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INTERVAL-VALUED INTUITIONISTIC FUZZY IDEALS OF A RING

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Abstract

In this paper, we discuss and study certain properties of interval-valued intuitionistic fuzzy sets and their level subsets. We introduce a special class of interval-valued intuitionistic fuzzy ideals of a ring having the same tip and prove that it forms a complete sublattice of the lattice of interval-valued intuitionistic fuzzy ideal of a ring. Further a sub-class of the interval-valued intuitionistic fuzzy ideals with the same tip is introduced and shows that it forms a complete modular sublattice of the lattice of interval-valued intuitionistic fuzzy ideal of a ring.

1. Introduction

Zadeh [15] in 1965 introduced the concept of a fuzzy set to describe vagueness mathematically in its very abstractness. In [14], Rosenfeld introduced the notion of fuzzy subgroups and fuzzy ideals. Ajmal and Thomas [1] formulated, the lattice structures of various fuzzy algebraic

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structures and their modularity. Zadeh [16] made an extension of the concept of a fuzzy subset by an interval-valued fuzzy subset, i.e., a fuzzy subset with an interval-valued membership function. In [9], Lee and others applied interval-valued fuzzy sets to ring theory. Atanassov [2] introduced the idea of intuitionistic fuzzy sets which is a generalization of fuzzy sets. The notion of interval-valued intuitionistic fuzzy sets was introduced by Atanassov and Gargov [3]. Interval-valued fuzzy subsets have many application in several areas. Biswas [6] worked on Rosenfeld's fuzzy subgroups with interval-valued membership function. Several mathematicians [4, 5, 7, 8] applied the concept of interval-valued intuitionistic fuzzy sets to algebraic structures.

In our previous papers [10-13], we studied intuitionistic L-fuzzy sets on a ring. In this paper, we investigate the lattice structure of certain types of sublattice of the lattice of interval-valued intuitionistic fuzzy subring of a given ring. We prove that the set of interval-valued intuitionistic fuzzy ideals with sup property and the same tip forms a sublattice of interval-valued intuitionistic fuzzy ideal of a ring. In addition to this, the above sublattice is modular.

2. Preliminaries

Now we list some basic concepts which are applied in this paper.

Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0, 1]$. The elements of $D[0, 1]$ called *interval numbers* are generally denoted by $\hat{\alpha}$, where $\hat{\alpha} = [\alpha^L, \alpha^U]$, where $0 \leq \alpha^L \leq \alpha^U \leq 1$, where α^L and α^U are the lower and upper end points, respectively. The interval $[\alpha, \alpha]$ is identified with the number $\alpha \in [0, 1]$. For interval numbers $\hat{\alpha} = [\alpha^L, \alpha^U]$, $\hat{\beta} = [\beta^L, \beta^U] \in D[0, 1]$, we define

$$\hat{\alpha} \vee \hat{\beta} = [\alpha^L \vee \beta^L, \alpha^U \vee \beta^U],$$

$$\hat{\alpha} \wedge \hat{\beta} = [\alpha^L \wedge \beta^L, \alpha^U \wedge \beta^U],$$

$$\vee_{i \in I} \hat{\alpha}_i = [\vee_{i \in I} \alpha_i^L, \vee_{i \in I} \alpha_i^U],$$

$$\wedge_{i \in I} \hat{\alpha}_i = [\wedge_{i \in I} \alpha_i^L, \wedge_{i \in I} \alpha_i^U], \text{ where } \hat{\alpha}_i = [\alpha_i^L, \alpha_i^U].$$

For any two interval numbers $\hat{\alpha}, \hat{\beta},$ we define

(1) $\hat{\alpha} \leq \hat{\beta} \Leftrightarrow \alpha^L \leq \beta^L$ and $\alpha^U \leq \beta^U$

(2) $\hat{\alpha} = \hat{\beta} \Leftrightarrow \alpha^L = \beta^L$ and $\alpha^U = \beta^U$

(3) $\hat{\alpha} < \hat{\beta} \Leftrightarrow \alpha^L < \beta^L$ and $\alpha^U < \beta^U$.

Now we recall interval-valued intuitionistic fuzzy sets and list some of its basic concepts. We define interval-valued intuitionistic fuzzy subrings and ideals and discuss some of its properties. We introduce a subclass of interval-valued intuitionistic fuzzy ideals of a ring having the same tip. Let R denote a commutative ring with binary operations denoted by "+" and ".". In this paper, we assume that any two interval numbers are comparable.

Definition 2.1. An *interval-valued intuitionistic fuzzy set* in X $(\text{IVIFS}(X))$ is defined as an expression of the form $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in X\}$, where $\hat{\mu}_A : X \to D[0, 1]$, $\hat{\nu}_A : X \to D[0, 1]$ with $\hat{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ and $\hat{\nu}_A(x) = [\nu_A^L(x), \nu_A^U(x)]$ such that $0 \leq \mu_A^L(x) + \nu_A^L(x) \leq 1$ and $0 \leq \mu_A^U(x) + \nu_A^U(x) \leq 1$.

Definition 2.2. Let $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in X\}$ and $B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / x \in X\}$ be $\text{IVIFS}(X)$.

Let $\{A_\alpha\}_{\alpha \in I} \in \text{IVIFS}(X)$. Then

(a) $A \subseteq B \Leftrightarrow \hat{\mu}_A(x) \leq \hat{\mu}_B(x)$ and $\hat{\nu}_A(x) \geq \hat{\nu}_B(x)$ for all $x \in X$,

(b) $A = B \Leftrightarrow \hat{\mu}_A(x) = \hat{\mu}_B(x)$ and $\hat{\nu}_A(x) = \hat{\nu}_B(x)$ for all $x \in X$,

(c) $A^c = \{\langle x, \hat{\mu}_A^c(x), \hat{\nu}_A^c(x)\rangle / x \in X\}$, where

$$\hat{\mu}_A^c(x) = [1 - \mu_A^U(x), 1 - \mu_A^L(x)], \; \hat{\nu}_A^c(x) = [1 - \nu_A^U(x), 1 - \nu_A^L(x)],$$

(d) $A \bigcup B = \{\langle x, (\hat{\mu}_A \vee \hat{\mu}_B)(x), (\hat{\nu}_A \wedge \hat{\nu}_B)(x)\rangle x \in X\}$, where

$$(\hat{\mu}_A \vee \hat{\mu}_B)(x) = [\mu_A^L(x) \vee \mu_B^L(x), \mu_A^U(x) \vee \mu_B^U(x)]$$

and

$$(\hat{\nu}_A \wedge \hat{\nu}_B)(x) = [\nu_A^L(x) \wedge \nu_B^L(x), \nu_A^U(x) \wedge \nu_B^U(x)],$$

(e) $A = \bigcap_{i \in I} A_i = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in X\}$, where

$$\hat{\mu}_A(x) = [\wedge_{i \in I} \mu_{A_i}^L(x), \wedge_{i \in I} \mu_{A_i}^U(x)]$$

and

$$\hat{\nu}_A(x) = [\vee_{i \in I} \nu_{A_i}^L(x), \vee_{i \in I} \nu_{A_i}^U(x)].$$

Definition 2.3. Let $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ be an $\mathrm{IVIFS}(R)$. Then A is called an *interval-valued intuitionistic fuzzy subring* of R $(\mathrm{IVIFSR}(R))$ if for $x, y \in R$,

(i) $\hat{\mu}_A(x - y) \geq \hat{\mu}_A(x) \wedge \hat{\mu}_A(y)$

(ii) $\hat{\mu}_A(xy) \geq \hat{\mu}_A(x) \wedge \hat{\mu}_A(y)$

(iii) $\hat{\nu}_A(x - y) \leq \hat{\mu}_A(x) \vee \hat{\nu}_B(y)$

(iv) $\hat{\nu}_A(xy) \leq \hat{\nu}_A(x) \vee \hat{\nu}_A(y)$.

Proposition 2.4. *Let* $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ *be an* $\mathrm{IVIFSR}(R)$. *Then*

(i) $\hat{\mu}_A(0) \geq \hat{\mu}_A(x)$ *and* $\hat{\nu}_A(0) \leq \hat{\nu}_A(x)$

(ii) $A(-x) = A(x)$ *for all* $x \in R$.

Definition 2.5. Let $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x) / x \in R\rangle\}$ be an $\mathrm{IVIFSR}(R)$. Then

$$\hat{\mu}_A(0) = \sup_{x \in R} \hat{\mu}_A(x), \quad \hat{\nu}_A(0) = \inf_{x \in R} \hat{\nu}_A(x)$$

is called the *tip* of A.

Example 2.6. Consider the commutative ring, $R = (Z_4, +, \cdot)$, where $Z_4 = \{0, 1, 2, 3\}$. Let $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)/x \in Z_4\rangle\}$ be given by

$$\{\langle 0, [0.4, 0.6], [0.1, 0.2]\rangle, \langle 1, [0.2, 0.3], [0.3, 0.5]\rangle,$$

$$\langle 2, [0.4, 0.5], [0.2, 0.3]\rangle, \langle 3, [0.2, 0.3], [0.3, 0.5]\rangle\}.$$

Then A is an $\mathrm{IVIFSR}(R)$.

Proposition 2.7. *Let* $A_i = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)/x \in R\rangle\}$, $i \in I$, *be an* $\mathrm{IVIFSR}(R)$. *Then* $\bigcap_{i\in I} A_i \in IVIFSR(R)$.

Proof. Let $A = \bigcap_{i\in I} A_i = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)/x \in R\rangle\}$, where $\hat{\mu}_A(x) = \wedge_{i\in I} \hat{\mu}_{A_i}(x)$, $\hat{\nu}_A(x) = \vee_{i\in I} \hat{\nu}_{A_i}(x)$.

For $x, y \in R$,

$$\begin{aligned} \hat{\mu}_A(x-y) &= \wedge_{i\in I} \hat{\mu}_{A_i}(x-y) \\ &\geq \wedge_{i\in I}(\hat{\mu}_{A_i}(x) \wedge \hat{\mu}_{A_i}(y)) \\ &= (\wedge_{i\in I}(\hat{\mu}_{A_i}(x))) \wedge (\wedge_{i\in I}(\hat{\mu}_{A_i}(y))) \\ &= \hat{\mu}_A(x) \wedge \hat{\mu}_A(y) \end{aligned}$$

and

$$\begin{aligned} \hat{\mu}_A(xy) &= \wedge_{i\in I} \hat{\mu}_{A_i}(xy) \\ &\geq \wedge_{i\in I}(\hat{\mu}_{A_i}(x) \wedge \hat{\mu}_{A_i}(y)) \\ &= (\wedge_{i\in I}(\hat{\mu}_{A_i}(x))) \wedge (\wedge_{i\in I}(\hat{\mu}_{A_i}(y))) \\ &= \hat{\mu}_A(x) \wedge \hat{\mu}_A(y). \end{aligned}$$

For $x, y \in R$,

$$\begin{aligned} \hat{\nu}_A(x-y) &= \vee_{i\in I} \hat{\nu}_{A_i}(x-y) \\ &\leq \vee_{i\in I}(\hat{\nu}_{A_i}(x) \vee \hat{\nu}_{A_i}(y)) \end{aligned}$$

$$= (\vee_{i \in I} (\hat{\nu}_{A_i}(x))) \vee (\vee_{i \in I} (\hat{\nu}_{A_i}(y)))$$

$$= \hat{\nu}_A(x) \vee \hat{\nu}_A(y)$$

and

$$\begin{aligned} \hat{\nu}_A(xy) &= \vee_{i \in I} \hat{\nu}_{A_i}(xy) \\ &\leq \vee_{i \in I} (\hat{\nu}_{A_i}(x) \vee \hat{\nu}_{A_i}(y)) \\ &= (\vee_{i \in I} (\hat{\nu}_{A_i}(x))) \vee (\vee_{i \in I} (\hat{\nu}_{A_i}(y))) \\ &= \hat{\nu}_A(x) \vee \hat{\nu}_A(y). \end{aligned}$$

Hence $A \in \text{IVIFSR}(R)$. □

Definition 2.8. Let $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ be an $\text{IVIFSR}(R)$. Let $\{A_\alpha\}_{\alpha \in I}$ be the family of $\text{IVIFSR}(R)$ containing A. Then $\bigcap_{\alpha \in I} A_\alpha$ containing A is called the $\text{IVIFSR}(R)$ *generated by* A and denoted by $\langle A \rangle$. It is the smallest $\text{IVIFSR}(R)$ containing A.

The set of $\text{IVIFSR}(R)$ is a poset with respect to $\subseteq$. Define two operations $\vee, \wedge$ on $\text{IVIFSR}(R)$ as follows: for $A, B \in \text{IVIFSR}(R)$, $A \wedge B = A \cap B$ and $A \vee B = \langle A \cup B \rangle$.

Proposition 2.9. *The set* $(\text{IVIFSR}(R), \vee, \wedge)$ *is a complete lattice under the ordering of interval value of intuitionistic fuzzy set*.

Definition 2.10. Let $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ be an $\text{IVIFSR}(R)$. Then A is called an *interval-valued intuitionistic fuzzy ideal* of R $(\text{IVIFI}(R))$ if for $x, y \in R$,

(i) $\hat{\mu}_A(x - y) \geq \hat{\mu}_A(x) \wedge \hat{\mu}_A(y)$

(ii) $\hat{\mu}_A(xy) \geq \hat{\mu}_A(x)$

(iii) $\hat{\nu}_A(x - y) \leq \hat{\nu}_A(x) \vee \hat{\nu}_A(y)$

(iv) $\hat{\nu}_A(xy) \leq \hat{\nu}_A(x)$.

The set of $\mathrm{IVIFI}(R)$ with the same tip is denoted by $\mathrm{IVIFI}_0(R)$. The following result is straightforward.

Proposition 2.11. *Let* $\{A_i\}_{i\in I}$ *be* $\mathrm{IVIFI}(R)$. *Then* $\bigcap_{i\in I} A_i \in \mathrm{IVIFI}(R)$.

Example 2.12. Consider the commutative ring $R = (Z_4, +, \cdot)$, where $Z_4 = \{0, 1, 2, 3\}$. Let $B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / x \in Z_4\}$ be given by

$$\{\langle 0, [0.5, 0.6], [0.1, 0.1]\rangle, \langle 1, [0.2, 0.2], [0.1, 0.2]\rangle,$$

$$\langle 2, [0.3, 0.4], [0.1, 0.1]\rangle, \langle 3, [0.2, 0.2], [0.1, 0.2]\rangle\}.$$

Then B is an $\mathrm{IVIFI}(R)$.

Proposition 2.13. *The set of* $\mathrm{IVIFI}(R)$ *is a complete sublattice of* $\mathrm{IVIFSR}(R)$.

Definition 2.14. Let X be a set and $A \in \mathrm{IVIFS}(X)$. Then A is said to have the *sup property* if for each subset Y of X, there exists a $y_0 \in Y$ such that

$$\hat{\mu}_A(y_0) = \vee_{y\in Y} \hat{\mu}_A(y)$$

and

$$\hat{\nu}_A(y_0) = \wedge_{y\in Y} \hat{\nu}_A(y).$$

Definition 2.15. Let $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ and $B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / x \in R\}$ be IVIFS of R. Then the sum of A and B is defined as follows: for $x, y \in R$,

$$A + B = \{\langle z, \hat{\mu}_{A+B}(z), \hat{\nu}_{A+B}(z)\rangle / z \in R\},$$

where

$$\begin{aligned}\hat{\mu}_{A+B}(z) &= [\mu^L_{A+B}(z), \mu^U_{A+B}(z)] \\ &= [\vee_{z=x+y} (\mu^L_A(x) \wedge \mu^L_B(y)), \vee_{z=x+y} (\mu^U_A(x) \wedge \mu^U_B(y))]\end{aligned}$$

and

$$\hat{\nu}_{A+B}(z) = [\nu^L_{A+B}(z),\ \nu^U_{A+B}(z)]$$

$$= [\wedge_{z=x+y}\ (\nu^L_A(x) \vee \nu^L_B(y)),\ \wedge_{z=x+y}\ (\nu^U_A(x) \vee \nu^U_B(y))].$$

Definition 2.16. Let $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ and $B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / x \in R\}$ be IVIFS(R). Then the composition of A and B is defined as follows: for $x, y \in R$,

$$A \circ B = \{\langle z, \hat{\mu}_{A\circ B}(z), \hat{\nu}_{A\circ B}(z)\rangle / z \in R\},$$

where

$$\hat{\mu}_{A\circ B}(z) = [\mu^L_{A\circ B}(z),\ \mu^U_{A\circ B}(z)]$$

$$= [\vee_{z=xy}\ (\mu^L_A(x) \wedge \mu^L_B(y)),\ \vee_{z=xy}\ (\mu^U_A(x) \wedge \mu^U_B(y))]$$

and

$$\hat{\nu}_{A\circ B}(z) = [\nu^L_{A\circ B}(z),\ \nu^U_{A\circ B}(z)]$$

$$= [\wedge_{z=xy}\ (\nu^L_A(x) \vee \nu^L_B(y)),\ \wedge_{z=xy}\ (\nu^U_A(x) \vee \nu^U_B(y))].$$

Definition 2.17. Let $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ be an IVIFS(R). Then for $\hat{\lambda}, \hat{\mu} \in D[0, 1]$, the set

(i) $A_{[\hat{\lambda}, \hat{\mu}]} = \{x \in X / \hat{\mu}_A(x) \geq \hat{\lambda}$ and $\hat{\nu}_A(x) \leq \hat{\mu}\}$ is called the $[\hat{\lambda}, \hat{\mu}]$ *level subset* of A.

(ii) The set $A_{(\hat{\lambda}, \hat{\mu})} = \{x \in X / \hat{\mu}_A(x) > \hat{\lambda}$ and $\hat{\nu}_A(x) < \hat{\mu}\}$ is called the *strong level subset* of A.

3. Relation Between Ideals and Interval-valued Intuitionistic Fuzzy Ideals

We study some properties of the level subsets of interval-valued intuitionistic fuzzy sets of R and also give its relation with $\text{IVIFSR}(R)$ and $\text{IVIFI}(R)$. Moreover, some properties of interval-valued intuitionistic fuzzy ideals under ring homomorphism are discussed.

Theorem 3.1. *Let* $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ *be an* $\text{IVIFS}(R)$. *Then* $A \in \text{IVIFI}(R)$ $(\text{IVIFSR}(R))$ *iff* $A_{[\hat{\lambda}, \hat{\mu}]}$ *is an ideal* (*subring*) *of* R *for each* $\hat{\lambda}, \hat{\mu} \in D[0, 1]$.

Proof. Let $A \in \text{IVIFI}(R)$. For $x, y \in R$, let $x, y \in A_{[\hat{\lambda}, \hat{\mu}]}$. Then

$$\begin{aligned}\hat{\mu}_A(x + y) &\geq \hat{\mu}_A(x) \wedge \hat{\mu}_A(y)\\ &\geq \hat{\lambda}\end{aligned}$$

and

$$\begin{aligned}\hat{\nu}_A(x + y) &\leq \hat{\nu}_A(x) \vee \hat{\nu}_A(y)\\ &\leq \hat{\mu}.\end{aligned}$$

Hence $x + y \in A_{[\hat{\lambda}, \hat{\mu}]}$.

For $r \in R$ and $x \in A_{[\hat{\lambda}, \hat{\mu}]}$,

$$\hat{\mu}_A(xr) \geq \hat{\mu}_A(x) \geq \hat{\lambda}$$

and

$$\hat{\nu}_A(xr) \leq \hat{\nu}_A(x) \leq \hat{\mu}.$$

Hence $xr \in A_{[\hat{\lambda}, \hat{\mu}]}$. Therefore, $A_{[\hat{\lambda}, \hat{\mu}]}$ is an ideal of R.

Conversely, assume that $A_{[\hat{\lambda}, \hat{\mu}]}$ is an ideal of R. For each $x, y \in R$, let

$$\hat{\lambda} = [\mu_A^L(x) \wedge \mu_A^L(y), \mu_A^U(x) \wedge \mu_A^U(y)]$$

and

$$\hat{\mu} = [\nu_A^L(x) \vee \nu_A^L(y), \nu_A^U(x) \vee \nu_A^U(y)].$$

Let $x, y \in A_{[\hat{\lambda}, \hat{\mu}]}$. Then $x + y \in A_{[\hat{\lambda}, \hat{\mu}]}$. Hence

$$\hat{\mu}_A(x + y) \geq \hat{\lambda} = [\mu_A^L(x) \wedge \mu_A^L(y), \mu_A^U(x) \wedge \mu_A^U(y)]$$

and

$$\hat{\nu}_A(x + y) \leq \hat{\mu} = [\nu_A^L(x) \vee \nu_A^L(y), \nu_A^U(x) \vee \nu_A^U(y)].$$

Let $\hat{\lambda} = \hat{\mu}_A(x)$ and $\hat{\mu} = \hat{\nu}_A(x)$. For $r \in R$,

$$\hat{\mu}_A(xr) \geq \hat{\lambda} = \hat{\mu}_A(x) \text{ and } \hat{\nu}_A(xr) \leq \hat{\mu} = \hat{\nu}_A(x).$$

Hence $A \in \mathrm{IVIFI}(R)$. □

Corollary 3.2. *Let* $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ *be an* $\mathrm{IVIFS}(R)$. *Then A is an* $\mathrm{IVIFI}(R)$ $(\mathrm{IVIFSR}(R))$ *if and only if for all* $\hat{\lambda}, \hat{\mu} \in D[0, 1]$ *with* $\hat{\lambda} + \hat{\mu} \leq 1$, $A_{(\hat{\lambda}, \hat{\mu})}$ *is an ideal (subring) of R.*

$A \in \mathrm{IVIFI}(R)$ $(\mathrm{IVIFSR}(R))$

Theorem 3.3. *If* $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ *is an* $\mathrm{IVIFI}(R)$, *then* $A_{[\hat{\alpha}, \hat{\beta}]} \subseteq A_{[\hat{\nu}, \hat{\delta}]}$ *if* $\hat{\alpha} \geq \hat{\nu}$ *and* $\hat{\beta} \leq \hat{\delta}$, *where* $\hat{\alpha}, \hat{\beta}, \hat{\nu}, \hat{\delta} \in D[0, 1]$.

Proof. Let $x \in A_{[\hat{\alpha}, \hat{\beta}]}$. Then $\hat{\mu}_A(x) \geq \hat{\alpha}$ and $\hat{\nu}_A(x) \leq \hat{\beta}$. But $\hat{\alpha} \geq \hat{\nu}$ and $\hat{\beta} \leq \hat{\delta}$, then $\hat{\mu}_A(x) \geq \hat{\nu}$ and $\hat{\nu}_A(x) \leq \hat{\delta}$. Then $x \in A_{[\hat{\nu}, \hat{\delta}]}$. □

Theorem 3.4. *If* $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / x \in R\}$ *are* $\mathrm{IVIFI}(R)$, *then*

$$(A \cap B)_{[\hat{\alpha}, \hat{\beta}]} = A_{[\hat{\alpha}, \hat{\beta}]} \cap B_{[\hat{\alpha}, \hat{\beta}]}, \text{ where } \hat{\alpha}, \hat{\beta} \in D[0, 1].$$

Proof. We have

$$(A \cap B)_{[\hat{\alpha}, \hat{\beta}]} = \{x \in R / \hat{\mu}_{A \cap B}(x) \geq \hat{\alpha} \text{ and } \hat{\nu}_{A \cap B}(x) \leq \hat{\beta}\}.$$

Now,

$$x \in (A \cap B)_{[\hat{\alpha}, \hat{\beta}]} \Leftrightarrow \hat{\mu}_{A \cap B}(x) \geq \hat{\alpha} \text{ and } \hat{\nu}_{A \cap B}(x) \leq \hat{\beta}$$

$$\Leftrightarrow (\hat{\mu}_A \wedge \hat{\mu}_B)(x) \geq \hat{\alpha} \text{ and } (\hat{\nu}_A \vee \hat{\nu}_B)(x) \leq \hat{\beta}$$

$$\Leftrightarrow \hat{\mu}_A(x) \wedge \hat{\mu}_B(x) \geq \hat{\alpha} \text{ and } \hat{\nu}_A(x) \vee \hat{\nu}_B(x) \leq \hat{\beta}$$

$$\Leftrightarrow \hat{\mu}_A(x) \geq \hat{\alpha},\ \hat{\mu}_B(x) \geq \hat{\alpha} \text{ and } \hat{\nu}_A(x) \leq \hat{\beta},\ \hat{\gamma}_B(x) \leq \hat{\beta}$$

$$\Leftrightarrow x \in A_{[\hat{\alpha}, \hat{\beta}]} \text{ and } x \in B_{[\hat{\alpha}, \hat{\beta}]}$$

$$\Leftrightarrow x \in A_{[\hat{\alpha}, \hat{\beta}]} \cap B_{[\hat{\alpha}, \hat{\beta}]}.$$

Therefore, $(A \cap B)_{[\hat{\alpha}, \hat{\beta}]} = A_{[\hat{\alpha}, \hat{\beta}]} \cap B_{[\hat{\alpha}, \hat{\beta}]}$. □

Theorem 3.5. *Let* $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / x \in R\}$ *be* $\mathrm{IVIFI}(R)$ *such that* $A \subseteq B$. *Then* $A_{[\hat{\alpha}, \hat{\beta}]} \subseteq B_{[\hat{\alpha}, \hat{\beta}]}$, *where* $\hat{\alpha}, \hat{\beta} \in D[0, 1]$.

Proof. Let $x \in A_{[\hat{\alpha}, \hat{\beta}]}$. Then $\hat{\mu}_A(x) \geq \hat{\alpha}$ and $\hat{\nu}_A(x) \leq \hat{\beta}$. Since $A \subseteq B$,

$$\hat{\mu}_B(x) \geq \hat{\mu}_A(x) \geq \hat{\alpha} \quad \text{and} \quad \hat{\nu}_B(x) \leq \hat{\nu}_A(x) \leq \hat{\beta}.$$

Hence $x \in B_{[\hat{\alpha}, \hat{\beta}]}$. Therefore, $A_{[\hat{\alpha}, \hat{\beta}]} \subseteq B_{[\hat{\alpha}, \hat{\beta}]}$. □

Theorem 3.6. *Let* $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / x \in R\}$ *be* $\mathrm{IVIFI}(R)$. *Then* $(A \cup B)_{[\hat{\alpha}, \hat{\beta}]} \supseteq A_{[\hat{\alpha}, \hat{\beta}]} \cup B_{[\hat{\alpha}, \hat{\beta}]}$, *where* $\hat{\alpha}, \hat{\beta} \in D[0, 1]$.

Proof. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$,

$$A_{[\hat{\alpha}, \hat{\beta}]} \subseteq (A \cup B)_{[\hat{\alpha}, \hat{\beta}]} \quad \text{and} \quad B_{[\hat{\alpha}, \hat{\beta}]} \subseteq (A \cup B)_{[\hat{\alpha}, \hat{\beta}]}.$$

Hence $(A \cup B)_{[\hat{\alpha}, \hat{\beta}]} \supseteq A_{[\hat{\alpha}, \hat{\beta}]} \cup B_{[\hat{\alpha}, \hat{\beta}]}$. □

Remark 3.7. Reverse of inclusion in the above theorem does not always hold. Consider the ring $R = \{\mathbb{Z}_4, +, \times\}$. Define two $\text{IVIFI}(R)$

$$A = \{\langle 0, [0.4, 0.6], [0.1, 0.2]\rangle, \langle 1, [0.2, 0.3], [0.3, 0.5]\rangle,$$

$$\langle 2, [0.4, 0.5], [0.2, 0.3]\rangle, \langle 3, [0.2, 0.3], [0.3, 0.5]\rangle\}$$

and

$$B = \{\langle 0, [0.5, 0.6], [0.1, 0.1]\rangle, \langle 1, [0.2, 0.2], [0.1, 0.2]\rangle,$$

$$\langle 2, [0.3, 0.4], [0.1, 0.1]\rangle, \langle 3, [0.2, 0.2], [0.1, 0.2]\rangle\},$$

$$A \cup B = \{\langle 0, [0.5, 0.6], [0.1, 0.1]\rangle, \langle 1, [0.2, 0.3], [0.1, 0.2]\rangle,$$

$$\langle 2, [0.4, 0.5], [0.1, 0.1]\rangle, \langle 3, [0.2, 0.3], [0.1, 0.2]\rangle\}.$$

For $\hat{\alpha} = [0.1, 0.3]$ and $\hat{\beta} = [0.1, 0.2]$, $A_{[\hat{\alpha}, \hat{\beta}]} = \{0\}$, $B_{[\hat{\alpha}, \hat{\beta}]} = \{0, 2\}$ and $(A \cup B)_{[\hat{\alpha}, \hat{\beta}]} = \{0, 1, 2, 3\}$. Hence

$$(A \cup B)_{[\hat{\alpha}, \hat{\beta}]} \not\subseteq A_{[\hat{\alpha}, \hat{\beta}]} \cup B_{[\hat{\alpha}, \hat{\beta}]}.$$

Definition 3.8. Let $f : X \to Y$ be a mapping and $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in X\}$ and $B = \{\langle y, \hat{\mu}_B(y), \hat{\nu}_B(y)\rangle / y \in Y\}$ be IVIFS of X and Y. Then the image

$$f(A) = \{\langle y, f(\hat{\mu}_A)(y), f(\hat{\nu}_A)(y)\rangle / y \in Y\}$$

is defined as

$$f(\hat{\mu}_A)(y) = \begin{cases} \vee (\hat{\mu}_A(x) / x \in X, \ f(x) = y) & \text{if } f^{-1}(y) \neq \phi \\ [0, 0] & \text{otherwise} \end{cases}$$

and

$$f(\hat{\nu}_A)(y) = \begin{cases} \wedge (\hat{\nu}_A(x) / x \in X, \ f(x) = y) & \text{if } f^{-1}(y) \neq \phi \\ [1, 1] & \text{otherwise.} \end{cases}$$

Similarly inverse image

$$f^{-1}(B) = \{\langle x,\ f^{-1}(\hat{\mu}_B)(x),\ f^{-1}(\hat{\nu}_B)(x)\rangle / x \in X\}$$

is defined by

$$f^{-1}(\hat{\mu}_B)(x) = \hat{\mu}_B(f(x))$$

and

$$f^{-1}(\hat{\nu}_B)(x) = \hat{\nu}_B(f(x)) \text{ for all } x \in X.$$

Lemma 3.9. *If* $f : R \to R'$ *is a ring epimorphism and* $A = \{\langle x,\ \hat{\mu}_A(x),\ \hat{\nu}_A(x)\rangle / x \in R\}$ *be an* $\mathrm{IVIFI}(R)$. *Then* $f(A_{(\hat{\alpha}, \hat{\beta})}) = (f(A))_{(\hat{\alpha}, \hat{\beta})}$, *where* $\hat{\alpha}, \hat{\beta} \in D[0, 1]$.

Proof. Let $y \in f(A_{(\hat{\alpha}, \hat{\beta})})$. Then there exists $x_0 \in A_{(\hat{\alpha}, \hat{\beta})}$ such that $y = f(x_0)$. Now $\hat{\mu}_A(x_0) > \hat{\alpha}$ and $\hat{\nu}_A(x_0) < \hat{\beta}$. Hence

$$f(\hat{\mu}_A)(y) = \vee\{\hat{\mu}_A(x) / x \in R,\ f(x) = y\} > \hat{\alpha}$$

and

$$f(\hat{\nu}_A)(y) = \wedge\{\hat{\nu}_A(x) / x \in R,\ f(x) = y\} < \hat{\beta}.$$

Hence $y \in (f(A))_{(\hat{\alpha}, \hat{\beta})}$. Therefore,

$$f(A_{(\hat{\alpha}, \hat{\beta})}) \subseteq (f(A))_{(\hat{\alpha}, \hat{\beta})}.$$

For the reverse inclusion, let $y \in (f(A))_{(\hat{\alpha}, \hat{\beta})}$. Then

$$\begin{aligned} f(\hat{\mu}_A)(y) &= \vee\{\hat{\mu}_A(x) / x \in R,\ f(x) = y\} \\ &> \hat{\alpha} \end{aligned}$$

and

$$\begin{aligned} f(\hat{\nu}_A)(y) &= \wedge\{\hat{\nu}_A(x) / x \in R,\ f(x) = y\} \\ &< \hat{\beta}. \end{aligned}$$

Hence there exists $x_0 \in f^{-1}(y)$ such that $\hat{\mu}_A(x_0) > \hat{\alpha}$ and $\hat{\nu}_A(x_0) < \hat{\beta}$. Hence $x_0 \in A_{(\hat{\alpha}, \hat{\beta})}$ and $y = f(x_0) \in f(A_{(\hat{\alpha}, \hat{\beta})})$. Therefore, $(f(A))_{(\hat{\alpha}, \hat{\beta})} \subseteq f(A_{(\hat{\alpha}, \hat{\beta})})$. Therefore,

$$(f(A))_{(\hat{\alpha}, \hat{\beta})} = f(A_{(\hat{\alpha}, \hat{\beta})}).$$ □

Lemma 3.10. *Let* $f : R \to R'$ *be a ring homomorphism and*

$$B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / x \in R'\}$$

be IVIFI(R'). *Then* $f^{-1}(B_{(\hat{\alpha}, \hat{\beta})}) = (f^{-1}(B))_{(\hat{\alpha}, \hat{\beta})}$, *where* $\hat{\alpha}, \hat{\beta} \in D[0, 1]$.

Proof. Let $x \in R$. Then

$$\begin{aligned} x \in f^{-1}(B_{(\hat{\alpha}, \hat{\beta})}) &\Leftrightarrow f(x) \in B_{(\hat{\alpha}, \hat{\beta})} \\ &\Leftrightarrow \hat{\mu}_B(f(x)) > \hat{\alpha} \text{ and } \hat{\nu}_B(f(x)) < \hat{\beta} \\ &\Leftrightarrow f^{-1}(\hat{\mu}_B)(x) > \hat{\alpha} \text{ and } f^{-1}(\hat{\nu}_B)(x) < \hat{\beta} \\ &\Leftrightarrow x \in (f^{-1}(B))_{(\hat{\alpha}, \hat{\beta})}. \end{aligned}$$

Hence the result. □

Lemma 3.11. *Let* $f : R \to R'$ *be a ring epimorphism and* A, A' *be ideals of R and* R', *respectively. Then* $f(A)$ *and* $f^{-1}(A')$ *are ideals of* R' *and R, respectively*.

Theorem 3.12. *Let* $f : R \to R'$ *be a ring epimorphism and*

$$A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$$

be an IVIFI(R) (IVIFSR(R)) *and* $B = \{\langle y, \hat{\mu}_B(y), \hat{\nu}_B(y)\rangle / y \in R'\}$ *be an IVIFI* (R') (*IVIFSR*(R')). *Then*

(i) $f(A) \in$ *IVIFI* (R')(*IVIFSR* (R'))

(ii) $f^{-1}(B) \in$ *IVIFI* (R)(*IVIFSR* (R)).

Proof. Let $A \in \text{IVIFI}(R)$. Then $A_{(\hat{\alpha}, \hat{\beta})}$ is an ideal of R for $\hat{\alpha}, \hat{\beta} \in D[0, 1]$. Then $f(A_{(\hat{\alpha}, \hat{\beta})})$ is an ideal of R'. But

$$f(A_{(\hat{\alpha}, \hat{\beta})}) = (f(A))_{(\hat{\alpha}, \hat{\beta})}.$$

Hence $(f(A))_{(\hat{\alpha}, \hat{\beta})}$ is an ideal of R'. Therefore, $f(A) \in \text{IVIFI}(R')$.

(ii) Let $B \in \text{IVIFI}(R')$. Then $B_{(\hat{\alpha}, \hat{\beta})}$ is an ideal of R', where $\hat{\alpha}, \hat{\beta} \in D[0, 1]$. Then $f^{-1}(B_{(\hat{\alpha}, \hat{\beta})})$ is an ideal of R. But $f^{-1}(B_{(\hat{\alpha}, \hat{\beta})}) = (f^{-1}(B))_{(\hat{\alpha}, \hat{\beta})}$. Hence $(f^{-1}(B))_{(\hat{\alpha}, \hat{\beta})}$ is an ideal of R. Therefore, $f^{-1}(B) \in \text{IVIFI}(R)$. □

4. Lattice of Interval-valued Intuitionistic Fuzzy Ideal

Here we prove that $\text{IVIFI}_0(R)$ forms a complete sublattice of $\text{IVIFI}(R)$. The sup properties of $\text{IVIFI}(R)$ are studied. We consider an interesting subclass of $\text{IVIFI}_0(R)$ which constitutes a sublattice of $\text{IVIFI}_0(R)$. Here we prove the results for the non-membership function of $\text{IVIFS}(R)$.

Lemma 4.1. *Let* $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / y \in R\}$ *be IVIFI* (R). *If A and B have the sup property, then*

$$(A + B)_{[\hat{\alpha}, \hat{\beta}]} = A_{[\hat{\alpha}, \hat{\beta}]} + B_{[\hat{\alpha}, \hat{\beta}]}$$

for all $\hat{\alpha}, \hat{\beta} \in D[0, 1]$.

Proof. Let $z \in R$. For $z \in (A + B)_{[\hat{\alpha}, \hat{\beta}]}$,

$$\hat{\nu}_{A+B}(z) = [\wedge_{z=x+y} (\nu_A^L(x) \vee \nu_B^L(y)), \wedge_{z=x+y} (\nu_A^U(x)) \vee \nu_B^U(y)] \leq \hat{\beta}. \quad (4.1)$$

Now, for each decomposition $z = x + y$, we have either $\nu_A^L(x) \geq \nu_B^L(y)$ and $\nu_A^U(x) \geq \nu_B^U(y)$ or $\nu_B^L(y) \geq \nu_A^L(x)$ and $\nu_B^U(y) \geq \nu_A^U(x)$.

Hence, we define the following subsets of R:

$$P(z) = \{x \in R,\ z = x + y \text{ for some } y \in R \text{ such that}$$

$$\nu_A^L(x) \geq \nu_B^L(y) \text{ and } \nu_A^U(x) \geq \nu_B^U(y)\},$$

$$Q(z) = \{y \in R,\ z = x + y \text{ for some } x \in R \text{ such that}$$

$$\nu_B^L(y) \geq \nu_A^L(x) \text{ and } \nu_B^U(y) \geq \nu_A^U(x)\},$$

$$P^*(z) = \{x \in R : z = x + y \text{ for some } x \in R \text{ such that}$$

$$\nu_B^L(y) \geq \nu_A^L(x),\ \nu_B^U(y) \geq \nu_A^U(x)\}.$$

Clearly $R = P(z) \cup P^*(z)$, since A, B have the sup property, there exist $x_0 \in P(z)$ and $y_0 \in Q(z)$ such that

and
$$\left.\begin{array}{l} \nu_A^L(x_0) = \wedge_{x \in P(z)} \nu_A^L(x), \quad \nu_A^U(x_0) = \wedge_{x \in P(z)} \nu_A^U(x) \\ \nu_B^L(y_0) = \wedge_{y \in Q(z)} \nu_B^L(y), \quad \nu_B^U(y_0) = \wedge_{y \in Q(z)} \nu_B^U(y) \end{array}\right\}. \tag{4.2}$$

Since $x_0 \in P(z)$, there exists $y_0' \in R$ with $z = x_0 + y_0'$ such that $\nu_A^L(x_0) \geq \nu_B^L(y_0')$ and $\nu_A^U(x_0) \geq \nu_B^U(y_0')$.

Since $y_0 \in Q(z)$, there exists $x_0' \in R$ with $z = x_0' + y_0$ such that $\nu_B^L(y_0) \geq \nu_A^L(x_0')$ and $\nu_B^U(y_0) \geq \nu_A^U(x_0')$.

But for $x_0 \in P(z)$, $y_0 \in Q(z)$, we have either $\nu_A^L(x_0) \geq \nu_B^L(y_0)$ and $\nu_A^U(x_0) \geq \nu_B^U(y_0)$ or $\nu_B^L(y_0) \geq \nu_A^L(x_0)$ and $\nu_B^U(y_0) \geq \nu_A^U(x_0)$.

Case i. Suppose $\nu_A^L(x_0) \geq \nu_B^L(y_0)$ and $\nu_A^U(x_0) \geq \nu_B^U(y_0)$. Then

$$\wedge_{z=x+y} (\nu_A^L(x) \vee \nu_B^L(y))$$

$$= \wedge_{x \in R}(\nu_A^L(x) \vee \nu_B^L(z - x))$$

$$= (\wedge_{x \in P(z)} (\nu_A^L(x) \vee \nu_B^L(z - x))) \wedge (\wedge_{y \in Q(z)} (\nu_A^L(x) \vee \nu_B^L(y)))$$

$$= (\wedge_{x \in P(z)} (\nu_A^L(x))) \wedge (\wedge_{y \in Q(z)} (\nu_B^L(y)))$$

$$= \nu_A^L(x_0) \wedge \nu_B^L(y_0) \text{ (By (4.2))}$$

$$= \nu_B^L(y_0).$$

Similarly

$$\wedge_{z=x+y} [\nu_A^U(x) \vee \nu_A^U(y)] = \nu_B^U(y_0).$$

By (4.1), $\nu_B^L(y_0) \leq \beta^L$ and $\nu_B^U(y_0) \leq \beta^U$.

So $y_0 \in B_{[\hat{\alpha}, \hat{\beta}]}$. Since $\nu_A^L(x_0') \leq \nu_B^L(y_0) \leq \beta^L$ and

$$\nu_A^U(x_0') \leq \nu_B^U(y_0) \leq \beta^U \Rightarrow x_0' \in A_{[\hat{\alpha}, \hat{\beta}]}.$$

Thus $z = x_0' + y_0 \in A_{[\hat{\alpha}, \hat{\beta}]} + B_{[\hat{\alpha}, \hat{\beta}]}$.

Case ii. Suppose $\nu_A^L(x_0) \leq \nu_B^L(y_0)$ and $\nu_A^U(x_0) \leq \nu_B^U(y_0)$. Then as above, it follows that $x_0 \in A_{[\hat{\alpha}, \hat{\beta}]}$ and $y_0' \in B_{[\hat{\alpha}, \hat{\beta}]}$. Thus

$$z = x_0 + y_0' \in A_{[\hat{\alpha}, \hat{\beta}]} + B_{[\hat{\alpha}, \hat{\beta}]}.$$

Hence

$$(A + B)_{[\hat{\alpha}, \hat{\beta}]} \subseteq A_{[\hat{\alpha}, \hat{\beta}]} + B_{[\hat{\alpha}, \hat{\beta}]}.$$

Let $z \in A_{[\hat{\alpha}, \hat{\beta}]} + B_{[\hat{\alpha}, \hat{\beta}]}$. Then there exist $x_0 \in A_{[\hat{\alpha}, \hat{\beta}]}$ and $y_0 \in B_{[\hat{\alpha}, \hat{\beta}]}$ such that $z = x_0 + y_0$. Then $\hat{\nu}_A(x_0) \leq \hat{\beta}$ and $\hat{\nu}_B(y_0) \leq \hat{\beta}$.

For $z \in R$,

$$\hat{\nu}_{A+B}(z) = [\wedge_{z=x+y} (\nu_A^L(x) \vee \nu_B^L(y)), \wedge_{z=x+y} (\nu_A^U(x) \vee \nu_B^U(y))]$$

$$\leq [\beta^L, \beta^U]$$

$$= \hat{\beta}$$

$$\Rightarrow z \in (A+B)_{[\hat{\alpha}, \hat{\beta}]}.$$

Hence $A_{[\hat{\alpha}, \hat{\beta}]} + B_{[\hat{\alpha}, \hat{\beta}]} \subseteq (A+B)_{[\hat{\alpha}, \hat{\beta}]}$. Therefore,

$$(A+B)_{[\hat{\alpha}, \hat{\beta}]} = A_{[\hat{\alpha}, \hat{\beta}]} + B_{[\hat{\alpha}, \hat{\beta}]}. \qquad \square$$

Lemma 4.2. *Let* $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / x \in R\}$ *be IVIFS* (*R*). *Let* $\hat{\alpha}, \hat{\beta} \in D[0, 1]$. *Then*

$$(A+B)_{(\hat{\alpha}, \hat{\beta})} = A_{(\hat{\alpha}, \hat{\beta})} + B_{(\hat{\alpha}, \hat{\beta})}.$$

Proof. Let $z \in (A+B)_{(\hat{\alpha}, \hat{\beta})}$. Then

$$\hat{\nu}_{A+B}(z) = [\wedge_{z=x+y} (\nu_A^L(x) \vee \nu_B^L(y)), \wedge_{z=x+y} (\nu_A^U(x) \vee \nu_B^U(y))]$$

$$< \hat{\beta}.$$

Then there exist $x_0, y_0 \in R$ with $z = x_0 + y_0$ such that $\nu_A^L(x_0) \vee \nu_B^L(y_0) < \beta^L$ and $\nu_A^U(x_0) \vee \nu_B^U(y_0) < \beta^U$. Hence $\nu_A^L(x_0) < \beta^L$, $\nu_B^L(y_0) < \beta^L$ and $\nu_A^U(x_0) < \beta^U$, $\nu_B^U(y_0) < \beta^U$.

Then $x_0 \in A_{(\hat{\alpha}, \hat{\beta})}$ and $y_0 \in B_{(\hat{\alpha}, \hat{\beta})}$. Then $z = x_0 + y_0 \in A_{(\hat{\alpha}, \hat{\beta})} + B_{(\hat{\alpha}, \hat{\beta})}$. Hence

$$(A+B)_{(\hat{\alpha}, \hat{\beta})} \subset A_{(\hat{\alpha}, \hat{\beta})} + B_{(\hat{\alpha}, \hat{\beta})}.$$

Let $z \in A_{(\hat{\alpha}, \hat{\beta})} + B_{(\hat{\alpha}, \hat{\beta})}$. Then there exist $x_0 \in A_{(\hat{\alpha}, \hat{\beta})}$ and $y_0 \in B_{(\hat{\alpha}, \hat{\beta})}$ such that $z = x_0 + y_0$ and

$$\hat{\nu}_{A+B}(z) = [\wedge_{z=x+y} (\nu_A^L(x) \vee \nu_B^L(y)), \wedge_{z=x+y} (\nu_A^U(x) \vee \nu_B^U(y))]$$

$$\leq [\nu_A^L(x_0) \vee \nu_B^L(y_0), \nu_A^U(x_0) \vee \nu_B^U(y_0)]$$

$$< \hat{\beta}.$$

Then $z \in (A+B)_{(\hat{\alpha}, \hat{\beta})}$. Hence

$$A_{(\hat{\alpha}, \hat{\beta})} + B_{(\hat{\alpha}, \hat{\beta})} \subset (A+B)_{(\hat{\alpha}, \hat{\beta})}.$$

Therefore,

$$(A+B)_{(\hat{\alpha}, \hat{\beta})} = A_{(\hat{\alpha}, \hat{\beta})} + B_{(\hat{\alpha}, \hat{\beta})}. \qquad \square$$

Proposition 4.3. *Let* $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / x \in R\}$ *be* $IVIFI_0(R)$*. Then* $A \vee B = A + B$*.*

Proof. Let $\hat{\mu}_A(0) = \hat{\mu}_B(0) = \hat{\alpha}_0$ and $\hat{\nu}_A(0) = \hat{\nu}_B(0) = \hat{\beta}_0$, where $\hat{\alpha}, \hat{\beta}_0 \in D[0, 1]$. Let $z \in R$. Then

$$\begin{aligned}\hat{\nu}_{A+B}(z) &= \wedge_{z=x+y}[\hat{\nu}_A(x) \vee \hat{\nu}_B(y)] \\ &\geq [\nu_A^L(0) \vee \nu_B^L(0), \nu_A^U(0) \vee \nu_B^U(0)] \\ &= \hat{\beta}_0.\end{aligned}$$

Hence $\wedge_{z \in R}\, \hat{\nu}_{A+B}(z) \geq \hat{\beta}_0$. Also,

$$\begin{aligned}\wedge_{z \in R}\, (\hat{\nu}_{A+B}(z)) &\leq \hat{\nu}_{A+B}(0) \\ &= \wedge_{0=x+y}[\hat{\nu}_A(x) \vee \hat{\nu}_B(y)] \\ &= [\wedge_{0=x+y}\, (\nu_A^L(x) \vee \nu_B^L(y)), \wedge_{0=x+y}\, (\nu_A^U(x) \vee \nu_B^U(y))] \\ &\leq [\nu_A^L(0) \vee \nu_B^L(0), \nu_A^U(0) \vee \nu_B^U(0)] \\ &= \hat{\beta}_0.\end{aligned}$$

Hence $\hat{\nu}_{A+B}(0) = \hat{\beta}_0$.

For each $\hat{\alpha}, \hat{\beta} \in D[0, 1]$ with $\hat{\alpha} < \hat{\alpha}_0$ and $\hat{\beta} > \hat{\beta}_0$, $(A+B)_{(\hat{\alpha}, \hat{\beta})} \neq \phi$. By Lemma 4.2,

$$(A+B)_{(\hat{\alpha}, \hat{\beta})} = A_{(\hat{\alpha}, \hat{\beta})} + B_{(\hat{\alpha}, \hat{\beta})}.$$

Since $A, B \in \text{IVIFI}(R)$, $A_{(\hat{\alpha}, \hat{\beta})}$, $B_{(\hat{\alpha}, \hat{\beta})}$ are ideals of R. Hence $(A+B)_{(\hat{\alpha}, \hat{\beta})}$ is an ideal of R. Therefore, $(A+B) \in \text{IVIFI}(R)$. Moreover, $A+B \in \text{IVIFI}_0(R)$.

Let $z \in R$. Then

$$\nu^L_{A+B}(z) = \wedge_{z=x+y}(\nu^L_A(x) \vee \nu^L_B(y))$$

$$\leq \nu^L_A(z) \vee \nu^L_B(0)$$

$$= \nu^L_A(z)$$

and

$$\nu^U_{A+B}(z) = \wedge_{z=x+y}(\nu^U_A(x) \vee \nu^U_B(y))$$

$$\leq \nu^U_A(z) \vee \nu^U_B(0)$$

$$= \nu^U_A(z).$$

Hence $A \subseteq A+B$, similarly $B \subseteq A+B$. Let $C \in \text{IVIFI}_0(R)$ such that $A \subseteq C$ and $B \subseteq C$. Let $z \in R$. Then

$$\hat{\nu}_{A+B}(z) = [\wedge_{z=x+y}(\nu^L_A(x) \vee \nu^L_B(y)), \wedge_{z=x+y}(\nu^U_A(x) \vee \nu^U_B(y))]$$

$$\geq [\wedge_{z=x+y}(\nu^L_C(x) \vee \nu^L_C(y)), \wedge_{z=x+y}(\nu^U_C(x) \vee \nu^U_C(y))]$$

$$= \hat{\nu}_C(z).$$

Thus $A+B \subseteq C$. Therefore, $A \vee B = A+B$.

The following result is straightforward. □

Proposition 4.4. *Let* $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / x \in R\}$ *be* $IVIFI_0(R)$. *Then* $A \cap B \in IVIFI_0(R)$.

The set of $\text{IVIFI}_0\,(R)$ is a poset with respect to "$\subseteq$". Define two operations $\vee,\ \wedge$ on $\text{IVIFI}_0\,(R)$ as follows: for $A,\ B \in \text{IVIFI}_0\,(R)$,

$$A \wedge B = A \cap B \text{ and } A \vee B = A + B.$$

Theorem 4.5. *The set* $(IVIFI_0(R), \vee, \wedge)$ *is a complete sublattice of* $IVIFI\,(R)$.

Lemma 4.6. *Let* $A = \{\langle x,\ \hat{\mu}_A(x),\ \hat{\nu}_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x,\ \hat{\mu}_B(x),\ \hat{\nu}_B(x)\rangle / x \in R\}$ *be* $IVIFI\,(R)$. *If A and B have sup property, then*

(i) $A + B$ *has the sup property*.

(ii) $A \cap B$ *has the sup property*.

Proof. (i) Let S be any subset of R. Then

$$\wedge_{z\in S}\ \hat{\nu}_{A+B}(z)$$

$$= \wedge_{z\in S}[\wedge_{z=x+y}\ (\nu_A^L(x) \vee \nu_B^L(y)),\ \wedge_{z=x+y}\ (\nu_A^U(x) \vee \nu_B^U(y))]$$

$$= \wedge_{z=x+y}[(\nu_A^L(x) \vee \nu_B^L(y)),\ \wedge_{z\in x+y}\ (\nu_A^U(x) \vee \nu_B^U(y))].$$

This leads us to define two subsets $P(S)$ and $Q(S)$ of R by

$$P(S) = \{x \in R : z \in S,\ z = x + y \text{ for some } y \in R$$
$$\text{such that } \nu_A^L(x) \geq \nu_B^L(y),\ \nu_A^U(x) \geq \nu_B^U(y)\}$$

and

$$Q(S) = \{y \in R : z \in S,\ z = x + y \text{ for some } x \in R$$
$$\text{such that } \nu_A^L(x) \leq \nu_B^L(y),\ \nu_A^U(x) \leq \nu_B^U(y)\}.$$

Since A and B have sup property, there exist $x' \in P(S)$ and $y'' \in Q(S)$ such that

$$\hat{\mu}_A(x') = [\vee_{x \in P(S)}\ \mu_A^L(x),\ \vee_{x \in P(S)}\ \mu_A^U(x)],$$

$$\hat{\nu}_A(x') = [\wedge_{x \in P(S)}\ \nu_A^L(x),\ \wedge_{x \in P(S)}\ \nu_A^U(x)]$$

and

$$\hat{\mu}_B(y'') = [\vee_{y \in Q(S)}\ \mu_B^L(y),\ \vee_{y \in Q(S)}\ \mu_B^U(y)],$$

$$\hat{\nu}_B(y'') = [\wedge_{y \in Q(S)}\ \nu_B^L(y),\ \wedge_{y \in Q(S)}\ \nu_B^U(y)].$$

Since $x' \in P(S)$, there exists $z' \in S$ such that $z' = x' + y'$ for some $y' \in R$ satisfying $\nu_A^L(x') \geq \nu_B^L(y')$ and $\nu_A^U(x') \geq \nu_B^L(y')$. Also, since $y'' \in Q(S)$, there exists $z'' \in S$ such that $z'' = x'' + y''$ for some $x'' \in R$ satisfying

$$\nu_A^L(x'') \leq \nu_B^L(y'') \text{ and } \nu_A^U(x'') \leq \nu_B^U(y'').$$

But we have either

$$\nu_A^L(x') \leq \nu_B^L(y'') \text{ and } \nu_A^U(x') \leq \nu_B^U(y'')$$

or

$$\nu_A^L(x') \geq \nu_B^L(y'') \text{ and } \nu_A^U(x') \geq \nu_B^U(y'').$$

Case i. Suppose

$$\nu_A^L(x') \leq \nu_B^L(y'') \text{ and } \nu_A^U(x') \leq \nu_B^U(y'').$$

Then

$$\wedge_{\substack{z \in S \\ z = x+y}} (\nu_A^L(x) \vee \nu_B^L(y))$$

$$= (\wedge_{x \in P(S)} (\nu_A^L(x) \vee \nu_B^L(y))) \wedge (\wedge_{x \in Q(S)} (\nu_A^L(x) \vee \nu_B^L(y)))$$

$$= (\wedge_{x\in P(S)} (\nu_A^L(x))) \wedge (\wedge_{y\in Q(S)} (\nu_B^L(y)))$$

$$= \nu_A^L(x') \wedge \nu_B^L(y'')$$

$$= \nu_A^L(x').$$

Similarly we have

$$\wedge_{\substack{z\in S \\ z=x+y}} (\nu_A^U(x) \vee \nu_B^U(y)) = \nu_A^U(x').$$

Thus

$$\nu_A^L(x') = \wedge_{z\in S} \nu_{A+B}^L(z),$$

$$\nu_A^U(x') = \wedge_{z\in S} \nu_{A+B}^U(z). \tag{4.3}$$

Here we claim that for $z' \in R$,

$$\wedge_{z\in S} \nu_{A+B}^L(z) = \nu_{A+B}^L(z')$$

and

$$\wedge_{z\in S} \nu_{A+B}^U(z) = \nu_{A+B}^U(z').$$

For decomposition $z' = x_i' + y_i'$, we have

$$\nu_{A+B}^L(z') = \wedge_{z=x_i'+y_i'} (\nu_A^L(x_i') \vee \nu_B^L(y_i))$$

and

$$\nu_{A+B}^U(z') = \wedge_{z'=x_i'+y_i'} (\nu_A^U(x_i') \vee \nu_B^U(y_i)).$$

Again we construct subsets $P(z')$, $Q(z')$ of R as follows:

$$P(z') = \{x_i' \in R : z' = x_i' + y_i' \text{ for some } y_i' \in R$$

$$\text{such that } \nu_A^L(x_i') \geq \nu_B^L(y_i') \text{ and } \nu_A^U(x_i') \geq \nu_B^U(y_i')\},$$

$$Q(z') = \{y'_i \in R : z' = x'_i + y'_i \text{ for some } x'_i \in R$$

$$\text{such that } \nu^L_A(x'_i) \leq \nu^L_B(y'_i) \text{ and } \nu^U_A(x'_i) \leq \nu^U_B(y'_i)\}.$$

Then

$$\nu^L_{A+B}(z') = \wedge_{z'=x'_i+y'_i}(\nu^L_A(x'_i) \vee \nu^L_B(y'_i))$$

$$= (\wedge_{x'_i \in P(z')}(\nu^L_A(x'_i) \vee \nu^L_B(y'_i))) \wedge (\wedge_{y'_i \in Q(z')}(\nu^L_A(x'_i) \vee \nu^L_B(y'_i)))$$

$$= (\wedge_{x'_i \in P(z')}(\nu^L_A(x'_i))) \wedge (\wedge_{y'_i \in Q(z')} \nu^L_B(y'_i)).$$

Similarly we have

$$\nu^U_{A+B}(z') = \wedge_{z'=x'_i+y'_i}(\nu^U_A(x'_i) \vee \nu^U_B(y'_i))$$

$$= (\wedge_{x_i \in P(z')}(\nu^U_A(x'_i))) \wedge (\wedge_{y'_i \in Q(z')}(\nu^U_B(y'_i))).$$

Since $P(z') \subset P(S)$ and $x' \in P(z')$, we have

$$\nu^L_A(x') \geq \wedge_{x'_i \in P(z')}\nu^L_A(x'_i) \geq \wedge_{x \in P(S)}\nu^L_A(x) = \nu^L_A(x')$$

and

$$\nu^U_A(x') \geq \wedge_{x'_i \in P(z')}\nu^U_A(x'_i) \geq \wedge_{x \in P(S)}\nu^U_A(x) = \nu^U_A(x').$$

Thus

$$\wedge_{x'_i \in P(z')} \nu^L_A(x'_i) = \nu^L_A(x') \text{ and } \wedge_{x'_i \in P(z')} \nu^U_A(x'_i) = \nu^U_A(x'). \tag{4.4}$$

Also, since $Q(z') \subset Q(S)$, we have

$$\nu^L_B(y'') = \wedge_{y'_i \in Q(S)}\nu^L_B(y'_i) \leq \wedge_{y'_i \in Q(z')}\nu^L_B(y'_i)$$

and

$$\nu^U_B(y'') = \wedge_{y'_i \in Q(S)}\nu^U_B(y'_i) \leq \wedge_{y'_i \in Q(z')}\nu^U_B(y'_i).$$

By case (i), we have

$$\wedge_{x_i' \in P(z')} \nu_A^L(x_i') = \nu_A^L(x') \leq \nu_B^L(y'') \leq \wedge_{y_i' \in Q(z')} \nu_B^L(y_i')$$

and

$$\wedge_{x_i' \in P(z')} \nu_A^U(x_i') = \nu_A^U(x') \leq \nu_B^U(y'') \leq \wedge_{y_i' \in Q(z')} \nu_B^U(y_i').$$

Thus

$$\begin{aligned} \nu_{A+B}^L(z') &= (\wedge_{x_i' \in P(z')} (\nu_A^L(x_i')) \wedge (\wedge_{y_i' \in Q(z')} (\nu_B^L(y_i')))) \\ &= \wedge_{x_i' \in P(z')} \nu_A^L(x_i') \\ &= \nu_A^L(x') \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \nu_{A+B}^U(z') &= (\wedge_{x_i' \in P(z')} (\nu_A^U(x_i')) \wedge (\wedge_{y_i' \in Q(z')} (\nu_B^U(y_i')))) \\ &= \wedge_{x_i' \in P(z')} \nu_A^U(x_i') \\ &= \nu_A^U(x'). \end{aligned} \tag{4.6}$$

From (4.3), (4.5) and (4.6),

$$\nu_{A+B}^L(z') = \wedge_{z \in S} \nu_{A+B}^L(z) \text{ and } \nu_{A+B}^U(z') = \wedge_{z \in S} \nu_{A+B}^U(z).$$

Case ii. Suppose, let $\nu_A^L(x') \geq \nu_B^L(y'')$ and $\nu_A^U(x') \geq \nu_B^U(y'')$. We can verify that

$$\nu_{A+B}^L(z'') = \wedge_{z \in S} \nu_{A+B}^L(z) \text{ and } \nu_{A+B}^U(z'') = \wedge_{z \in S} \nu_{A+B}^U(z)$$

for all $z'' \in S$. Hence $A + B$ has the sup property.

(ii) Let S be any subset of R. Then

$$\hat{\nu}_{A \cap B}(x) = [\nu_A^L(x) \vee \nu_B^L(x),\ \nu_A^U(x) \vee \nu_B^U(x)].$$

Define subsets $P(S), Q(S)$ of R as follows:

$$P(S) = \{x \in R : \nu_A^L(x) \geq \nu_B^L(x), \nu_A^U(x) \geq \nu_B^U(x)\},$$

$$Q(S) = \{x \in R : \nu_B^L(x) \geq \nu_A^L(x), \nu_B^U(x) \geq \nu_A^U(x)\}.$$

Then $R = P(S) \bigcup Q(S)$. A and B have sup property, there exists $x_0 \in P(S)$ and $y_0 \in Q(S)$ such that

$$\nu_A^L(x_0) = \wedge_{x \in P(S)} \nu_A^L(x), \nu_A^U(x_0) = \wedge_{x \in P(S)} \nu_A^U(x)$$

and

$$\nu_B^L(y_0) = \wedge_{y \in Q(S)} \nu_B^L(y), \nu_B^U(y_0) = \wedge_{y \in Q(S)} \nu_B^U(y).$$

Case (i). Suppose $\nu_A^L(x_0) \leq \nu_B^L(y_0)$ and $\nu_A^U(x_0) \leq \nu_B^U(y_0)$. Then

$$\begin{aligned}
&\wedge_{x \in S} \nu_{A \bigcap B}^L(x) \\
&= \wedge_{x \in S} (\nu_A^L(x) \vee \nu_B^L(x)) \\
&= (\wedge_{x \in P(S)} (\nu_A^L(x) \vee \nu_B^L(x))) \wedge (\wedge_{x \in Q(S)} (\nu_A^L(x) \vee \nu_B^L(x))) \\
&= (\wedge_{x \in P(S)} (\nu_A^L(x))) \wedge (\wedge_{x \in Q(S)} (\nu_B^L(x))) \\
&= \nu_A^L(x_0) \wedge \nu_B^L(y_0) \\
&= \nu_A^L(x_0).
\end{aligned}$$

Similarly, we have

$$\wedge_{x \in S} \nu_{A \bigcap B}^U(x) = \wedge_{x \in S} (\nu_A^U(x) \vee \nu_B^U(x)) = \nu_A^U(x_0).$$

Thus

$$\wedge_{x \in S} \nu_{A \bigcap B}^L(x) = \nu_A^L(x_0) \quad \text{and} \quad \wedge_{x \in S} \nu_{A \bigcap B}^U(x) = \nu_A^U(x_0).$$

Because $x_0 \in P(S)$, therefore,

$$\nu_A^L(x_0) = \nu_A^L(x_0) \vee \nu_B^L(x_0) = \nu_{A \cap B}^L(x_0)$$

and

$$\nu_A^U(x_0) = \nu_A^U(x_0) \vee \nu_B^U(x_0) = \nu_{A \cap B}^U(x_0).$$

Then

$$\wedge_{x \in P(S)} \nu_{A \cap B}^L(x) = \nu_A^L(x_0) = \nu_{A \cap B}^L(x_0)$$

and

$$\wedge_{x \in P(S)} \nu_{A \cap B}^U(x) = \nu_A^U(x_0) = \nu_{A \cap B}^U(x_0).$$

Case ii. Suppose let $\nu_A^L(x_0) \geq \nu_B^L(y_0)$ and $\nu_A^U(x_0) \geq \nu_B^U(y_0)$. As above, we can verify that

$$\nu_{A \cap B}^L(y_0) = \wedge_{y \in Q(S)} \nu_{A \cap B}^L(y)$$

and

$$\nu_{A \cap B}^U(y_0) = \wedge_{y \in Q(S)} \nu_{A \cap B}^U(y).$$

Hence $A \cap B$ has the sup property. □

Let the set of all $\mathrm{IVIFI}(R)$ with sup property and the same tip be denoted as $\mathrm{IVIFI}_0\mathrm{s}(R)$.

Proposition 4.7. *$IVIFI_0 s(R)$ forms a sub-lattice of $IVIFI_0(R)$ and hence of $IVIFI(R)$.*

Proof. Let $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in R\}$ and $B = \{\langle x, \hat{\mu}_B(x), \hat{\nu}_B(x)\rangle / x \in R\}$ be $\mathrm{IVIFI}_0\mathrm{s}(R)$ with tip $\hat{\alpha}_0, \hat{\beta}_0 \in D[0, 1]$. For each $\hat{\alpha}, \hat{\beta} \in D[0, 1]$, $A_{[\hat{\alpha}, \hat{\beta}]}$, $B_{[\hat{\alpha}, \hat{\beta}]}$ are ideals of R. Then $A_{[\hat{\alpha}, \hat{\beta}]} + B_{[\hat{\alpha}, \hat{\beta}]}$ is an ideal of R.

Since A and B have sup property, by Lemma 4.1

$$A_{[\hat{\alpha},\hat{\beta}]} + B_{[\hat{\alpha},\hat{\beta}]} = (A+B)_{[\hat{\alpha},\hat{\beta}]}.$$

Hence $A + B \in \mathrm{IVIFI_0s}(R)$.

For $z \in S \subset R$,

$$\begin{aligned}\nu^L_{A+B}(z) &= \wedge_{z=x+y}(\nu^L_A(x) \vee \nu^L_B(y)) \\ &\leq \nu^L_A(z) \vee \nu^L_B(0) \\ &= \nu^L_A(z)\end{aligned}$$

and

$$\begin{aligned}\nu^U_{A+B}(z) &= \wedge_{z=x+y}(\nu^U_A(x) \vee \nu^U_B(y)) \\ &\leq \nu^U_A(z) \vee \nu^U_B(0) \\ &= \nu^U_A(z).\end{aligned}$$

Then $A \subseteq A + B$. Similarly $B \subseteq A + B$. Let $C \in \mathrm{IVIFI_0s}(R)$ such that $A \subseteq C$ and $B \subseteq C$. Let $z \in S$ such that $z = x + y$. Then

$$\nu^L_C(z) = \nu^L_C(x+y) \leq \nu^L_C(x) \vee \nu^L_C(y)$$

and

$$\nu^U_C(z) = \nu^U_C(x+y) \leq \nu^U_C(x) \vee \nu^U_C(y).$$

Thus

$$\begin{aligned}\nu^L_{A+B}(z) &= \wedge_{z=x+y}(\nu^L_A(x) \vee \nu^L_B(y)) \\ &\geq \wedge_{z=x+y}(\nu^L_C(x) \vee \nu^L_C(y)) \\ &= \nu^L_C(z)\end{aligned}$$

and

$$\begin{aligned} \nu_{A+B}^{U}(z) &= \wedge_{z=x+y}(\nu_{A}^{U}(x) \vee \nu_{B}^{U}(y)) \\ &\geq \wedge_{z=x+y}(\nu_{C}^{U}(x) \vee \nu_{C}^{U}(y)) \\ &= \nu_{C}^{U}(z). \end{aligned}$$

So $A + B \subseteq C$. Hence $A + B$ is the least $\text{IVIFI}_0\text{s}(R)$ containing A and B. Therefore, $A \vee B = A + B$ and $A \cap B \in \text{IVIFI}_0\text{s}(R)$. Thus the set of $\text{IVIFI}_0\text{s}(R)$ forms a sublattice of $\text{IVIFI}_0(R)$ and hence of $\text{IVIFI}(R)$. □

5. Modularity

In this section, we prove that the interval-valued intuitionistic fuzzy ideals on a ring with sup property and same tip form a complete modular lattice.

Lemma 5.1. *Let* $A = \{\langle x, \hat{\mu}_A(x), \hat{\nu}_A(x)\rangle / x \in X\}$ *be an* $\text{IVIFSR}(R)$. *If* $\hat{\mu}_A(x) < \hat{\mu}_A(y)$ *and* $\hat{\nu}_A(x) > \hat{\nu}_A(y)$ *for some* $x, y \in R$, *then*

$$\hat{\mu}_A(x+y) = \hat{\mu}_A(x) \text{ and } \hat{\nu}_A(x+y) = \hat{\nu}_A(x).$$

Proof. For $x, y \in R$,

$$\nu_A^L(x+y) \leq \nu_A^L(x) \vee \nu_A^L(y) = \nu_A^L(x)$$

and

$$\nu_A^U(x+y) \leq \nu_A^U(x) \vee \nu_A^U(y) = \nu_A^U(x).$$

Assume that $\nu_A^L(x+y) < \nu_A^L(x)$ and $\nu_{A+B}^U(x+y) < \nu_A^U(x)$. Then

$$\begin{aligned} \nu_A^L(x) &= \nu_A^L(x+y-y) \leq \nu_A^L(x+y) \vee \nu_A^L(y) \\ &< \nu_A^L(x) \end{aligned}$$

and

$$\nu_A^U(x) = \nu_A^U(x+y-y) \leq \nu_A^U(x+y) \vee \nu_A^U(y)$$

$$< \nu_A^U(x).$$

This is a contradiction. Hence

$$\nu_A^L(x+y) \geq \nu_A^L(x)$$

and

$$\nu_A^U(x+y) \geq \nu_A^U(x).$$

Therefore,

$$\hat{\nu}_A(x+y) = \hat{\nu}_A(x). \qquad \square$$

Theorem 5.2. *The sublattice $IVIFI_0s(R)$ of $IVIFI(R)$ is modular.*

Proof. Let $A, B, C \in \mathrm{IVIFI}_0\mathrm{s}(R)$ such that $B \subset A$. Then by modular inequality

$$B \vee (C \wedge A) \subset (B \vee C) \wedge A \tag{5.1}$$

holds. Assume that

$$B \vee (C \wedge A) \neq (B \vee C) \wedge A.$$

Let $z \in R$. Then

$$(\nu_B \vee (\nu_C \wedge \nu_A))^L(z) > ((\nu_B \vee \nu_C) \wedge \nu_A)^L(z)$$

and

$$(\nu_B \vee (\nu_C \wedge \nu_A))^U(z) > ((\nu_B \vee \nu_C) \wedge \nu_A)^U(z).$$

By Proposition 4.7,

$$(\nu_B + \nu_C)^L(z) \vee \nu_A^L(z) < (\nu_B + (\nu_C \vee \nu_A))^L(z)$$

and

$$(\nu_B + \nu_C)^U(z) \vee \nu_A^U(z) < (\nu_B + (\nu_C \vee \nu_A))^U(z).$$

So

$$\nu_A^L(z) < (\nu_B + (\nu_C \vee \nu_A))^L(z),$$

$$\nu_A^U(z) < (\nu_B + (\nu_C \vee \nu_A))^U(z) \tag{5.2}$$

and

$$(\nu_B + \nu_C)^L(z) < (\nu_B + (\nu_C \vee \nu_A))^L(z),$$

$$(\nu_B + \nu_C)^U(z) < (\nu_B + (\nu_C \vee \nu_A))^U(z).$$

Then there exists $x_0,\ y_0 \in R$ with $z = x_0 + y_0$ such that

$$\nu_B^L(x_0) \vee \nu_C^L(y_0) < (\nu_B + (\nu_C \vee \nu_A))^L(z)$$

and

$$\nu_B^U(x_0) \vee \nu_C^U(y_0) < (\nu_B + (\nu_C \vee \nu_A))^U(z).$$

Thus

$$\nu_B^L(x_0) < (\nu_B + (\nu_C \vee \nu_A))^L(z),$$

$$\nu_B^U(x_0) < (\nu_B + (\nu_C \vee \nu_A))^U(z) \tag{5.3}$$

and

$$\nu_C^L(y_0) < (\nu_B + (\nu_C \vee \nu_A))^L(z),$$

$$\nu_C^U(y_0) < (\nu_B + (\nu_C \vee \nu_A))^U(z). \tag{5.4}$$

For $z \in R$,

$$(\nu_B + (\nu_C \vee \nu_A))^L(z) = \wedge_{z=x+y}(\nu_B^L(x) \vee (\nu_C \vee \nu_A)^L(y))$$

$$= \wedge_{z=x+y}(\nu_B^L(x) \vee \nu_C^L(y) \vee \nu_A^L(y))$$

$$\leq \nu_B^L(x_0) \vee \nu_C^L(y_0) \vee \nu_A^L(y_0). \tag{5.5}$$

Similarly for $z \in R$,

$$(\nu_B + (\nu_C \vee \nu_A))^U(z) \leq \nu_B^U(x_0) \vee \nu_C^U(y_0) \vee \nu_A^U(y_0).$$

Hence (from (5.2), (5.3) and (5.4))

$$\nu_A^L(z),\ \nu_B^L(x_0),\ \nu_C^L(y_0) < \nu_B^L(x_0) \vee \nu_C^L(y_0) \vee \nu_A^L(y_0)$$

and

$$\nu_A^U(z),\ \nu_B^U(x_0),\ \nu_C^U(y_0) < \nu_B^U(x_0) \vee \nu_C^U(y_0) \vee \nu_A^U(y_0).$$

Now it follows that

$$\nu_A^L(y_0) = \nu_B^L(x_0) \vee \nu_C^L(y_0) \vee \nu_A^L(y_0)$$

and

$$\nu_A^U(y_0) = \nu_B^U(x_0) \vee \nu_C^U(y_0) \vee \nu_A^U(y_0). \tag{5.6}$$

Hence by (5.2), $\nu_A^L(z) < \nu_A^L(y_0)$ and $\nu_A^U(z) < \nu_A^U(y_0)$.

Therefore, $\nu_A^L(-y_0) = \nu_A^L(y_0) > \nu_A^L(z) = \nu_A^L(x_0 + y_0)$ and

$$\nu_A^U(-y_0) = \nu_A^U(y_0) > \nu_A^U(z) = \nu_A^U(x_0 + y_0).$$

Then by Lemma 5.1,

$$\nu_A^L(x_0) = \nu_A^L(y_0) \text{ and } \nu_A^U(x_0) = \nu_A^U(y_0) \tag{5.7}$$

which implies that

$$\nu_B^L(x_0) < \nu_A^L(x_0) \text{ and } \nu_B^U(x_0) < \nu_A^U(x_0).$$

This is a contradiction to the fact $B \subset A$. Hence

$$B \vee (C \wedge A) = (B \vee C) \wedge A.$$

Therefore, $\mathrm{IVIFI}_0\mathrm{s}(R)$ is modular. □

References

[1] N. Ajmal and K. V. Thomas, The lattices of fuzzy ideals of a ring, Fuzzy Sets and Systems 74 (1995), 371-379.

[2] K. T. Atanassov, Intuitionistic Fuzzy Sets, VII ITKR's Session, Sofia, June 1983. (Deposed in Central Sci.-Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian).

[3] K. T. Atanassov and G. Gargov, Interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31 (1989), 343-349.

[4] M. Akram and W. A. Dudek, Interval-valued intuitionistic fuzzy Lie ideals of Lie algebras, World Appl. Sci. J. 7 (2009), 812-819.

[5] M. Akram, K. H. Dar, B. L. Meng and K. P. Shum, Interval-valued intuitionistic fuzzy ideals of *K*-algebras, WSEAS Trans. Math. 7(9) (2008), 559-568.

[6] R. Biswas, Rosenfeld's fuzzy subgroups with interval-valued membership functions, Fuzzy Sets and Systems 63 (1994), 87-90.

[7] H. Hedayati, Interval-valued intuitionistic $(S; T)$-fuzzy substructures in semi-rings, Int. Math. Forum 4(6) (2009), 293-301.

[8] J. G. Lee and K. H. Kim, Interval-valued intuitionistic $(S; T)$-fuzzy ideals of subtraction algebras, Int. Math. Forum 4(46) (2009), 2281-2291.

[9] K. C. Lee, Kul Hur and Pyung Ki Lim, The lattice of interval-valued fuzzy ideals of a ring, Honam Math. J. 34(3) (2012), 351-373.

[10] K. Meena and K. V. Thomas, Intuitionistic *L*-fuzzy subrings, IMF 6(52) (2011), 2561-2572.

[11] K. Meena and K. V. Thomas, Intuitionistic *L*-fuzzy rings, GJSFR(F) 12(14) (2012), 14-F, Art. 2.

[12] K. Meena and K. V. Thomas, Intuitionistic *L*-fuzzy congruence on a ring, Notes on Intuitionistic Fuzzy Sets 19(1) (2013), 7-20.

[13] K. Meena and K. V. Thomas, Intuitionistic L-fuzzy ideals and congruences on a ring (communicated).

[14] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971), 512-517.

[15] L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965), 338-353.

[16] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning I, Inform. Sci. 8 (1975), 199-249.