16]. Then, by (1), we get $m + r(c-1) \geq 2r + r(c-3) + x + k$, which gives $m \geq x + k$. Since $\exp\frac{G}{L_1} \leq |Z|$, we get $a \geq m$. Therefore, $a \geq m \geq x + k$ and so $A \geq a + n - x - k \geq n$ as $a \geq x + k$.

Theorem 2. *Let G be a group of order* p^n *and class c. If the center Z of G is elementary abelian, then G is an LA-group.*

Proof. We may assume that G has more than two generators. Then $t \geq 3$, where t is the number of invariants of $\frac{G}{L_1}$. Since $k_1 = 1$, $k_i = 1$ for all $i = 1, 2, \ldots, s$, by Lemma 2(ii),

$$a \geq kt \geq 3k. \tag{1}$$

Let $G = Z_c \geq Z_{c-1} \geq Z_{c-2} \geq \cdots \geq Z_1 > Z_0 = 1$ be the upper central series of G, where $Z_1 = Z$ is the center of G. Since $\frac{G}{Z_{c-1}}$ cannot by cyclic, we have $\left|\frac{G}{Z_{c-1}}\right| \geq p^2$.

Also, $\left|\frac{Z_{c-i}}{Z_{c-i-1}}\right| \geq p$ for all $i = 1, 2, \ldots, c-3$.

Then we have $\left|\frac{G}{Z_2}\right| = \left|\frac{G}{Z_{c-1}}\right| \cdot \left|\frac{Z_{c-1}}{Z_2}\right| \geq p^2 \cdot p^{c-3} = p^{c-1}$.

Hence

$$b \geq c - 1 \geq 2, \tag{2}$$

where $\left|\frac{G}{Z_2}\right| = p^b$.

Also, if $k = 1$, then $|A(G)| = p|I(G)| = p\left|\frac{G}{Z}\right| = |G|$ and G is an LA-group.

Assume that $k \geq 2$. By (1) and (2), we get $A \geq a + b \geq 3k + 2 \geq 8$.

Hence, for $n \leq 8$, G is an LA-group.

Also, $k \geq 2$ and so $a \geq 3k \geq 2k + 2$. Then $A \geq a + b \geq 2k + 4 \geq n$ for $k \geq \frac{n-4}{2}$.

Since $A \geq a + b \geq 2k + 2 + c - 1 = 2k + c + 1 \geq n$ for $c \geq n - 5$, we get $A \geq n$ for $c \geq n - 5$.

Hence, if it holds either $n \leq 8$, $k \geq \frac{n-4}{2}$ or $c \geq n-5$, then G is an *LA-group*.

Therefore, we may assume that the following conditions (3), (4) and (5) hold simultaneously:

$$n \geq 9, \tag{3}$$

$$k < \frac{n-4}{2} \tag{4}$$

and

$$c < n-5. \tag{5}$$

Also, $A \geq a + b \geq 2k + c + 1$ and we have to prove that $2k + c + 1 \geq n$.

By (4) and (5), we get $2k + c < 2n - 9$. So $2k + c + 1 < 2n - 8$.

If $2k + c + 1 < n$, then $2n - 8 < n$ which gives $n < 8$, a contradiction to (3).

Hence $2k + c + 1 \geq n$ and so $A \geq a + b \geq 2k + c + 1 \geq n$ and G is an LA-group.

This proves Theorem 2.

Theorem 3. *Any finite non-cyclic abelian group of order* p^n, $n > 2$ *is an LA-group*.

Proof. Otto proved in [24] that any finite non-cyclic abelian group of order p^n, $n > 2$ is an LA-group. Also, Exarchakos [10] gives a formula, by which we can determine the number of automorphisms of an abelian p-group of a variety of types of abelian p-groups. This formula makes Otto's result a special case. Also, this formula makes special cases all until now known results about the number of automorphisms of abelian p-groups. Here we state this formula without proof. Details could be found in [10, Theorem 2]. In that paper, Exarchakos proves:

Let G be a finite abelian group of order p^n, $n > 2$. Let $r_1 \geq r_2 \geq \cdots \geq r_s \geq 1$ be the invariants of G and $|AG| = p^m$. Then we have:

I. If G is elementary abelian, then G is isomorphic to a vector space over a field of characteristic p. And so,

$$|A(G)| = p^{\frac{1}{2}n(n-1)}(p^n - 1)(p^{n-1} - 1)\cdots(p - 1). \tag{1}$$

II. Let G be not elementary abelian. Then

1. If G is homocyclic, then $m \geq \frac{1}{2}s(2n - s - 1)$.

2. If $s = 2$, then $m = n + 2(r_2 - 1)$.

3. If $s > 2$, then $m \geq n + \frac{1}{2}s(s + 1)$, except the particular Cases (a), (b) and (c):

(a) $(i, j, 1, ..., 1)$ with $i > j > 1$.

(b) $(i, ..., i, 1, 1, ..., 1)$ with $a_i > p$.

(c) $(i, 1, 1, ..., 1)$.

In these particular cases, we have:

For the type (a), $m \geq n + \frac{1}{2}s(s + 1) - 1$.

For the type (b), $m \geq n + \frac{1}{2}s(s - 1)$.

For the type (c), $m \geq n + \frac{1}{2}s(s - 1) - 1$.

It is easily seen in all above cases we have $m \geq n$ and so we have the corollary:

Corollary. *Any finite non-cyclic abelian group of order* p^n, $n > 2$, *is an LA-group*.

Theorem 4. *Let G be a group of order p^n and class $c > 2$. Then if* $\left|\frac{G}{Z_2}\right| = p^b$ *and* $b \leq 2c - 3$, *then G is an LA-group.*

Proof. We may assume that G has more than two generators. Then $t \geq 3$, where t is the number of invariants of $\frac{G}{L_1}$. Let $G = L_0 \geq L_1 \geq \cdots \geq L_{c-1} > L_c = 1$ and $G = Z_c \geq Z_{c-1} \geq \cdots \geq Z_1 > Z_0 = 1$ be the lower and the upper central series of G, where $L_1 = G' = [G, G]$ is the commutator subgroup of G and $Z_1 = Z$ is the center of G. Since $\frac{G}{Z_{c-1}}$ is not cyclic, we get that $\left|\frac{G}{Z_{c-1}}\right| \geq p^2$. If $\left|\frac{G}{Z_{c-1}}\right| = p^2$, then $\exp \frac{G}{Z_{c-1}} = p$. Then $\exp Z = p$ and by Theorem 2, G is an LA-group. Hence we may assume that $\left|\frac{G}{Z_{c-1}}\right| \geq p^3$. Also, if $\frac{Z_{c-i}}{Z_{c-i-1}} = p$ for some i, $1 \leq i \leq c - 3$, then $\exp \frac{Z_{c-i}}{Z_{c-i-1}} = p$ and so $\exp Z = p$ and G is an LA-group by Theorem 2. Therefore, we may assume that $\left|\frac{G}{Z_{c-1}}\right| \geq p^3$ and $\left|\frac{Z_{c-i}}{Z_{c-i-1}}\right| \geq p^2$ for all i, $2 \leq i \leq c - 3$.

Then

$$\left|\frac{G}{Z_2}\right| = \left|\frac{G}{Z_{c-1}}\right| \cdot \left|\frac{Z_{c-1}}{Z_2}\right| \geq p^3 \cdot p^{2(c-3)} = p^{2c-3}. \tag{1}$$

Consider the following cases:

(a) Let $b < 2c - 3$. Then either $\left|\frac{G}{Z_{c-1}}\right| \leq p^2$ or $\left|\frac{Z_{c-i}}{Z_{c-i-1}}\right| = p$ for at least one i, $1 \leq i \leq c - 3$. Then we have $\exp \frac{G}{Z_{c-1}} = p$ or $\exp \frac{Z_{c-i}}{Z_{c-i-1}} = p$. In each case, we get $\exp Z = p$ and so by Theorem 2, G is an LA-group.

(b) Let $b = 2c - 3$. Then $\left|\frac{G}{Z_{c-1}}\right| = p^3$ and $\left|\frac{Z_{c-i}}{Z_{c-i-1}}\right| = p^2$ for all i. Since $\frac{G}{Z_{c-1}}$ cannot be cyclic, we have $\exp\frac{G}{Z_{c-1}} \leq p^2$. If $\exp\frac{G}{Z_{c-1}} = p$, then $\exp Z = p$ and so G is an LA-group. Therefore, we may assume that $\exp\frac{G}{Z_{c-1}} = p^2$. Then $\exp\frac{Z_{c-i}}{Z_{c-i-1}} = p^2$ and so $\frac{Z_{c-i}}{Z_{c-i-1}}$ is cyclic of order p^2 for all $i = 1, 2, ..., c-3$. Then, by [16], $Z_{c-2} < L_1 \leq Z_{c-1}$. This gives that $\left|\frac{Z_{c-1}}{L_1}\right| \leq p$. Then $p^m = \left|\frac{G}{L_1}\right| \leq p^4$ and $m \leq 4$. Since G has more than two generators, $\frac{G}{L_1}$ has more than two invariants. Hence $3 \leq t \leq 4$.

If $t = 4$ and $\left|\frac{G}{L_1}\right| = p^4$, then $\exp\frac{G}{L_1} = p$ and so $m_i = 1$ for all i. Then $m_2 = 1$ and so $k_1 \leq m_2 = 1$. Then $k_1 = 1$, $\exp Z = p$ and by Theorem 2, G is an LA-group.

If $t = 3$, let $m_1 \geq m_2 \geq m_3 \geq 1$ be the invariants of $\frac{G}{L_1}$. Then $m_1 + m_2 + m_3 = 4$. This gives that $m_3 = 1$. Hence $m_1 + m_2 = 3$, which gives $m_2 = 1$. Then $k_1 = 1$ and Z is elementary abelian. Hence, by Theorem 2, G is an LA-group. This proves Theorem 4.

Theorem 5. *Let G be a group of order p^n and class c. Also, let $|A(G)|_p = p^A$, $|A_c(G)| = p^a$, $\exp\frac{G}{L_1} = p^{m_1}$, $\exp Z = p^{k_1}$ and $|Z| = p^k$. Then, if $k_1 \geq \frac{k}{3}$, G is an LA-group.*

Proof. We may assume that $t \geq 3$ and $k_1 \geq 2$ (Theorem 2). Also, by Theorem 4, we may assume that $b \geq 2c - 2 \geq 4$ also for $k_1 \geq \frac{k}{3}$, we get

$b \geq k_1 c - 2k_1 + 1 \geq \frac{k}{3} + 1.$ If $\frac{k}{3} + 1 \leq 2c - 3$ by Theorem 4, then G is an LA-group.

Assume that $\frac{k}{3} + 1 \geq 2c - 2 \geq 4.$ Then

$$k \geq 9. \tag{1}$$

Since $m_2 \geq k_1,$ Lemma 2(ii) gives $a \geq 2k + (t-2)s \geq 2k + s,$ as $t \geq 3.$ Hence

$$a \geq 2k + 1. \tag{1}$$

Since $b \geq 4,$ we get

$$A \geq a + b \geq 2k + 1 + 4 = 2k + 5 \geq 23. \tag{2}$$

Also, $A \geq 2k + 5 \geq n$ for

$$k \geq \frac{n-5}{2}. \tag{3}$$

Hence, for $n \leq 23$ or $k \geq \frac{n-5}{2},$ G is an *LA*-*group*. Also, since $b \geq 2c - 2,$ we get $A \geq 2k + 1 + 2c - 2 = 2k + 2c - 1 \geq n$ for $c \geq \frac{n-17}{2}.$ Hence, for $c \geq \frac{n-17}{2},$ G is an *LA*-*group*. Therefore, we may assume that the following conditions (4), (5) and (6) hold simultaneously:

$$n \geq 24, \tag{4}$$

$$k < \frac{n-5}{2}, \tag{5}$$

$$c < \frac{n-17}{2}. \tag{6}$$

Also, $A \geq 2k + 2c - 1$ and we have to prove that $2k + 2c - 1 \geq n.$

By the Cases (5) and (6), we have $k + c < \frac{2n-22}{2},$ so $k + c - \frac{1}{2} < \frac{2n-23}{2}.$ If $k + c - \frac{1}{2} < \frac{n}{2},$ then we have $\frac{2n-23}{2} < \frac{n}{2}$ which gives $n < 23,$

a contradiction to (4). Hence $k + c - \frac{1}{2} \geq \frac{n}{2}$ which gives $2k + 2c - 1 \geq n$. Therefore, $A \geq 2k + 2c - 1 \geq n$ and G is an *LA-group*.

This proves Theorem 5.

Corollary. *Let G be a group of order p^n and class c. Then G is an LA-group under anyone of the following conditions*:

(i) *If G has cyclic center*.

(ii) *If* $|Z| = p^k$ *and* $k \leq 6$.

Proof. (i) If Z is cyclic, $k_1 = k$ and by Theorem 5, G is an LA-group.

(ii) If $k_1 = 1$, then Z is elementary abelian and by Theorem 2, G is an LA-group. If $k_1 \geq 2$, then $3k_1 \geq 6 \geq k$ and so by Theorem 5, G is an LA-group.

Theorem 6. *Any finite non-cyclic group of order p^n and class two is an LA-group*.

Proof[1]**.** Since G has class two, $\frac{G}{Z}$ is abelian and so $I(G) \leq A_c(G)$ as $A_c(G)$ is the centralizer of $I(G)$ in $A(G)$. Also, since G has class two, $L_1 \leq Z_1$, where $L_1 = G' = [G, G]$ is the commutator subgroup of G.

Also, $\exp Z = p^{k_1}$ gives $\exp \frac{Z_2}{Z} = \exp Z = p^{k_1}$ (Lemma 1(e)). By [14, Lemma 2], we have $\exp \frac{G}{L_1} = \exp \frac{Z_2}{L_1} = p^{k_1}$. This gives $m_1 = k_1$. Since $m_1 \geq m_2$ and by Lemma 1(f), $m_2 \geq k_1$, we get

$$m_1 = m_2 = k_1. \tag{1}$$

[1]Schenkman in 1955 proved that theorem [26]. But in that paper, some lemmas were incorrect. Later in 1968, Faudree proved by another way the incorrect lemmas and then he proved the theorem [15]. Here we prove the theorem by a completely different way. This proof has been done by elementary calculation.

By Theorem 2, we may assume that $k_1 \geq 2$. Also, by Theorem 5, we may assume that $k > 3k_1$. By Lemma 2(v), we have $a \geq k + m + s - m_1 - 1$ $= k + m + s - k_1 - 1$. Since $k > 3k_1$ and $k_1 s \geq k$, we get $k_1 s \geq k > 3k_1$, which gives $s > 3$ and since s is an integer $s \geq 4$. Also, since $m_2 \geq k_1$, Lemma 2(ii) gives $a \geq 2k + (t - 2)s \geq 2k + s$, as $t \geq 3$. Hence

$$a \geq 2k + 4. \tag{2}$$

We claim that

$$s > k_1. \tag{3}$$

Since $k_1 \geq 2$, for $k_1 = 2$, we get $2s \geq k > 3k_1$ and so $s > k_1$.

For $k_1 \geq 3$, we have $3s \geq k > 3k_1$, which gives $s > k_1$ and our claim has been established. Hence $s > k_1$ or $s \geq k_1 + 1$.

Therefore, $a \geq k + m + s - k_1 - 1 \geq k + m \geq k + n - k = n$ and so $a \geq n$ and G is an LA-group. This proves Theorem 6.

Now we proceed to prove Theorem 7, which is the main result of this paper.

Theorem 7. *Any finite non-cyclic group of order* p^n *and class* $c = 3$ *is an LA-group*.

Proof. We may assume that G has more than two generators. Then $t \geq 3$, where t is the number of invariants of $\frac{G}{L_1}$. Let $G = L_0 \geq L_1 \geq L_2 \geq L_3 = 1$ and $G = Z_3 \geq Z_2 \geq Z_1 > Z_0 = 1$ be the lower and the upper central series of G, respectively, where $L_1 = G' = [G, G]$ is the commutator subgroup of G and $Z_1 = Z$ is the center of G. Then we have $L_1 \leq Z_2$ and $L_2 \leq Z$. Also, $\exp \frac{G}{L_1} = p^{m_1}$ and $\exp Z = p^{k_1}$. Then $\exp \frac{Z_2}{Z} = p^{k_1}$. Also, $\exp \frac{G}{Z_2} = p^{k_1}$,

$\exp \frac{Z_2}{L_1} = p^{k_1}$. Thus, $\exp \frac{G}{L_1} \leq p^{2k_1}$. By Theorem 5, we may assume that $k > 3k_1$ and so

$$m_1 < k. \tag{1}$$

Let $\left| \frac{G}{L_1} \right| = p^m$ and $|Z| = p^k$. Since $m_1 < k$, we get that $a \geq k + m + s - m_1 - 1$ (Lemma 2(vi)). Hence

$$a \geq k + m + s - 2k_1 - 1. \tag{2}$$

Let $|L_1| = p^l$, $\left| \frac{Z_2}{L_1} \right| = p^w$. Then $|Z_2| = p^{l+w}$. Also, let $\left| \frac{Z_2}{Z} \right| = p^x$. Then $|Z_2| = p^{x+k}$. Hence

$$w + l = x + k. \tag{3}$$

Since $L_1 \leq Z_2$, we get $l \leq x + k$. So $n - l \geq n - x - k$. Hence $m = n - l \geq n - x - k = b$. Also, since $m_1 \leq 2k_1 < k$, we have $a \geq m \geq n - x - k$. Hence $a + x + k \geq n$. Also, $b + x + k = n$ and so $a + b + 2(x + k) \geq 2n$. This gives $a + b \geq n$ for $x + k \leq \frac{1}{2}n$. Then $A \geq a + b \geq n$ for $x + k \leq \frac{1}{2}n$. Therefore, we may assume that

$$x + k \geq \frac{1}{2}n. \tag{3}$$

If $l \leq k + 1$, then $n - l \geq n - k - 1$ and so $m = n - l \geq n - k - 1$. By (2), $a \geq k + m + s - 2k_1 - 1$ and $b \geq k_1c - 2k_1 + 1 \geq k_1 + 1$.

Also, by (3) in Theorem 6, we have $s > k_1$. Hence $b + s \geq 2k_1 + 2$. Then

$$A \geq a + b \geq k + m + s + b - 2k_1 - 1 \geq k + m + 1 \geq k + n - k - 1 + 1 = n$$

and G is an LA-group. Therefore, we may assume that $l \geq k + 2$. If $x \leq k + 1$, then $x + k \leq 2k + 1$. Then $a \geq 2k + 4 \geq x + k$ and $A \geq a + b \geq a + n - x - k \geq n$. Hence we may assume that $x \geq k + 2$. Then $x + k \geq$

$2k+2$ and by (3), $2k+2 \geq \frac{1}{2}n$. So $k \geq \frac{n-4}{4}$ and $x \geq k+2 \geq \frac{k+4}{4}$. Therefore, $k+x \geq \frac{n-4}{4}+\frac{n+4}{4}=\frac{n}{2}$. Also, for $k \geq \frac{n-4}{4}$, we get $a \geq 2k+4 \geq \frac{n+4}{2} > \frac{n}{2}$ and so $a \geq x+k$. Then $A \geq a+b=\alpha+n-x-k \geq n$. This proves Theorem 7.

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