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STABILITY AND PRACTICAL STABILITY OF IMPULSIVE INTEGRO-DIFFERENTIAL SYSTEMS BY CONE-VALUED LYAPUNOV FUNCTIONS

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Abstract

Stability and practical stability comparison criteria of impulsive integro-differential systems with fixed moments of impulse effects are established by cone-Lyapunov functions through comparing with impulsive ordinary differential equations.

1. Introduction

Impulsive integro-differential systems which are an important embranchment of nonlinear impulsive differential systems [3], arise from extensive applications in nature-science such as mathematic models of circuit

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simulation in physics and neuronal networks in biology. Consequently, there are some results about stability of such systems by vector Lyapunov functions coupled with Razumikhin techniques [4, 5]. However, it is difficult to choose a right vector Lyapunov function because of the restrictive conditions. At the same time, the method of cone-valued Lyapunov functions is well known to be advantageous in applications [2]. Hence, the stability results for impulsive integro-differential systems could be improved via the method of cone-valued Lyapunov functions.

In [1], the author considered the comparison principle by cone-valued Lyapunov functions for a class of integro-differential systems without impulses. But it was not proved and also cannot be applied to impulsive integro-differential systems. In this paper, we shall firstly prove the comparison principle. Then, by employing cone-valued Lyapunov functions, a new comparison principle for impulsive integro-differential systems with fixed moments of impulse effects is established, which is compared with impulsive differential systems whose stability is relatively easy to solve. Finally, the relevant new comparison criteria of stability and practically stability [6] of impulsive integro-differential systems are obtained too.

The remainder of this paper is organized as follows. In Section 2, we describe impulsive integro-differential systems and introduce some notions and concepts. In Section 3, we get some comparison results of stability and practically stability of the impulsive integro-differential systems with fixed moments of impulse effects by using the method of cone-Lyapunov functions.

2. Preliminaries

Consider the following impulsive integro-differential systems of the form:

$$\begin{cases} x' = f(t,\ x,\ Tx), & t \neq t_k, \\ x(t_k) = J_k(x(t_k^-)), & k \in N, \\ x(t_0^+) = x_0, \end{cases} \tag{1}$$

where N is the set of all positive integers, $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $t_k \to \infty (k \to \infty)$. $f \in C([t_k, t_{k+1}) \times S(\rho) \times R^n, R^n)$ $(k \in N)$, $f(t, 0, 0) \equiv 0$, where $S(\rho) = \{x : |x| < \rho, x \in R^n\}$. $Tx = \int_{t_0}^{t} K(t, s, x(s))ds$, where $K \in C([t_k, t_{k+1}) \times [t_k, t_{k+1}) \times S(\rho), R^n)$, $K(t, t, 0) \equiv 0$. $J_k(x) : S(\rho) \to R^n$, $J_k(0) \equiv 0$ $(\forall k \in N)$ and there exists $\rho_1 : 0 < \rho_1 \le \rho$ such that $x \in S(\rho_1)$ implies that $J_k(x) \in S(\rho)$ for all $k \in N$.

In addition, we always assume that f, J_k satisfy certain conditions such that the solution of system (1) exists on $[t_0, +\infty]$ and is unique. We denote by $x(t) = x(t, t_0, x_0)$ the solution of system (1) with initial value (t_0, x_0). Since $f(t, 0, 0) = 0$, $J_k(0) = 0$, $k \in N$, $x(t) = 0$ is a solution of (1), which is called the *trivial solution*. Note that the solutions $x(t)$ of (1) are right continuous, satisfying $x(t_k^+) = x(t_k) = J_k(x(t_k^-))$.

Let $t_0 = 0$. Then the following sets are introduced:

$$G_k = \{(t, x) \in R_+ \times S(\rho) : t_{k-1} < t < t_k\}, \quad G = \bigcup_{k=1}^{\infty} G_k.$$

For convenience, we define the following classes of functions:

$$K = \{a \in C[R_+, R_+] : \text{strictly increasing and } a(0) = 0\},$$

$$CK = \{a \in C[R_+^2, R_+] : \text{for every } t \in R_+, a(t, s) \in K\},$$

$$S(h, \rho) = \{(t, x) \in R_+ \times R^n : h(t, x) < \rho\}.$$

In addition, we introduce some definitions as follows:

Definition 1. Let $Z \subseteq R^n$ be a cone, that is, Z is closed, convex with $\lambda Z \subset Z$, $\lambda \ge 0$ and $Z \cap (-Z) = \{0\}$ with interior $Z^0 \ne \emptyset$. For any $x, y \in R^n$, we let $x \le y$ if $y - x \in Z$ and for any functions $u, v : R_+ \to R^n$,

$u \le v$ if $u(t) \le v(t)$ on R_+. Also, let $Z^* = \{\varphi \in R^n,\ \varphi(x) \ge 0,$ for all $x \in Z\}$ and $Z_0^* = Z^* - \{0\}$, where $\varphi(x) = \sum_{i=1}^{n} \varphi_i x_i$.

Definition 2. A function $F : R^n \to R^n$ is said to be *quasi-monotone nondecreasing* relative to the cone $Z \subseteq R^n$ if $x \le y$ and $\varphi(y - x) = 0$ for all $\varphi \in Z_0^*$ implies $\varphi(F(y) - F(x)) \ge 0$.

Definition 3. We shall say that the function $V_0 : R_+ \times R^n \to Z$ belongs to the class of cone-Lyapunov if: (i) $V \in V_0$ is continuous on $G_k \times R^n$ and the following limit exists $\lim_{(t, y) \to (t_k^-, x)} V(t, y)$, $k \in N$; (ii) $V(t, x) \in V_0$ is locally Lipschitz in *x* relative to *Z*.

Derivative of $V(t, x) \in V_0$ along system (1) is defined:

$$D^- V(t, x(t)) = \limsup_{h \to 0^-} \frac{1}{h} [V(t + h, x(t) + hf(t, x(t), Tx(t))) - V(t, x(t))].$$

Definition 4. The trivial solution of (1) is said to be:

(i) *stable*, if for any $\varepsilon > 0$, every $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $|x_0| < \delta$ implies $|x(t, t_0, x_0)| \le \varepsilon$ for all $t \ge t_0$;

(ii) *uniformly stable*, if δ in (i) is independent of t_0;

(iii) *uniformly asymptotically stable*, if it is uniformly stable, and there exists a $\delta > 0$ such that for any $\varepsilon > 0$ and every $t_0 \in R_+$ there is a $T = T(\varepsilon) > 0$ such that $|x_0| < \delta$ implies $|x(t, t_0, x_0)| \le \varepsilon$ for all $t \ge t_0 + T$.

Definition 5. The trivial solution of (1) is said to be:

(i) *practically stable*, if for given number pair (λ, A) with $0 < \lambda < A$, we have $|x_0| < \lambda$ implies $|x(t)| < A$, $t \ge t_0$ for some $t_0 \in R_+$;

(ii) *uniformly practically stable*, if (i) holds for every $t_0 \in R_+$;

(iii) *practically quasi-stable*, if for given number $(\lambda, B, T) > 0$ and some $t_0 \in R_+$, we have $|x_0| < \lambda$ implies $|x(t)| < B$ for $t \geq t_0 + T$;

(iv) *uniformly practically quasi-stable*, if (iii) holds for every $t_0 \in R_+$;

(v) *strongly practically stable*, if (i) and (iii) hold together;

(vi) *strongly uniformly practically stable*, if (ii) and (iv) hold together.

We also consider the comparison differential system:

$$\begin{cases} u' = g(t, u), & t \neq t_k, \\ u(t_k) = \Psi_k(u(t_k^-)), & k \in N, \\ u(t_0^+) = u_0, \end{cases} \tag{2}$$

where $g \in C([t_k, t_{k+1}) \times Z, R^n), k \in N$.

In addition, we always assume that g, Ψ_k satisfy certain conditions such that the solution of system (2) exists on $[t_0, +\infty]$ and is unique. We denote by $u(t) = u(t, t_0, u_0)$ the solution of system (2) with initial value (t_0, u_0). Note that the solutions $u(t)$ of (2) are right continuous, satisfying $u(t_k^+) = u(t_k) = \Psi_k(x(t_k^-))$.

Definition 6. The trivial solution of (2) is said to be:

(i) ϕ_0-*stable*, if for any $\varepsilon > 0$, every $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $(\phi_0, u_0) < \delta$ implies $(\phi_0, u(t)) < \varepsilon$, for all $t \geq t_0$, where $\phi_0 \in Z_0^*$;

(ii) ϕ_0-*uniformly stable*, if δ in (i) is independent of t_0;

(iii) ϕ_0-*attractive*, if for any $\varepsilon > 0$, every $t_0 \in R_+$, there exists a $\delta = \delta(t_0) > 0$, $T = T(t_0, \varepsilon) > 0$, such that $(\phi_0, u_0) < \delta$ implies $(\phi_0, u(t)) < \varepsilon$, for all $t \geq t_0 + T$, where $\phi_0 \in Z_0^*$;

(iv) ϕ_0-*uniformly attractive*, if δ, T in (iii) are independent of t_0;

(v) ϕ_0-*asymptotically stable*, if (i) and (iii) hold together;

(vi) ϕ_0-*uniformly asymptotically stable*, if (ii) and (iv) hold together.

Definition 7. The trivial solution of (2) is said to be:

(i) ϕ_0-*practically stable*, if for given number pair (λ, A) with $0 < \lambda < A$, we have $(\phi_0, u_0) < \lambda$ implies $(\phi_0, u(t)) < A$, $t \geq t_0$ for some $t_0 \in R_+$, $\phi_0 \in Z_0^*$, where $\phi_0 \in Z_0^*$;

(ii) ϕ_0-*uniformly practically stable*, if (i) holds for every $t_0 \in R_+$;

(iii) ϕ_0-*practically quasi-stable*, if for given number $(\lambda, B, T) > 0$ and some $t_0 \in R_+$, $\phi_0 \in Z_0^*$, we have $(\phi_0, u_0) < \lambda$ implies $(\phi_0, u(t)) < B$, for all $t \geq t_0 + T$;

(iv) ϕ_0-*uniformly practically quasi-stable*, if (iii) holds for every $t_0 \in R_+$;

(v) ϕ_0-*strongly practically stable*, if (i) and (iii) hold together;

(vi) ϕ_0-*strongly uniformly practically stable*, if (ii) and (iv) hold together.

3. Main Results

Lemma 1. *Assume that*

(i) $g \in C[R_+ \times Z, R_+^N]$, $g(t, u)$ *is quasi-monotone nondecreasing in u for each fixed t on Z and* $r(t, t_0, u_0)$ *is the maximal solution of the system*

$$\begin{cases} u'(t) = g(t, u), \\ u(t_0) = u_0; \end{cases}$$

(ii) $V \in C[R_+ \times R^n, Z]$, $V(t, x)$ *is locally Lipschitz in x relative to the cone Z and* $D^- V(t, x(t)) \leq_Z g(t, V(t, x(t)))$ $(t \geq t_0)$ *for any solution* $x(t) = x(t, t_0, x_0)$ *of* $\begin{cases} x' = f(t, x, Tx), \\ x(t_0) = x_0. \end{cases}$

Then $V(t_0, x_0) \leq_Z u_0$ *implies* $V(t, x(t)) \leq_Z r(t, t_0, u_0)$, $t \geq t_0$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of system in (ii), satisfying $V(t_0, x_0) \leq_Z u_0$.

Set $m(t) = V(t, x(t))$, for small enough $h < 0$, from (ii), $V(t, x)$ is locally Lipschitz in x relative to the cone Z, therefore, $m(t + h) - m(t) \leq_Z L\| x(t + h) - x(t) - hf(t, x, Tx) \| + V(t + h, x + hf(t, x, Tx)) - V(t, x(t))$, when $h \to 0$ we have $D^- m(t) \leq_Z D^- V(t, x(t))$, $D^- m(t) \leq_Z g(t, m(t))$.

For small enough $\forall \varepsilon > 0$, consider system $\begin{cases} u'(t) = g(t, u) + \varepsilon\eta, \\ u(t_0) = u_0, \ t_0 \in R_+, \end{cases}$ where $\eta \in Z$.

The solution of it is $u(t, \varepsilon) = u(t, t_0, u_0, \varepsilon)$, then we have $\lim_{\varepsilon \to 0} u(t, \varepsilon) = r(t, t_0, u_0)$.

To prove the conclusion, we only need to prove

$$m(t) \leq_Z u(t, \varepsilon), \qquad t \geq t_0. \tag{3.1}$$

If it is not true, then there exists $t_1 > t_0$ such that $u(t_1, \varepsilon) - m(t_1) \in \partial Z$, and $u(t, \varepsilon) - m(t) \in Z^0$, $t \in [t_0, t_1)$.

From (i), $g(t, u)$ is quasi-monotone nondecreasing in u for each fixed t on Z, so there exists $\varphi \in Z_0^*$, such that

$$\varphi(u(t_1, \varepsilon) - m(t_1)) = 0 \quad \text{and} \quad \varphi(g(t_1, u(t_1, \varepsilon)) - g(t_1, m(t_1))) \geq 0.$$

Set $\bar{u}(t) = \varphi(u(t, \varepsilon) - m(t)),\ t \in [t_0, t_1].$ Obviously,

$$\bar{u}(t) > 0,\ t \in [t_0, t_1) \quad \text{and} \quad \bar{u}(t_1) = 0.$$

Therefore, $D^{-}\bar{u}(t_1) < 0.$

But

$$\begin{aligned} D^{-}\bar{u}(t_1) &= \varphi(D^{-}u(t_1, \varepsilon) - D^{-}m(t_1)) \\ &> \varphi(g(t_1, u(t_1, \varepsilon)) + \varepsilon\eta - g(t_1, m(t_1))) \\ &> \varphi(g(t_1, u(t_1, \varepsilon)) - g(t_1, m(t_1))) \\ &\geq 0, \end{aligned}$$

a contradiction. So (3.1) holds, thus, Lemma 1 holds. □

Lemma 2. *Assume that*

(i) $g \in C[[t_k, t_{k+1}) \times Z, R_+^n],\quad k \in N,\quad g(t, u)$ *is quasi-monotone nondecreasing in u for each fixed t on the cone Z and* $r(t) = r(t, t_0, u_0)$ *is the maximal solution of system* (2) *on Z*;

(ii) $\Psi_k : Z \to Z(k \in N)$ *is strictly increasing on Z*;

(iii) *for any solution* $x(t) = x(t, t_0, x_0)$ *of system* (1) *and* $V \in V_0$,

$$D^{-}V(t, x(t)) \leq_Z g(t, V(t, x(t))),\quad t \neq t_k,\quad k \in N;$$

(iv) $V(t_k, J_k(x(t_k^{-}))) \leq_Z \Psi_k(V(t_k^{-}, x)).$

Then $V(t_0, x_0) \leq_Z u_0$ *implies* $V(t, x(t)) \leq_Z r(t, t_0, u_0),\ t \geq t_0.$

Proof. For any $t_0 \in R_+$, and $t_0 \in [t_k, t_{k+1})$, for some $k \geq 1$, we designate $t_i = t_{k+i},\ i = 1, 2, \ldots,$ for convenience, then for $t \in [t_0, t_1)$ from Lemma 1, we have $V(t, x(t)) \leq_Z r_1(t, t_0, u_0)$, where $r_1(t, t_0, u_0)$ is the maximal solution of system (2) on $[t_0, t_1)$ such that $r_1(t_0^{+}, t_0, u_0) = u_0$.

So $V(t_1^-, x) \leq_Z r_1(t_1^-, t_0, u_0)$.

Thus, from (iii), we have

$$\begin{aligned} V(t_1, x(t_1, t_0, x_0)) &= V(t_1, J_1(x(t_1^-))) \\ &\leq_Z \Psi_1(V(t_1^-, x)) \\ &\leq_Z \Psi_1(r_1(t_1^-, t_0, u_0)) \\ &= r_1(t_1, t_0, u_0) = r_1. \end{aligned}$$

Again, from Lemma 1, for $t \in [t_1, t_2)$, $V(t, x(t, t_0, x_0)) \leq_Z r_2(t, t_1, r_1)$, where $r_2(t, t_1, r_1)$ is the maximal solution of system (2) on $[t_1, t_2)$ such that $r_2(t_1^+, t_1, r_1) = r_1$.

Therefore, we have

$$V(t, x(t, t_0, x_0)) \leq_Z r_k(t, t_{k-1}, r_{k-1}),$$

where $r_k(t, t_{k-1}, r_{k-1})$ is the maximal solution of system (2) on $[t_{k-1}, t_k)$ such that $r_k(t_{k-1}^+, t_{k-1}, r_{k-1}) = r_{k-1}$.

So, if we define

$$u^*(t) = \begin{cases} u_0, & t = t_0, \\ r_1(t, t_0, u_0), & t \in [t_0, t_1), \\ r_2(t, t_1, r_1), & t \in [t_1, t_2), \\ & \ldots\ldots \\ r_k(t, t_{k-1}, r_{k-1}), & t \in [t_{k-1}, t_k),\ k = 3, 4, \ldots, \end{cases}$$

then $u^*(t)$ is the solution of system (2), and $V(t, x(t)) \leq_Z u^*(t)$, since $r(t, t_0, u_0)$ is the maximal solution of system (2) on Z, we get immediately

$$V(t, x(t)) \leq_Z r(t, t_0, u_0), \quad t \geq t_0. \qquad \square$$

Theorem 1. *Assume that* $\exists a,\, b \in K$ *such that*

(i) $b(|x|) \le (\phi_0, V(t, x)) \le a(|x|),\ x \in S(\rho)$;

(ii) $g \in C[[t_k, t_{k+1}) \times Z, R_+^n],\quad k \in N,\quad g(t, u)$ *is quasi-monotone nondecreasing in u for each fixed t on the cone Z and* $g(t, 0) \equiv 0,\ \ r(t) = r(t, t_0, u_0)$ *is the maximal solution of system* (2) *on Z*;

(iii) $\Psi_k : Z \to Z(k \in N)$ *is strictly increasing on* $Z,\ \Psi_k(0) \equiv 0$;

(iv) *for any solution* $x(t) = x(t, t_0, x_0) \in S(\rho)$ *of system* (1) *and* $V \in V_0$, *we have*

$$D^- V(t, x(t)) \le_Z g(t, V(t, x(t))),\quad t \ne t_k,\ \ k \in N;$$

(v) $V(t_k, J_k(x(t_k^-))) \le_Z \Psi_k(V(t_k^-, x)),\ x \in S(\rho_1)$.

Then the ϕ_0*-stability properties of the trivial solution of system* (2) *imply the corresponding stability properties of the trivial solution of system* (1).

Proof. For any $\varepsilon : 0 < \varepsilon < \rho_1$, every $t_0 \in R_+$, set $V(t) = V(t, x(t))$.

Let the trivial solution of system (2) be ϕ_0-stable. Then, for $b(\varepsilon) > 0$, every $t_0 \in R_+$, there exists $\delta_1 = \delta_1(t_0, \varepsilon)$ such that $0 \le (\phi_0, u_0) < \delta_1$ implies

$$(\phi_0, u(t, t_0, u_0)) < b(\varepsilon),\quad t \ge t_0.$$

Set $\delta_2 = \delta_2(\varepsilon)$ satisfying $a(\delta_2) < \delta_1$.

We define $\delta = \min\{\delta_1, \delta_2\}$.

Next, we claim that $|x_0| < \delta$ implies

$$|x(t)| < \varepsilon,\quad t \ge t_0. \tag{3.2}$$

If it is not true, then there exists a solution $x(t)$ of system (1) such that $x(t_0, t_0, x_0) = x_0$ with $|x_0| < \delta$, then there exists $t^* > t_0$ such that $t_k \le t^* < t_{k+1}(k \in N)$ satisfying $|x(t^*)| \ge \varepsilon$ and $|x(t)| < \varepsilon,\ t_0 \le t < t_k$.

From $0 < \varepsilon < \rho_1$, we have $|x(t_k)| = |J_k(x(t_k^-))| < \rho$, then there exists $\bar{t} \in [t_k, t^*]$ satisfying $\varepsilon \leq |x(\bar{t})| < \rho$.

From (iv), we have $D^- V(t) \leq_Z g(t, V(t)), t \in [t_0, \bar{t}]$.

Set $u_0 = V(t_0)$, thus, from Lemma 2, we get that $t \in [t_0, \bar{t}]$ implies

$$V(t) \leq_Z r(t, t_0, u_0),$$

where $r(t, t_0, u_0)$ is the maximal solution of system (2) such that $u(t_0, t_0, u_0) = u_0$.

And, from (i), $(\phi_0, u_0) = (\phi_0, V(t_0)) \leq a(|x_0|) < a(\delta) < \delta_1$, so from the ϕ_0-stability of the trivial solution of system (2), we have $(\phi_0, r(t, t_0, u_0)) < b(\varepsilon), t \geq t_0$.

Therefore, $b(\varepsilon) \leq b(|x(\bar{t})|) \leq (\phi_0, V(\bar{t})) < (\phi_0, r(\bar{t}, t_0, u_0)) < b(\varepsilon)$, a contradiction.

Then (3.2) holds, thus, the trivial solution of system (1) is stable.

If the trivial solution of system (2) is ϕ_0-uniformly stable, then it is clear that δ will be independent of t_0 and thus we get the uniform stability of the trivial solution of system (1).

Assume that the trivial solution of system (2) is ϕ_0-asymptotically stable, consequently, we get that the trivial solution of system (1) is stable, then, for $\varepsilon = \rho_1$, there exists $\delta_0^* = \delta(t_0, \rho_1)$ such that $|x_0| < \delta_0^*$ implies

$$|x(t)| < \rho_1, \quad t \geq t_0.$$

From (iv), we have $D^- V(t) \leq_Z g(t, V(t)), t \geq t_0$.

For any $\varepsilon : 0 < \varepsilon < \rho_1$, every $t_0 \in R_+$, from the ϕ_0-attractivity of the trivial solution of system (2), we can get that for $b(\varepsilon) > 0$, every $t_0 \in R_+$, there exists $\delta_{10} = \delta_{10}(t_0) > 0$ and $T = T(t_0, \varepsilon) > 0$ such that $0 \leq (\phi_0, u_0)$

$< \delta_{10}$ implies

$$(\phi_0, u(t, t_0, u_0)) < b(\varepsilon), \quad t \geq t_0 + T.$$

Set $\delta_0 = \min\{\delta_0^*, a^{-1}(\delta_{10})\}$.

For $|x_0| < \delta_0$, set $u_0 = V(t_0)$, then, from Lemma 2, we have $V(t) \leq_Z r(t, t_0, u_0), t \geq t_0$.

And, from (i), $(\phi_0, u_0) = (\phi_0, V(t_0)) \leq a(|x_0|) < a(\delta_0) < \delta_{10}$, so from the ϕ_0-attractivity of the trivial solution of system (2), we get that $(\phi_0, r(t, t_0, u_0)) < b(\varepsilon)$, $t \geq t_0 + T$, therefore, $b(|x(t)|) \leq (\phi_0, V(t)) \leq (\phi_0, r(t, t_0, u_0)) < b(\varepsilon)$, $t \geq t_0 + T$, thus, $|x(t)| < \varepsilon$, $t \geq t_0 + T$, so the trivial solution of system (1) is attractive.

Then the trivial solution of system (1) is asymptotically stable.

If the trivial solution of system (2) is ϕ_0-uniformly asymptotically stable, then it is clear that δ_0, T will be independent of t_0, and thus we get the uniform asymptotically stability of the trivial solution of system (1). □

Theorem 2. *Assume that Theorem* 1 (i)-(v) *hold and we have*:

(vi) *for given* $0 < \lambda < A \leq \rho_1$ *with* $a(\lambda) < b(A)$.

Then the ϕ_0*-practical stability properties of the trivial solution of system* (2) *with respect to* $(a(\lambda), b(A))$ *imply the corresponding practical stability properties of the trivial solution of system* (1) *with respect to* (λ, A).

Proof. Set $V(t) = V(t, x(t))$, suppose that the trivial solution of system (2) is ϕ_0-practically stable with respect to $(a(\lambda), b(A))$, then there exists $t_0 \in R_+$ such that $0 \leq (\phi_0, u_0) < a(\lambda)$ implies

$$(\phi_0, u(t, t_0, u_0)) < b(A), \quad t \geq t_0.$$

For above $t_0 \in R_+$, next we will prove that $|x_0| < \lambda$ implies

$$|x(t)| < A, \quad t \geq t_0. \tag{3.3}$$

If it is not true, then there exists a solution $x(t)$ of system (1) such that $x(t_0, t_0, x_0) = 0$ with $|x_0| < \delta$, then there exists $t^* > t_0$ such that $t_k \leq t^* < t_{k+1}$ $(k \in N)$ satisfying $|x(t^*)| \geq A$ and $|x(t)| < A$, $t_0 \leq t < t_k$.

Since $0 < A < \rho_1$, we have $|x(t_k)| = |J_k(x(t_k^-))| < \rho$, then there exists $\bar{t} \in [t_k, t^*]$ satisfying $A \leq |x(\bar{t})| < \rho$.

From (iv), we have $D^- V(t) \leq_Z g(t, V(t))$, $t \in [t_0, \bar{t}]$.

Set $u_0 = V(t_0)$, then from Lemma 2, we get that $t \in [t_0, \bar{t}]$ implies

$$V(t) \leq_Z r(t, t_0, u_0),$$

where $r(t, t_0, u_0)$ is the maximal solution of system (2) such that $u(t_0, t_0, u_0) = u_0$.

And, from (i), $(\phi_0, u_0) = (\phi_0, V(t_0)) \leq a(|x_0|) < a(\lambda)$, so from the ϕ_0-practical stability, we have $(\phi_0, r(t, t_0, u_0)) < b(A)$, $t \geq t_0$.

Then $b(A) \leq b(|x(\bar{t})|) \leq (\phi_0, V(\bar{t})) < (\phi_0, r(\bar{t}, t_0, u_0)) < b(A)$, a contradiction, thus, (3.3) holds, so the trivial solution of system (1) is practically stable.

Suppose that the trivial solution of system (2) is ϕ_0-uniformly practically stable with respect to $(a(\lambda), b(A))$, then it is clear that λ will be independent of t_0, and thus we get the trivial solution of system (1) is uniformly practically stable with respect to (λ, A).

Suppose that the trivial solution of system (2) is ϕ_0-strongly practically stable with respect to $(a(\lambda), b(A), b(B), T)$, consequently, we get that the trivial solution of system (1) is practically stable with respect to (λ, A), then there exists $t_0 \in R_+$ such that $|x_0| < \lambda$ implies

$$|x(t)| < A \leq \rho, \quad t \geq t_0.$$

From (iv), $D^{-}V(t) \leq_Z g(t, V(t)), t \geq t_0$.

And since the trivial solution of system (2) is ϕ_0-practically quasi-stable with respect to $(a(\lambda), b(B), T)$, we have that $0 \leq (\phi_0, u_0) < a(\lambda)$ implies

$$(\phi_0, u(t, t_0, u_0)) < b(B), \quad t \geq t_0 + T.$$

For $|x_0| < \lambda$, set $u_0 = V(t_0)$, then, from Lemma 2, we get $V(t) \leq_Z r(t, t_0, u_0), t \geq t_0$.

And, from (i), $(\phi_0, u_0) = (\phi_0, V(t_0)) \leq a(|x_0|) < a(\lambda)$, so from the ϕ_0-practically quasi-stability of the trivial solution of system (2), we have $(\phi_0, r(t, t_0, u_0)) < b(B)$, $t \geq t_0 + T$, then $b(|x(t)|) \leq (\phi_0, V(t)) \leq (\phi_0, r(t, t_0, u_0)) < b(B)$, $t \geq t_0 + T$, thus $|x(t)| < B$, $t \geq t_0 + T$, so the trivial solution of system (1) is practically quasi-stable with respect to (λ, B, T).

Therefore, the trivial solution of system (1) is strongly practically stable with respect to (λ, A, B, T).

If the trivial solution of system (2) is ϕ_0-strongly uniformly practically stable with respect to (λ, A, B, T), then it is clear that the above proof establishes for every t_0, therefore, we get the trivial solution of system (1) is strongly uniformly practically stable with respect to (λ, A, B, T). □

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